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NUMERICAL OSCILLATION OF NONLINEAR GENERALIZED DELAY SINGLE SPECIES POPULATION MODEL

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Abstract. In this paper, we mainly consider the oscillation of numerical solutions for a nonlinear delay differential equation which is generalized from a delay Lotka-Volterra type single species population growth model. By studying the corresponding difference scheme of the equation discretized by θ-method, forward Euler method and backward Euler method, some sufficient conditions under which the numerical solutions oscillate are obtained. Furthermore, we prove that the positive non-oscillatory numerical solutions tend to the equilibrium of the original differential equation. Finally, some numerical experiments are given to verify the theoretical results.

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Keywords: nonlinear differential equation, delay differential equation, θ-method, forward Euler method, backward Euler method, oscillation

1. Introduction

In recent years, the reason why the properties of solutions of nonlinear delay differential equations (NDDEs) are getting more and more attention is that this type of equation has been widely used to describe phenomena in various fields of science such as life science, biology, ecology and physics [1, 3, 7, 14, 15, 17]. Many researchers hold strong interest in all kinds of behaviors of solutions of NDDEs such as stability [24], periodicity [4], hopf bifurcation [25] and oscillation [11, 13]. Relative to the oscillation of analytic solutions, it is necessary to study the oscillation of the corresponding numerical solutions. Because only those numerical methods that preserve the inherent properties of the continuous model are meaningful and valuable. Nowadays, some results on the numerical oscillation of NDDEs have been published in [18–20]. Different from them, in this paper, we consider the following NDDEs

$$\dot{x}(t) = x(t)(a + bx^p(t-\tau) - cx^q(t-\tau)), \quad t \geq 0, \tag{1.1}$$

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which is called generalized single species population model. The parameters in (1.1) satify

$$a, c, p, q, \tau > 0, b \in \mathbb{R}, \quad p < q. \tag{1.2}$$

We consider the solutions of (1.1) with initial condition of the form

$$x(t) = \varphi(t), \quad -\tau \leq t \leq 0,$$

where $\varphi \in C([-\tau, 0], (0, \infty))$ and $\varphi(0) > 0$.

In [12], Ladas and Qian studied globally asymptotically stablity for every positive solution of (1.1). After that, some extended forms of (1.1) were considered. The global attractive and the oscillation of a unique positive periodic solution of delay periodic logistic equation were studied in [22], sufficient conditions were obtained for the existence and global attractivity of positive periodic solution of an impulsive DDE with Allee effect [23], the existence of a positive periodic solution of the delayed periodic logistic equation was established in [5], and some corresponding discrete models were studied also [6].

In particular, when $p = 1$ and $q = 2$, then (1.1) gives

$$\dot{x}(t) = x(t)(a + bx(t - \tau) - cx^2(t - \tau)), \tag{1.3}$$

which is a delay Lotka-Volterra type single species population growth model proposed by Gopalsamy and Ladas [9]. As a case in exhibition of the Allee effect [21], (1.3) means the positive feedback effects of aggregation and cooperation are dominated by density-dependent stabilizing negative feedback effects due to intraspecific competition when the density of the population is higher than a critical value. In this model, the per capita growth is expressed as a quadratic function, related either at present or in history, of the density. Specifically, this per-capita growth rate can be interpreted as a ”first order” nonlinear approximation of more general types of plausible nonlinear growth rates with single humps. As for the research on the properties of (1.3), the interested readers can refer to [8].

The definitions of oscillatory for solutions of differential equation and difference equation can be found in [2], next we list some helpful results which will be used in the upcoming analysis.

Theorem 1 ([10, Corollary 7.1.1])**.** *Consider the difference equation*

$$a_{n+1} - a_n + \sum_{j=-k}^{l} q_j a_{n+j} = 0, \tag{1.4}$$

assume that $k, l \in \mathbb{N}$ and $q_j \in \mathbb{R}$ for $j = -k, \ldots, l$. Then the following statements are equivalent:

(i) *Every solution of (1.4) oscillates.*
(ii) *The characteristic equation $r - 1 + \sum_{j=-k}^{l} q_j r^j = 0$ has no positive roots.*

Theorem 2 ([10, Theorem 7.2.1]). *For the difference equation*

$$a_{n+1} - a_n + l a_{n-k} = 0, \tag{1.5}$$

where $l \in \mathbb{R}$, $k \in \mathbb{Z}$. Then every solution of (1.5) oscillates if and only if one of the following conditions holds:

(i) $k = -1$ *and* $l \le -1$*;*
(ii) $k = 0$ *and* $l \ge 1$*;*
(iii) $k \in \{\dots, -3, -2\} \cup \{1, 2, \dots\}$ *and* $l(k+1)^{k+1}/k^k > 1$.

Lemma 1 ([12, Theorem 1]). *Assume that (1.2) holds, then every solution of (1.1) oscillates about x^* if and only if*

$$(qc(x^*)^q - pb(x^*)^p)\tau > \frac{1}{e}, \tag{1.6}$$

where x^* is the positive equilibrium of (1.1).

2. Oscillations of numerical solutions

In this section, we will apply the θ-method to the simplified form of (1.1), then by analyzing the characteristic equation of the corresponding difference scheme we get the conditions under which the numerical solutions oscillate for two different ranges of parameter θ.

At first, making the change of variable $x(t) = x^* e^{y(t)}$, (1.1) is reformulated as

$$\dot{y}(t) + (qc(x^*)^q - pb(x^*)^p) g(y(t-\tau)) = 0, \tag{2.1}$$

where

$$g(v) = \frac{c(x^*)^q(e^{qv} - 1) - b(x^*)^p(e^{pv} - 1)}{qc(x^*)^q - pb(x^*)^p},$$

so the linearized form of (2.1) is

$$\dot{z}(t) + (cq(x^*)^q - bp(x^*)^p) z(t-\tau) = 0. \tag{2.2}$$

In the rest paper, we mainly investigate the conditions under which the numerical solution of (1.1) is oscillatory when (1.6) is satisfied. In this situation, the θ-method preserves the oscillation of the analytic solution of (1.1). Therefore this numerical method is meaningful and available.

Let $h = \tau/m$ be a given stepsize with integer $m \in \mathbb{N}^+$. Application of the θ-method to (2.1) gives

$$\begin{aligned} y_{n+1} = y_n &- h\theta(qc(x^*)^q - pb(x^*)^p) g(y_{n+1-m}) \\ &- h(1-\theta)(qc(x^*)^q - pb(x^*)^p) g(y_{n-m}), \end{aligned} \tag{2.3}$$

where $\theta \in [0, 1]$, y_{n+1} and y_{n+1-m} are approximations to $y(t)$ and $y(t-\tau)$ of (2.1) at t_{n+1}, respectively.

Let $y_n = \ln(x_n/x^*)$, we have the difference scheme for (1.1)

$$\begin{aligned} x_{n+1} = x_n \exp\Big\{ & h\theta \left[c\left((x^*)^q - x_{n+1-m}^q\right) + b\left(x_{n+1-m}^p - (x^*)^p\right)\right] \\ & + h(1-\theta)\left[c\left((x^*)^q - x_{n-m}^q\right) + b\left(x_{n-m}^p - (x^*)^p\right)\right]\Big\}. \end{aligned} \tag{2.4}$$

Next, we consider whether the θ-method preserves the oscillation of (1.1). Thus, we study the oscillation of (2.3) to get the conditions under which (2.4) oscillates.

On the other hand, using the θ-method to (2.2) yields difference equation

$$\begin{aligned} z_{n+1} = z_n & - h\theta(qc(x^*)^q - pb(x^*)^p)z_{n+1-m} \\ & - h(1-\theta)(qc(x^*)^q - pb(x^*)^p)z_{n-m}, \end{aligned} \tag{2.5}$$

which is exactly the linearized form of (2.3).

For simplicity, let $\alpha = qc(x^*)^q$ and $\beta = pb(x^*)^p$, then (1.6) and (2.5) reduces to the following simplified inequality

$$(\alpha - \beta)\tau > \frac{1}{e} \tag{2.6}$$

and

$$z_{n+1} = z_n - h\theta(\alpha-\beta)z_{n+1-m} - h(1-\theta)(\alpha-\beta)z_{n-m}, \tag{2.7}$$

respectively.

Lemma 2. *The characteristic equation of (2.5) is given by*

$$r = S\left(-h(\alpha-\beta)r^{-m}\right). \tag{2.8}$$

Proof. Substituting $z_n = r^n z_0$ into (2.7), and according to the expression of the stability function of the θ-method, we can get this proof. □

The next lemma is for the first case of θ which is in the interval $[0, 0.5]$.

Lemma 3. *If (1.2) and (2.6) hold, then for $\theta \in [0, 0.5]$, (2.8) has no positive roots.*

Proof. We introduce $D(r) = r - S\left(-h(\alpha-\beta)r^{-m}\right)$. On the other hand, from Lemma 3 in [16] we have

$$S\left(-h(\alpha-\beta)r^{-m}\right) \le \exp\left(-h(\alpha-\beta)r^{-m}\right).$$

In the following, we shall prove $E(r) = r - \exp\left(-h(\alpha-\beta)r^{-m}\right) > 0$ for all $r > 0$. We consider it by using the proof by contradiction. Suppose that there exists a $r_0 > 0$ such that $E(r_0) \le 0$, then

$$r_0 \le \exp\left(-h(\alpha-\beta)r_0^{-m}\right),$$

hence

$$r_0^m \le \exp\left(-\tau(\alpha-\beta)r_0^{-m}\right),$$

which, rearranging terms, writes also as

$$\tau(\alpha-\beta)e \le \tau(\alpha-\beta)r_0^{-m}\exp\left(1-\tau(\alpha-\beta)r_0^{-m}\right).$$

We examine it in two cases:

- If $1-\tau(\alpha-\beta)r_0^{-m}=0$, then $\tau(\alpha-\beta)e\le 1$, which contradicts to (2.6).
- If $1-\tau(\alpha-\beta)r_0^{-m}\ne 0$, by the fact $e^x<1/(1-x)$ we have

$$\exp\left(1-\tau(\alpha-\beta)r_0^{-m}\right)<\frac{1}{\tau(\alpha-\beta)r_0^{-m}},$$

that is $\tau(\alpha-\beta)e\le 1$, which also contradicts to (2.6).

Therefore, the following inequality is true for all $r>0$,

$$D(r)=r-S\left(-h(\alpha-\beta)r^{-m}\right)\ge r-\exp\left(-h(\alpha-\beta)r^{-m}\right)=E(r)>0,$$

the proof is finished. □

For the second case: $\theta\in(0.5,1]$, we might as well assume $m>1$.

Lemma 4. *If (1.2) and (2.6) hold, then for $\theta\in(0.5,1]$, (2.8) has no positive roots for $h<h_0$, where*

$$h_0=\begin{cases}\infty, & (\alpha-\beta)\tau\ge 1;\\ \tau(1+\ln(\alpha-\beta)), & (\alpha-\beta)\tau<1.\end{cases}\tag{2.9}$$

Proof. We only to prove that $D(r)>0$. Note that, the function $S\left(-h(\alpha-\beta)r^{-m}\right)$ is increasing about θ if $r>0$, that is

$$S\left(-h(\alpha-\beta)r^{-m}\right)=\frac{1-h(1-\theta)(\alpha-\beta)r^{-m}}{1+h\theta(\alpha-\beta)r^{-m}}\le\frac{1}{1+h(\alpha-\beta)r^{-m}}.$$

Next, we are going to prove

$$r-\frac{1}{1+h(\alpha-\beta)r^{-m}}>0\tag{2.10}$$

holds conditionally. Rearranging terms on the left of (2.10) gives

$$r-\frac{1}{1+h(\alpha-\beta)r^{-m}}=\frac{r^{1-m}}{1+h(\alpha-\beta)r^{-m}}\left(r^m-r^{m-1}+h(\alpha-\beta)\right).$$

Denoting

$$G(r)=r^m-r^{m-1}+h(\alpha-\beta),$$

then $G(r)$ is the characteristic polynomial of the difference equation

$$v_{n+1}-v_n+h(\alpha-\beta)v_{n+1-m}=0.$$

Next we aim to prove $G(r)>0$ for $r>0$. By means of Theorems 1 and 2, we know that $G(r)$ has no positive roots if and only if

$$h(\alpha-\beta)\frac{m^m}{(m-1)^{m-1}}>1,$$

which can be restated as

$$\ln(\alpha-\beta)\tau+(m-1)\ln\frac{m}{m-1}>0.\tag{2.11}$$

Therefore we have the following two cases:

- If $(\alpha-\beta)\tau \geq 1$, then (2.11) holds naturally for $m > 1$;
- If $(\alpha-\beta)\tau < 1$, in view of (2.9) as well as the fact $\ln(1+x) > x/(1+x)$ we know that (2.11) also holds.

Thus (2.10) is true for $h < h_0$. Then we arrive at

$$D(r) = r - S\left(-h(\alpha-\beta)r^{-m}\right) > r - \frac{1}{1+h(\alpha+\beta)r^{-m}} > 0,$$

which indicates that (2.8) has no positive roots. This proof is completed. □

We therefore have the following theorem.

Theorem 3. *If (1.2) and (2.6) hold, then (2.4) is oscillatory for*

$$h < \begin{cases} \infty & \text{if } \theta \in [0, 0.5], \\ h_0 & \text{if } \theta \in (0.5, 1], \end{cases}$$

where h_0 is defined by (2.9).

3. Asymptotic behavior of non-oscillatory solution

In this section, we study the asymptotic behavior of non-oscillatory numerical solution of (1.1). That is, we aim to prove that the non-oscillatory solution of (2.4) approaches the equilibrium point.

Lemma 5 ([12, Lemma 4])**.** *Let $x(t)$ be a positive solution of (1.1) which does not oscillate about x^*, then $\lim\limits_{t\to\infty} x(t) = x^*$.*

Lemma 6. *Let y_n be a non-oscillatory solution of (2.3), then $\lim\limits_{n\to\infty} y_n = 0$.*

Proof. Assume $y_n > 0$ for n sufficiently large. The case of $y_n < 0$ is similar and be omitted. To begin with, we notice that

$$0 = a + b(x^*)^p - c(x^*)^q,$$

together with (1.2) which implies

$$\alpha - \beta = qc(x^*)^q - pb(x^*)^p > q(c(x^*)^q - b(x^*)^p) = qa > 0. \tag{3.1}$$

From (2.3) we have

$$y_{n+1} - y_n = -h\theta(\alpha-\beta)g(y_{n+1-m}) - h(1-\theta)(\alpha-\beta)g(y_{n-m}) < 0, \tag{3.2}$$

then $\{y_n\}$ is decreasing. Hence there is a $\zeta > 0$ such that

$$\lim_{n\to\infty} y_n = \zeta. \tag{3.3}$$

In order to prove that $\zeta = 0$, in contrast we assume $\zeta > 0$, then there exists $\widetilde{N} \in \mathbb{N}^+$ and any given $\varepsilon > 0$, such that for $n - m > \widetilde{N}$, $0 < \zeta - \varepsilon < y_n < \zeta + \varepsilon$. So $y_{n-m} > \zeta - \varepsilon$ and $y_{n+1-m} > \zeta - \varepsilon$. Thus (3.2) gives

$$\begin{aligned} y_{n+1} - y_n &= -h\theta(\alpha - \beta)g(y_{n+1-m}) - h(1-\theta)(\alpha - \beta)g(y_{n-m}) \\ &< -h\theta(\alpha - \beta)g(\zeta - \varepsilon) - h(1-\theta)(\alpha - \beta)g(\zeta - \varepsilon) \\ &= -h(\alpha - \beta)g(\zeta - \varepsilon) < 0, \end{aligned}$$

results in $y_{n+1} - y_n < H < 0$, where

$$H = h(b(x^*)^p(e^{p(\zeta-\varepsilon)} - 1) - c(x^*)^q(e^{q(\zeta-\varepsilon)} - 1)).$$

Thus $y_n \to -\infty$ as $n \to \infty$, which contradicts to (3.3). The proof is complete. $\square$

Based on Lemma 6, we obtain the following theorem.

Theorem 4. *Let x_n be a positive solution of (2.4), which does not oscillate about x^*, then $\lim_{n\to\infty} x_n = x^*$.*

4. The forward Euler method and the backward Euler method

In this section, we shall carry out the analysis of oscillation and non-oscillation of (1.1) in the case of discretization by the forward Euler method and the backward Euler method. The results for the numerical oscillation and the asymptotic behavior will be given in detail.

4.1. *The forward Euler method*

Let $h = \tau/m$ be a given stepsize with $m \in \mathbb{N}^+$. Apply the forward Euler method to (2.1) and (2.2) gives

$$y_{n+1} = y_n - h(qc(x^*)^q - pb(x^*)^p)g(y_{n-m}) \tag{4.1}$$

and

$$z_{n+1} = z_n - h(qc(x^*)^q - pb(x^*)^p)z_{n-m}, \tag{4.2}$$

respectively.

Let $y_n = \ln(x_n/x^*)$ in (4.1), we have the discrete scheme for (1.1)

$$x_{n+1} = x_n \exp\left\{h\left[c\left((x^*)^q - x_{n-m}^q\right) + b\left(x_{n-m}^p - (x^*)^p\right)\right]\right\}. \tag{4.3}$$

Lemma 7. *If (1.2) and (2.6) hold, then the characteristic equation of (4.2) has no positive roots.*

Proof. Set $z_n = r^n z_0$ in (4.2), we get the characteristic equation of (4.2) as follows

$$r = 1 - h(\alpha - \beta)r^{-m}. \tag{4.4}$$

Denoting

$$F(r) = r - \left(1 - h(\alpha - \beta)r^{-m}\right),$$

next we will discuss in two cases.

(i) When $r \geq 1$, from (3.1) we know that $F(r) > 0$, so (4.4) has no positive roots in $[1, +\infty)$.

(ii) When $0 < r < 1$, in view of $1 + x < e^x$ for all $x > 0$, we get

$$1 - h(\alpha - \beta)r^{-m} < e^{-h(\alpha-\beta)r^{-m}},$$

then

$$F(r) = r - \left(1 - h(\alpha - \beta)r^{-m}\right) > r - e^{-h(\alpha-\beta)r^{-m}},$$

in order to prove $F(r) > 0$, we only need to prove $r > e^{-h(\alpha-\beta)r^{-m}}$, that is

$$\ln r > -h(\alpha - \beta)r^{-m}.$$

Set

$$\Gamma(r) = \ln r + h(\alpha - \beta)r^{-m},$$

from

$$\frac{d\Gamma}{dr} = \frac{r^m - \tau(\alpha - \beta)}{r^{m+1}} = 0$$

we have

$$r = (\tau(\alpha - \beta))^{\frac{1}{m}}.$$

So it is easy to know that

$$\min_{0<r<1} \Gamma(r) = \Gamma((\tau(\alpha - \beta))^{\frac{1}{m}}) = \frac{1 + \ln(\tau(\alpha - \beta))}{m},$$

in terms of (2.6) we have $\Gamma(r) > 0$ for $0 < r < 1$, thus $F(r) > 0$ for $0 < r < 1$. So (4.4) has no positive roots in $(0, 1)$. The proof is finished.

□

Theorem 5. *If (1.2) and (2.6) hold, then (4.3) is oscillatory for any $h > 0$.*

Similar to Lemma 6 and Theorem 4 we have the following theorem.

Theorem 6. *Let x_n be a positive solution of (4.3), which does not oscillate about x^*, then $\lim\limits_{n\to\infty} x_n = x^*$.*

4.2. *The backward Euler method*

Using the similar technique in Section 4.1, we obtain the corresponding results for the backward Euler method.

Let $h = \tau/m$ ($m \in \mathbb{N}^+$), apply the backward Euler method to (2.1) and (2.2) gives

$$y_{n+1} = y_n - h(qc(x^*)^q - pb(x^*)^p)g(y_{n+1-m}) \tag{4.5}$$

and

$$z_{n+1} = z_n - h(qc(x^*)^q - pb(x^*)^p)z_{n+1-m}, \tag{4.6}$$

respectively.

Let $y_n = \ln(x_n/x^*)$ in (4.5), we have the discrete scheme for (1.1)

$$x_{n+1} = x_n \exp\left\{h\left[c\left((x^*)^q - x_{n+1-m}^q\right) + b\left(x_{n+1-m}^p - (x^*)^p\right)\right]\right\}. \tag{4.7}$$

Lemma 8. *If (1.2) and (2.6) hold, then the characteristic equation of (4.6) has no positive roots for $h < \bar{h}$, here*

$$\bar{h} = \tau - \frac{1}{(\alpha - \beta)e}. \tag{4.8}$$

Theorem 7. *If (1.2) and (2.6) hold, then (4.7) is oscillatory for $h < \bar{h}$, here $\bar{h}$ is defined in (4.8).*

Theorem 8. *Let x_n be a positive solution of (4.7), which does not oscillate about x^*, then $\lim\limits_{n\to\infty} x_n = x^*$.*

5. Numerical experiments

In this section, we present some examples to verify the above theoretical results. First of all, we consider the equation

$$\dot{x}(t) = x(t)\left(3 - 10x(t-0.27) - 10x^2(t-0.27)\right), \quad t \geq 0 \tag{5.1}$$

with initial condition $\varphi(t) = 1$ for $-0.27 \leq t \leq 0$. Since $a = 3$, $b = -10$, $c = 10$, $p = 1$, $q = 2$ and $\tau = 0.27$, then we compute that $x^* \approx 0.2416$ and $1/e < (\alpha - \beta)\tau \approx 0.9676 < 1$, so (2.6) is satisfied, which indicates the analytic solution of (5.1) is oscillatory. From Theorem 3 we know that the numerical solution of (5.1) should be oscillatory if we choose $\theta = 0.4 \in [0, 0.5], m = 5$ and $h = \tau/m = 0.0540 < \infty$. At the same time, the numerical solution of (5.1) should also be oscillatory if we choose $\theta = 0.7 \in (0.5, 1]$, $m = 7$ and $h = \tau/m \approx 0.0385 < h_0 = \tau(1 + \ln(\alpha - \beta)) \approx 0.6146$. We draw the figures of the analytic solution and numerical solution of (5.1) in each of Figures 1 and 2 with the parameters mentioned above. From these figures we can

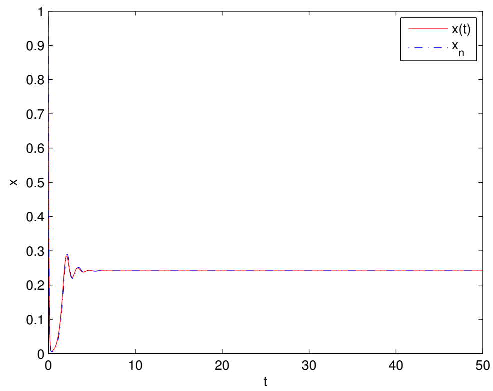

FIGURE 1. The analytic solution and the numerical solution of (5.1) with $m = 5$ and $\theta = 0.4$.

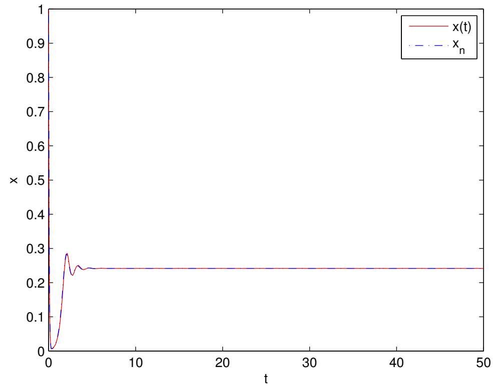

FIGURE 2. The analytic solution and the numerical solution of (5.1) with $m = 7$ and $\theta = 0.7$.

see that the numerical solution of (5.1) oscillates about x^*. That is, the numerical phenomena are consistent with Theorem 3.

Secondly, for another equation

$$\dot{x}(t) = x(t)\left(\frac{1}{11} + 2x(t-2) - 7x^2(t-2)\right), \quad t \geq 0 \tag{5.2}$$

with initial condition $\varphi(t) = 0.5$ for $-2 \leq t \leq 0$. It is easy to get that $x^* \approx 0.3256$ and $1/e < 1 < (\alpha - \beta)\tau \approx 1.6660$, so (2.6) holds, then the analytic solution of (5.2) is oscillatory. We set $\theta = 0.5 \in [0, 0.5]$, $m = 10$, $h = 0.2 < \infty$ in Figure 3 and $\theta = 0.55 \in (0.5, 1]$, $m = 20$, $h = 0.1 < h_0 = \infty$ in Figure 4, respectively. From the two

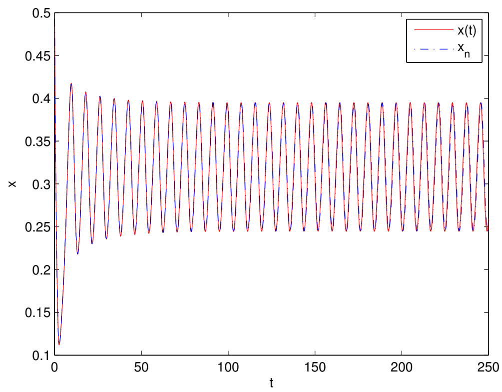

FIGURE 3. The analytic solution and the numerical solution of (5.2) with $m = 10$ and $\theta = 0.5$.

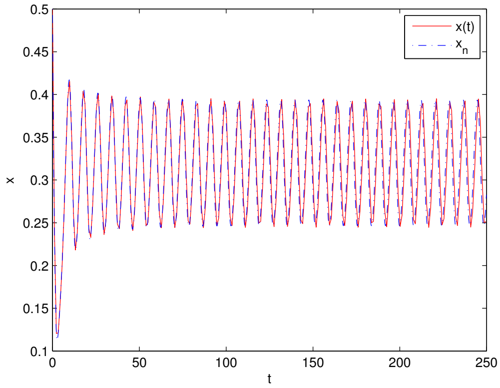

FIGURE 4. The analytic solution and the numerical solution of (5.2) with $m = 20$ and $\theta = 0.55$.

figures we see that the numerical solution of (5.2) oscillates about x^*, which are in agreement with Theorem 3.

Thirdly, we give an example to verify the asymptotic behavior. Consider the following equation

$$\dot{x}(t) = x(t)\left(\frac{1}{13} + \frac{1}{7}x(t-0.5) - \frac{1}{5}x^2(t-0.5)\right), \quad t \geq 0 \tag{5.3}$$

with initial condition $\varphi(t) = 2$ for $-0.5 \leq t \leq 0$. So we have $x^* \approx 1.0728$ and $(\alpha - \beta)\tau \approx 0.1536 < 1/e$, then (2.6) is not fulfilled. Thus the analytic solution of (5.3) is non-oscillatory. In Figure 5 we draw the figures of the analytic solution and

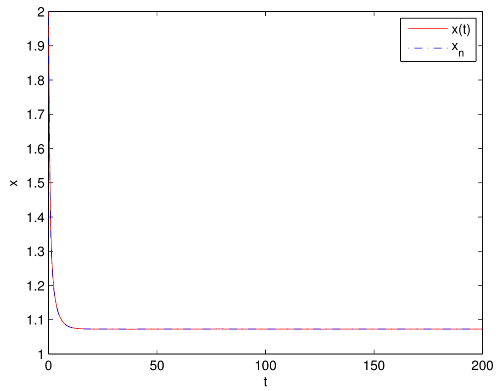

FIGURE 5. The analytic solution and the numerical solution of (5.3) with $m = 5$ and $\theta = 0.4$.

the numerical solution of (5.3) with $\theta = 0.4, m = 5$. We can see that $x(t) \to x^*$ as $t \to \infty$ and $x_n \to x^*$ as $n \to \infty$. The numerical results are identical with Theorem 4.

Finally, for the following four equations

$$\begin{aligned} &\dot{x}(t) = x(t)\left(0.5 - 13x^2(t-1.2) - 5x^5(t-1.2)\right), \quad t \geq 0,\\ &\varphi(t) = \frac{1}{3}, \quad -1.2 \leq t \leq 0, \end{aligned} \tag{5.4}$$

$$\begin{aligned} &\dot{x}(t) = x(t)\left(\frac{4}{21} - x^3(t-0.6) - \frac{1}{13}x^7(t-0.6)\right), \quad t \geq 0,\\ &\varphi(t) = 0.4, \quad -0.6 \leq t \leq 0, \end{aligned} \tag{5.5}$$

$$\begin{aligned} &\dot{x}(t) = x(t)\left(0.25 - 2x^2(t-2.2) - 13x^9(t-2.2)\right), \quad t \geq 0,\\ &\varphi(t) = 0.6, \quad -2.2 \leq t \leq 0, \end{aligned} \tag{5.6}$$

$$\begin{aligned} &\dot{x}(t) = x(t)\left(\frac{1}{17} - 4x^5(t-0.7) - 18x^8(t-0.7)\right), \quad t \geq 0,\\ &\varphi(t) = 0.12, \quad -0.7 \leq t \leq 0, \end{aligned} \tag{5.7}$$

we can verify Theorems 5 and 6 for the forward Euler method with (5.4) and (5.5) (see Figures 6, 7), Theorems 7 and 8 for the backward Euler method with (5.6) and (5.7) (see Figures 8, 9) in the similar way.

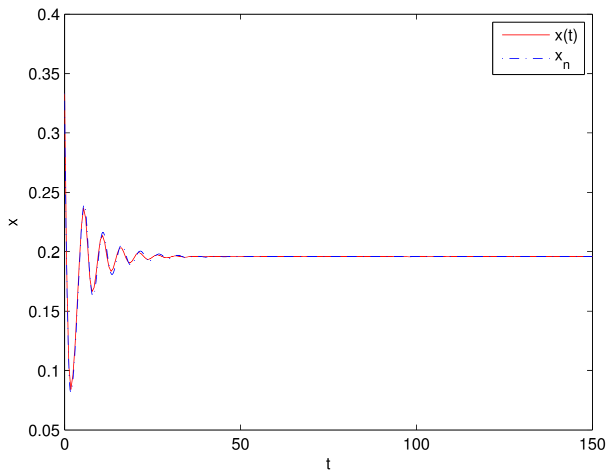

FIGURE 6. The analytic solution and the forward Euler numerical solution of (5.4) with $m = 20$.

From all these figures we can see that the numerical methods inherit the corresponding oscillation and asymptotic property of the above equations.

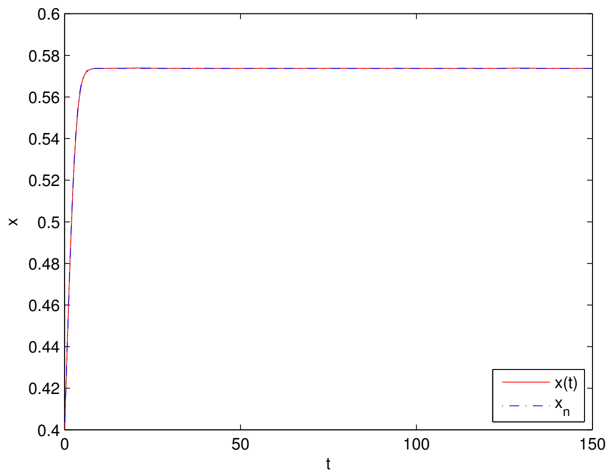

FIGURE 7. The analytic solution and the forward Euler numerical solution of (5.5) with $m = 10$.

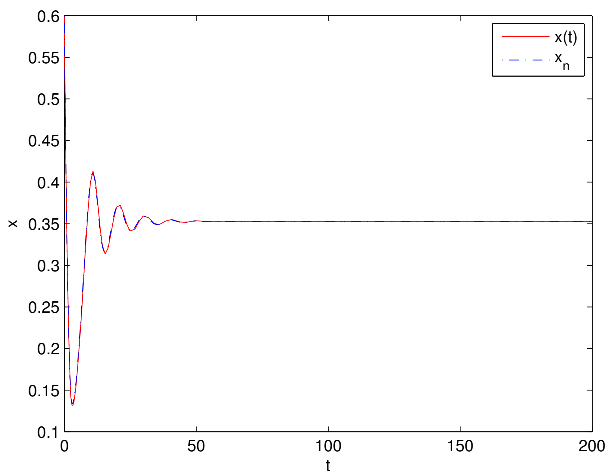

FIGURE 8. The analytic solution and the backward Euler numerical solution of (5.6) with $m = 35$.

6. CONCLUSION

In this paper, we investigate the oscillation and the asymptotic behavior of the numerical solution of nonlinear generalized delay single species population model with three kinds of numerical methods: the θ-method, the forward Euler method and the

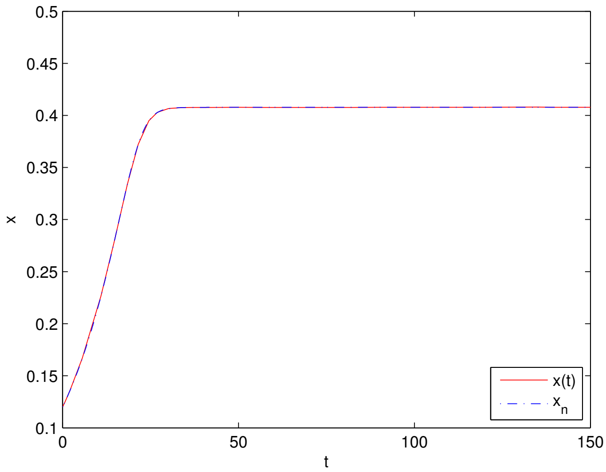

FIGURE 9. The analytic solution and the backward Euler numerical solution of (5.7) with $m = 14$.

backward Euler method. Under the condition that the analytic solution is oscillatory, several conditions under which the numerical solution oscillates are obtained. Furthermore, we confirmed that the non-oscillatory numerical solution approaches to the steady state of the equation. In our future work, we will consider the higher order numerical method, fractional order problem and variable exponents problem including the equation $\dot{x}(t) = x(t)(a + bx^{p(x)}(t-\tau) - cx^{q(x)}(t-\tau))$.

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