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# Parabolic class number of a subgroup of the normalizer of $\Gamma_0(m)$

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# PARABOLIC CLASS NUMBER OF A SUBGROUP OF THE NORMALIZER OF $\Gamma_0(m)$

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*Abstract.* In this study, we introduce a subgroup of the normalizer of $\Gamma_0(m)$ and we calculate parabolic class number of this group.

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*Keywords:* normalizer, parabolic class number, Fuchsian group, orbit

## 1. INTRODUCTION

Let $m$ be a positive integer and let $N(\Gamma_0(m))$ be the normalizer of $\Gamma_0(m)$ in $PSL(2,\mathbb{R})$. The normalizer $N(\Gamma_0(m))$ was studied for the first time by Newman in [8]. A complete description of the elements of $N(\Gamma_0(m))$ is given in [2]. Especially, a necessary and sufficient condition for $N(\Gamma_0(m))$ to act transitively on $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ was given in [1]. Moreover, the signatures and therefore the parabolic class numbers of some subgroups of $N(\Gamma_0(m))$ were calculated in [1], [5], and [6] . In this study, we introduce a subgroup of $N(\Gamma_0(m))$ and we calculate the parabolic class number of this subgroup when $m$ is a square free positive integer.

## 2. PRELIMINARIES

Let $\Lambda$ be a discrete subgroup of $PSL(2,\mathbb{R})$. When $x \in \mathbb{R} \cup \{\infty\}$ is a fixed point of a parabolic element of $\Lambda$, we say that $x$ is a parabolic point of $\Lambda$. We also call a parabolic point of $\Lambda$, a cusp of $\Lambda$.

The proof the following theorem is easy and can be found in [7] or [10].

**Theorem 1.** *Let $\Lambda$ be a discrete subgroup of $PSL(2,\mathbb{R})$ and assume that $\Lambda^*$ is a subgroup of finite index in $\Lambda$. Let $P$ be the set of the parabolic points of $\Lambda$. Then $P$ is the set of the parabolic points of $\Lambda^*$ and the parabolic class number of $\Lambda^*$ is the number of orbits of $\Lambda^*$ on $P$.*

By a Fuchsian group $\Lambda$ we will mean a finitely generated discrete subgroup of $PSL(2,\mathbb{R})$ the group of conformal homeomorphisms of the upper-half plane. The most general presentation for $\Lambda$ is

Generators:

$$a_1, b_1, \cdots, a_g, b_g \ Hyperbolic$$

$$x_1, x_2, \cdots, x_r \ Elliptic$$

$$p_1, p_2, \cdots, p_s \ Parabolic$$

with the relations

$$x_1^{m_1} = x_2^{m_2} = ..... = x_r^{m_r} = \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{r} x_j \prod_{k=1}^{s} p_k = 1.$$

We then say $\Lambda$ has signature ( see [4] )

$$(g, m_1, m_2, ......, m_r; s).$$

$s$ is the number of parabolic classes, i.e., of parabolic generators, and $r$ is the number of elliptic generators. The $m_i$ are the periods of $\Lambda$.

A complete description of the elements of $N(\ \Gamma_0(m))$ is given in [2]. If we represent the elements of $N(\ \Gamma_0(m))$ by the associated matrices, then the normalizer consists exactly of the matrices

$$\begin{pmatrix} ae & b/h \\ cm/h & de \end{pmatrix}$$

where $e \mid (m/h^2)$ such that $(e, m/h^2e) = 1$ and $h$ is the largest divisor of 24 for which $h^2 \mid m$ with the understanding that the determinant of the matrix is $e > 0$. If $e \mid n$ and $(e, n/e) = 1$, we represent this as $e \parallel n$ and we say that $e$ is exact divisor of $n$. Thus we have

$$N(\Gamma_0(m)) = \left\{ \begin{pmatrix} a\sqrt{q} & b/h\sqrt{q} \\ cm/h\sqrt{q} & d\sqrt{q} \end{pmatrix} : \det = 1; 1 \le q; q \parallel \frac{m}{h^2}; a, b, c, d \in \mathbb{Z} \right\}.$$

The following theorem is given in [1] and [11].

**Theorem 2.** *Let $m$ have prime power decomposition $2^{\alpha_1} 3^{\alpha_2} p_3^{\alpha_3} .... p_r^{\alpha_r}$. Then $N(\Gamma_0(m))$ acts transitively on $\hat{\mathbb{Q}}$ if and only if $\alpha_1 \le 7, \alpha_2 \le 3, \alpha_i \le 1, i = 3, 4, ..., r$.*

If $m$ is a square free positive integer, then

$$N(\Gamma_0(m)) = \left\{ \begin{pmatrix} a\sqrt{q} & b/\sqrt{q} \\ cm/\sqrt{q} & d\sqrt{q} \end{pmatrix} : \det = 1; 1 \le q; q \parallel m; a, b, c, d \in \mathbb{Z} \right\}.$$

Let $\Gamma$ be modular group. In [3], we calculated parabolic class number of the subgroup $\Gamma_0(n)\cap\Gamma^N$ such that

$$\Gamma^N=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma : a\equiv d\equiv 0(\bmod\ N)\,\text{or}\, b\equiv c\equiv 0(\bmod\ N)\right\}$$

and

$$\Gamma_0(n)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma : c\equiv 0(\bmod\ n)\right\},$$

where $(N,n)=1$ and $N$ is a power of a prime number. And it is shown that parabolic class number of this group is

$$N\sum_{d|n}\varphi\left(\left(d,\frac{n}{d}\right)\right).$$

Let $(m,n)=(m,N)=(n,N)=1$ and let $\Gamma_0^*(n)$ be defined by

$$\Gamma_0^*(n)=\left\{\begin{pmatrix} a\sqrt{q} & b/h\sqrt{q}\\ cm/h\sqrt{q} & d\sqrt{q}\end{pmatrix}\in N(\Gamma_0(m)) : c\equiv 0(\bmod n)\right\}$$

and let $\Gamma(m,N)$ be defined by

$$\Gamma(m,N)=\{A\in N(\Gamma_0(m)) : a\equiv d\equiv 0(\bmod N)\ or\ b\equiv c\equiv 0(\bmod N)\}.$$

Moreover, we define $\Gamma_0^*(m,n,N)$ by $\Gamma_0^*(m,n,N)=\Gamma(m,N)\cap\Gamma_0^*(n)$. It is clear that $\Gamma_0^*(m,n,1)=\Gamma_0^*(n)$. In this paper, we will show that the parabolic class number of $\Gamma_0^*(m,n,N)$ is

$$N\sum_{d|n}\varphi\left(\left(d,\frac{n}{d}\right)\right)$$

when $(m,n)=(m,N)=(n,N)=1$ and $N$ is a power of a prime number. In the next section, we give five lemmas, which are necessary for our main theorem.

## 3. Main Theorems

From now on, otherwise stated, we will assume that $m$ is a square free positive integer and $(m,n)=(m,N)=(n,N)=1$.

**Lemma 1.** $\left|N(\Gamma_0(m)):\Gamma_0^*(m,n,N)\right|=\frac{|\Gamma_0(m):\Gamma_0(mnN)|}{2}$

*Proof.* Let $q|m$. Since $m$ is square free, $(q,m/q)=1$. Thus it follows that $(qN^2,(m/q)n)=1$. Therefore there exist two integers $x_1$ and $y_1$ such that $qN^2x_1-(mn/q)y_1=1$. Let

$$A_q=\begin{pmatrix} N\sqrt{q} & y_1/\sqrt{q}\\ mn/\sqrt{q} & Nx_1\sqrt{q}\end{pmatrix}.$$

Then $AA_q^{-1}\in\Gamma_0(mnN)$ for any

$$A = \begin{pmatrix} aN\sqrt{q} & b/\sqrt{q} \\ cmn/\sqrt{q} & dN\sqrt{q} \end{pmatrix} \in \Gamma_0^*(m,n,N).$$

Thus $A \in \Gamma_0(mnN)A_q$. On the other hand, since $(q, n(m/q)N^2) = 1$, there exist two integers $x_2$ and $y_2$ such that $qx_2 - n(m/q)N^2 y_2 = 1$. Let

$$B_q = \begin{pmatrix} x_2\sqrt{q} & y_2N/\sqrt{q} \\ mnN/\sqrt{q} & \sqrt{q} \end{pmatrix}.$$

Then we see that $BB_q^{-1} \in \Gamma_0(mnN)$ for any

$$B = \begin{pmatrix} a\sqrt{q} & bN/\sqrt{q} \\ cmnN/\sqrt{q} & d\sqrt{q} \end{pmatrix} \in \Gamma_0^*(m,n,N).$$

Thus it follows that $B \in \Gamma_0(mnN)B_q$. It can be seen that $\Gamma_0(mnN)A_{q_1} \neq \Gamma_0(mnN)A_{q_2}$ and $\Gamma_0(mnN)B_{q_1} \neq \Gamma_0(mnN)B_{q_2}$ for $q_1 \neq q_2$. Moreover,

$$\Gamma_0(mnN)A_{q_1} \neq \Gamma_0(mnN)B_{q_2}$$

for $q_1|m$ and $q_2|m$. Thus it follows that $|\Gamma_0^*(m,n,N) : \Gamma_0(mnN)| < \infty$ and it can be shown that $|\Gamma_0^*(m,n,N) : \Gamma_0(mnN)| = 2^{t+1}$ where $t$ is the number of the prime divisor of $m$. Also we have

$$\begin{aligned} |N(\Gamma_0(m)) : \Gamma_0^*(m,n,N)| &= \frac{|N(\Gamma_0(m)) : \Gamma_0(mnN)|}{|\Gamma_0^*(m,n,N) : \Gamma_0(mnN)|} \\ &= \frac{|N(\Gamma_0(m)) : \Gamma_0(m)||\Gamma_0(m) : \Gamma_0(mnN)|}{2^{t+1}} \end{aligned}$$

Since $|N(\Gamma_0(m)) : \Gamma_0(m)| = 2^t$, the proof follows. □

In view of the above lemma, it is seen that $\Gamma_0^*(m,n,N)$ is finitely generated and therefore it is a Fuchsian group.

**Lemma 2.** *Let $N = p^r$ where $p$ is a prime number and let $k/s \in \hat{\mathbb{Q}}$ with $p|s$. Then there exist some $T \in \Gamma_0^*(m,n,N)$ such that $T(k/s) = k_1/s_1$ with $(s_1, N) = 1$.*

*Proof.* Since $(m,N) = (n,N) = 1$, we see that $(mn, N^2) = 1$ and therefore there exist two integers $x$ and $y$ such that $N^2x - mny = 1$. Let

$$T = \begin{pmatrix} xN & y \\ mn & N \end{pmatrix}.$$

Then $T \in \Gamma_0^*(m,n,N)$ and $T(k/s) = \frac{xNk+ys}{mnk+sN} = k_1/s_1$ with $s_1 = mnk + sN$. It can be seen easily that $(s_1, N) = 1$. □

**Lemma 3.** *Let $(k,s) = (s,N) = 1$. Then there exist some $T \in \Gamma_0^*(m,n,N)$ such that $T(k/s) = k_1/s_1$ with $s_1|n$.*

*Proof.* Let $s_1 = (s,n)$ and $q = (s,m)$. Then $s_1 = (s,kn) = (s,knN)$. Moreover $(s,m/q) = 1$, since $m$ is square free. Therefore $(s,knN\frac{m}{q}) = s_1$ and thus there exist two integers $c_1$ and $d_1$ such that $sd_1 + kn\frac{m}{q}Nc_1 = s_1$. Since $(d_1,(knm/s_1q)N) = 1$, there exist some integer $k_0$ such that

$$(d_1 - \frac{knm}{s_1q}Nk_0,(m/q)nN^2) = 1.$$

Let

$$d = d_1 - \frac{knm}{s_1q}Nk_0$$

and

$$c = c_1 + \frac{s}{s_1}k_0.$$

Then $kn(m/q)Nc + sd = s_1$. On the other hand, $(dq,cN^2(m/q)n) = 1$ since $(d,c) = (d,(m/q)nN^2) = 1$. It can be seen easily that $(q,c) = 1$. Therefore $(dq,c(m/q)nN^2) = 1$. Thus there exist two integers $x$ and $y$ such that $xdq - c(m/q)nN^2y = 1$. Let

$$T = \begin{pmatrix} x\sqrt{q} & yN/\sqrt{q} \\ cmnN/\sqrt{q} & d\sqrt{q} \end{pmatrix}.$$

Then $T \in \Gamma_0^*(m,n,N)$ and

$$T(k/s) = \frac{qxk + sNy}{cknmN + dsq} = \frac{qxk + sNy}{q(ckn(m/q)N + ds)} = \frac{xk + N(s/q)y}{(ckn(m/q)N + ds)} = \frac{k_1}{s_1}.$$

□

**Lemma 4.** *Let* $(a_1,d_1) = (a_2,d_1) = 1$ *and* $d_1|n$. *Then* $a_1/d_1$ *is conjugate to* $a_2/d_1$ *under* $\Gamma_0^*(m,n,N)$ *if and only if* $a_1 \equiv a_2 (\bmod Nt)$ *where* $t = (d_1,n/d_1)$.

*Proof.* Suppose that $a_1/d_1$ is conjugate to $a_2/d_1$ under $\Gamma_0^*(m,n,N)$. Then there exist some $T \in \Gamma_0^*(m,n,N)$ such that $T(a_1/d_1) = a_2/d_1$. Assume that

$$T = \begin{pmatrix} a\sqrt{q} & bN/\sqrt{q} \\ cmnN/\sqrt{q} & d\sqrt{q} \end{pmatrix},$$

where $q|m$. Then we have

$$\frac{aa_1q + bNd_1}{cnmNa_1 + dd_1q} = \frac{a_2}{d_1}$$

and

$$adq - bcn(m/q)N^2 = 1.$$

Since

$$a(cmnNa_1 + dd_1q) - \frac{cmnN}{q}(aa_1q + bd_1N) = d_1,$$

$$dq(aa_1q + bd_1N) - bN(cmnNa_1 + dd_1q) = a_1q,$$

and $(d_1,a_1q)=1$, we see that $(aa_1q+bd_1N, cmnNa_1+dd_1q)=1$. Therefore there exist some $u=\pm1$ such that

$$ua_2=aa_1q+bNd_1$$

and

$$ud_1=cmnNa_1+dd_1q.$$

Since $q(cn(m/q)Na_1+dd_1)=ud_1$ and $(q,d_1)=1$, it follows that $q=1$. Thus

$$ua_2=aa_1+bNd_1,$$

$$ud_1=cmnNa_1+dd_1,$$

and

$$ad-bcnmN^2=1$$

From the above equations, we see that $ua_2\equiv aa_1(\bmod tN)$, $u\equiv d(\bmod tN)$, and $ad\equiv 1(\bmod tN)$. Therefore $a_1\equiv u^2a_2(\bmod tN)$, which implies that $a_1\equiv a_2(\bmod tN)$. Now assume that

$$T=\begin{pmatrix} aN\sqrt{q} & b/\sqrt{q} \\ cmn/\sqrt{q} & dN\sqrt{q} \end{pmatrix},$$

where $adN^2q-bcn(m/q)=1$. Then we obtain

$$\frac{aa_1Nq+bd_1}{cnma_1+dd_1qN}=\frac{a_2}{d_1}.$$

Similarly, it is seen that $(cnma_1+dd_1qN, aa_1Nq+bd_1)=1$. Thus there exist some $u=\pm1$ such that

$$ua_2=aa_1Nq+bd_1$$

and

$$ud_1=cmna_1+dd_1qN.$$

Since $ud_1=q(cn(m/q)a_1+dd_1N)$ and $(q,d_1)=1$, it follows that $q=1$. Thus

$$ua_2=aa_1N+bd_1,$$

$$ud_1=cmna_1+dd_1N,$$

and

$$adN^2-bcnm=1.$$

From the above equations, we get

$$ua_2\equiv bd_1(\bmod N),$$

$$ud_1\equiv cmna_1(\bmod N),$$

and

$$bcmn\equiv 1(\bmod N).$$

This implies that $a_1\equiv a_2(\bmod N)$. In the same way we see that

$$ua_2\equiv aa_1N(\bmod t),$$

$$u \equiv dN (\bmod t),$$

and

$$adN^2 \equiv 1 (\bmod t).$$

Thus we have $a_1 \equiv a_2 (\bmod t)$. Since$(t,N) = 1$, it follows that $a_1 \equiv a_2 (\bmod tN)$.

Now suppose that $a_1 \equiv a_2 (\bmod tN)$, where $t = (d_1, n/d_1)$. Let $n_1 = n/d_1$. Then $t = (d_1, n_1)$ and $(a_1 a_2, d_1) = 1$. Therefore $t = (a_1 a_2 n_1, d_1)$. This shows that $tN = (d_1 N, a_1 a_2 n_1 N)$. Since $tN | a_2 - a_1$, there exist two integers $x$ and $y$ such that $n_1 a_1 a_2 N x + d_1 N y = a_2 - a_1$. Thus we obtain $a_1(n_1 a_2 N x + 1) + d_1 N y = a_2$. If we take $b = y$ and $a = n_1 a_2 N x + 1$, we have $aa_1 + d_1 N b = a_2$. Let $c = n_1 d_1 x = nx$ and $d = 1 - N n_1 a_1 x$. Then $cNa_1 + dd_1 = d_1$. On the other hand, we see that

$$\begin{aligned} ad - bcN^2 &= a(1 - N n_1 a_1 x) - b n_1 d_1 x N^2 \\ &= a - (aa_1 + d_1 N b) N n_1 x = a - n_1 a_2 N x = 1. \end{aligned}$$

Let

$$T(z) = \frac{az + bN}{cmNz + d}.$$

Then $n|c$ and therefore $T \in \Gamma_0^*(m,n,N)$. Moreover, we get

$$T(a_1/d_1) = \frac{aa_1 + bd_1 N}{cNa_1 + dd_1} = a_2/d_1.$$

□

The proof of the following lemma is easy and will be omitted.

**Lemma 5.** *Let $(a_1, d_1) = (a_2, d_1) = 1$, $d_1 | n$, $d_2 | n$, where $d_1$ and $d_2$ are positive integers. If $a_1/d_1$ is conjugate to $a_2/d_2$ under $\Gamma_0^*(m,n,N)$, then $d_1 = d_2$.*

In the following, we give a lemma without proof, which we use later. The lemma appears in [9], page 73, as problem.

**Lemma 6.** *Let $m$ and $k$ be positive integers. Then the number of the positive integers $\leq mk$ that are prime to $m$ is $k\varphi(m)$.*

**Theorem 3.** *Let $p$ be a prime number and $N = p^r$. Then the parabolic class number of $\Gamma_0^*(m,n,N)$ is*

$$N \sum_{d|n} \varphi\left(\left(d, \frac{n}{d}\right)\right)$$

*Proof.* It suffices to calculate the number of orbits of $\Gamma_0^*(m,n,N)$ on $\hat{\mathbb{Q}}$. By Lemma 4 and Lemma 6, there exist $N\varphi((d, n/d))$ different orbits for each $d | n$. Then from Lemma 2 to Lemma 5, it follows that the number of orbits of $\Gamma_0^*(m,n,N)$ on $\hat{\mathbb{Q}}$ is

$$\sum_{d|n} N\varphi\left(\left(d, \frac{n}{d}\right)\right) = N \sum_{d|n} \varphi\left(\left(d, \frac{n}{d}\right)\right)$$

□

**Corollary 1.** *Let $p$ be a prime number. Then the parabolic class number of $\Gamma(m, p^k)$ is $p^k$.*

**Corollary 2.** *The parabolic class number of $\Gamma_0^*(n)$ is*

$$\sum_{d|n} \varphi\left(\left(d, \frac{n}{d}\right)\right).$$

*Let $r$ be a natural number and let*

$$\Gamma_0^+(r) = \left\{ \begin{pmatrix} a\sqrt{q} & b/\sqrt{q} \\ cr/\sqrt{q} & d\sqrt{q} \end{pmatrix} : adq - bcr/q = 1; 1 \leq q;\ q \parallel r; a,b,c,d \in \mathbb{Z} \right\}.$$

*Then $\Gamma_0^+(r)$ is a subgroup of the normalizer of $\Gamma_0(r)$. Moreover, any element of $\Gamma_0^+(r)$ is an Atkin-Lehner involution of $\Gamma_0(r)$. Recall that an Atkin-Lehner involution $w_q$ of $\Gamma_0(r)$ is an element of determinant 1 of the form*

$$w_q = \begin{pmatrix} a\sqrt{q} & b/\sqrt{q} \\ cr/\sqrt{q} & d\sqrt{q} \end{pmatrix}$$

*for some exact divisor $q$ of $r$. If $h = 1$, then $\Gamma_0^+(r)$ is equal to $N(\Gamma_0(r))$, where $h$ is the largest divisor of 24 such that $h^2|r$. The following lemmas are well known and their proofs can be found in [11].*

**Lemma 7.** *$\Gamma_0^+(r)$ acts transitively on the set $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ if and only if $r$ is a square-free positive integer.*

**Lemma 8.** *Let $r = m/h^2$. Then $N(\Gamma_0(m))$ acts transitively on the set $\hat{\mathbb{Q}}$ if and only if $\Gamma_0^+(r)$ acts transitively on the set $\hat{\mathbb{Q}}$.*

Let $r = m/h^2$. Then, it is well-known and easy to see that

$$N(\Gamma_0(m)) = \begin{pmatrix} 1/\sqrt{h} & 0 \\ 0 & \sqrt{h} \end{pmatrix} \Gamma_0^+(r) \begin{pmatrix} 1/\sqrt{h} & 0 \\ 0 & \sqrt{h} \end{pmatrix}^{-1}.$$

Assume that $N(\Gamma_0(m))$ acts transitively on the set $\hat{\mathbb{Q}}$. Taking $r = m/h^2$ and

$$X = \begin{pmatrix} 1/\sqrt{h} & 0 \\ 0 & \sqrt{h} \end{pmatrix},$$

it is immediate that $XHX^{-1}$ is equal to $\Gamma_0^*(m,n,N)$, provided that $H = \Gamma_0^*(r,m,N)$. Therefore, we can give the following corollary easily.

**Corollary 3.** *Suppose that $N(\Gamma_0(m))$ acts transitively on the set $\hat{\mathbb{Q}}$ and $N = p^k$ for some prime $p$. Then parabolic class number of $\Gamma_0^*(m,n,N)$ is*

$$N \sum_{d|n} \varphi\left(\left(d, \frac{n}{d}\right)\right).$$

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