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ASYMPTOTIC PROPERTIES OF KNESER TYPE SOLUTIONS FOR THIRD ORDER HALF-LINEAR NEUTRAL DIFFERENCE EQUATIONS

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Abstract. The authors examine properties of positive solutions of the third order half-linear neutral difference equation

$$\Delta(a_n\Delta(b_n(\Delta z_n)^{\alpha}))+q_n y^{\alpha}_{n+1}=0,$$

where $z_n=y_n+p_n y_{\sigma(n)}$. They show that the positive solutions are in fact Kneser type solutions and they provide upper and lower bounds that yield the rate of convergence to zero for such solutions. Examples are provided to illustrate the main results.

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1. INTRODUCTION

In this paper, we are concerned with the asymptotic properties of solutions of the third order neutral difference equation

$$\Delta(a_n\Delta(b_n(\Delta z_n)^{\alpha}))+q_n y^{\alpha}_{n+1}=0, \quad n\geq n_0\geq 0, \tag{1.1}$$

where $z_n=y_n+p_n y_{\sigma(n)}$, α is the ratio of odd positive integers, and the following conditions are assumed to hold throughout:

(H_1) $\{a_n\}$, $\{b_n\}$, and $\{q_n\}$ are positive real sequences for all $n\geq n_0$;
(H_2) $\{p_n\}$ is a nonnegative real sequence with $0\leq p_n\leq p<1$;
(H_3) $\{\sigma(n)\}$ is a sequence of integers such that $\sigma(n)\geq n$ for all $n\geq n_0$;
(H_4) $\sum_{n=n_0}^{\infty}\frac{1}{a_n}=+\infty$ and $\sum_{n=n_0}^{\infty}\frac{1}{b_n^{1/\alpha}}=+\infty$.

By a *solution* of equation (1.1), we mean a nontrivial real sequence $\{y_n\}$ defined for all $n\geq n_0$ and satisfying equation (1.1). A solution $\{y_n\}$ of equation (1.1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative, and *nonoscillatory* otherwise.

Oscillatory and asymptotic properties of solution of equation (1.1) and its special cases have been an active area of investigation in recent years; see for example [2–5, 7–19] and the references contained therein. The well known discrete version of

Kiguradze's theorem [1] can be used to describe the structure of the solution space for the nonoscillatory solutions. For example, for the ordinary difference equation (see [13])

$$\Delta^2((\Delta y_n)^\alpha)+q_n y_{n+1}^\alpha=0, \quad n\geq n_0, \tag{1.2}$$

the set K of all positive solutions has the decomposition

$$K=K_0\cup K_2$$

where

$$y_n\in K_0 \text{ if and only if } y_n>0,\ \Delta y_n<0,\ \Delta((\Delta y_n)^\alpha)>0,\ \Delta^2((\Delta y_n)^\alpha)<0, \tag{1.3}$$

and

$$y_n\in K_2 \text{ if and only if } y_n>0,\ \Delta y_n>0,\ \Delta((\Delta y_n)^\alpha)>0,\ \Delta^2((\Delta y_n)^\alpha)<0. \tag{1.4}$$

A positive solution $\{y_n\}$ of equation (1.2) is said to be of the *Kneser type* if $\{y_n\}\in K_0$. In the study of the asymptotic behavior of the nonoscillatory solutions, many results were directed at obtaining criteria for $K_2=\varnothing$, that is, showing that the only possible nonoscillatory solutions are the Kneser type ones. We will also say that equation (1.2) has property (A) if every positive solution $\{y_n\}$ belongs to K_0 and $\lim_{n\to\infty} y_n=0$.

The aim of this paper is to conduct an analogous study for equation (1.1). That is, we wish to give sufficient conditions for the nonoscillatory solutions to be of the Kneser type. In addition, we wish to obtain upper and lower bounds for such solutions. These estimates allow us to determine the rate of convergence of Kneser type solutions to zero. The results presented in this paper are new and complement those in [4–8, 11–19].

2. Main Results

We begin with the following lemma that gives the basic properties of nonoscillatory solutions of equation (1.1). We present it for the case of positive solutions but clearly analogous statements hold for the negative solutions. As a part of this lemma, we define classes of solutions of equation (1.1) that are analogous to the sets K_0 and K_2 for equation (1.2).

Lemma 1. *Assume that $\{y_n\}$ is a positive solution of equation (1.1). Then the corresponding sequence $\{z_n\}$ belongs to one of the following classes:*

$$S_0=\left\{z_n : z_n>0,\ \Delta z_n<0,\ \Delta(b_n(\Delta z_n)^\alpha)>0,\ \Delta^2(a_n\Delta(b_n(\Delta z_n)^\alpha))<0\right\}$$
$$S_2=\left\{z_n : z_n>0,\ \Delta z_n>0,\ \Delta(b_n(\Delta z_n)^\alpha)>0,\ \Delta^2(a_n\Delta(b_n(\Delta z_n)^\alpha))<0\right\}$$

eventually.

Proof. The proof is similar to that of [16, Lemma 2.1] and so we omit the details. □

We will say that the neutral difference equation (1.1) has property (A), if every positive solution $\{y_n\}$ satisfies $\lim_{n\to\infty} y_n = 0$ and the corresponding sequence $\{z_n\} \in S_0$.

In the following, we first present sufficient conditions for the equation (1.1) to have only Kneser type solutions. Our approach will be to first obtain some properties that solutions of equation (1.1) that belong to the class S_2 must satisfy, and use these facts to obtain sufficient conditions for the class S_2 to in fact be empty. Then we give upper and lower estimate for such solutions.

To simplify our notation we let:

$$A_n = \sum_{s=N}^{n-1} \frac{1}{a_s}, \qquad B_n = \sum_{s=N}^{n-1} \frac{1}{b_s^{1/\alpha}},$$

$$C_n = \sum_{s=N}^{n-1} \frac{A_s}{b_s^{1/\alpha}}, \qquad E_n = \left(1 - p_n \frac{C_{\sigma(n)}}{C_n}\right) > 0,$$

$$Q_n = \left[\frac{1}{b_n} \sum_{s=n}^{\infty} \frac{1}{a_s} \sum_{t=s}^{\infty} q_t\right]^{1/\alpha}, \qquad \phi_n = \prod_{s=N}^{n-1} (1 + Q_s),$$

$$R_n = (1-p) \frac{\phi_n}{b_n^{1/\alpha}} \left(\sum_{s=n}^{\infty} \frac{1}{a_s} \sum_{t=s}^{\infty} \frac{q_t}{\phi_{t+1}^{\alpha}}\right)^{1/\alpha},$$

where $N \geq n_0$.

Lemma 2. *Let $\{y_n\}$ be a positive solution of equation (1.1) with the corresponding sequence $\{z_n\} \in S_2$ for $n \geq N \geq n_0$ and assume that*

$$\sum_{n=N}^{\infty} \frac{1}{a_n} \sum_{s=n}^{\infty} q_s E_{s+1}^{\alpha} B_{s+1}^{\alpha} = \infty. \tag{2.1}$$

Then:

(i) $\{\frac{z_n}{C_n}\}$ *is decreasing for all $n \geq N$;*

(ii) $\{\frac{b_n^{1/\alpha} \Delta z_n}{A_n^{1/\alpha}}\}$ *is decreasing for all $n \geq N$;*

(iii) $\{\frac{z_n}{B_n}\}$ *is increasing for all $n \geq N$.*

Proof. Let $\{y_n\}$ be a positive solution of equation (1.1) with the corresponding sequence $\{z_n\} \in S_2$ for all $n \geq N$. Since $a_n \Delta(b_n (\Delta z_n)^{\alpha})$ is decreasing, we have

$$b_n (\Delta z_n)^{\alpha} \geq \sum_{s=N}^{n-1} \frac{a_s \Delta(b_s (\Delta z_s)^{\alpha})}{a_s} \geq A_n a_n \Delta(b_n (\Delta z_n)^{\alpha}), \quad n \geq N.$$

From the last inequality, we obtain

$$\Delta\left(\frac{b_n (\Delta z_n)^{\alpha}}{A_n}\right) = \frac{A_n \Delta(b_n (\Delta z_n)^{\alpha}) - b_n (\Delta z_n)^{\alpha} \frac{1}{a_n}}{A_n A_{n+1}} \leq 0$$

for all $n \geq N \geq n_0$. Thus, $\{\frac{b_n(\Delta z_n)^\alpha}{A_n}\}$ is decreasing for all $n \geq N$, so (ii) holds and

$$z_n \geq \sum_{s=N}^{n-1} \frac{A_s^{1/\alpha} b_s^{1/\alpha} \Delta z_s}{A_s^{1/\alpha}} \geq \frac{b_n^{1/\alpha} \Delta z_n}{A_n^{1/\alpha}} C_n, \quad n \geq N. \tag{2.2}$$

Hence,

$$\Delta\left(\frac{z_n}{C_n}\right) = \frac{C_n \Delta z_n - z_n \frac{A_n^{1/\alpha}}{b_n^{1/\alpha}}}{C_n C_{n+1}} \leq 0,$$

which implies that $\{\frac{z_n}{C_n}\}$ is decreasing for all $n \geq N$, so (i) holds.

Since $b_n^{1/\alpha} \Delta z_n$ is positive and strictly increasing for any $n \geq N$, it is easy to see that for all $n \geq N_1 \geq N$,

$$\begin{aligned} z_n &\leq z_{N_1} + b_n^{1/\alpha} \Delta z_n \sum_{s=N_1}^{n-1} \frac{1}{b_s^{1/\alpha}} \\ &= z_{N_1} - b_n^{1/\alpha} \Delta z_n \sum_{s=N}^{N_1-1} \frac{1}{b_s^{1/\alpha}} + b_n^{1/\alpha} \Delta z_n \sum_{s=N}^{n-1} \frac{1}{b_s^{1/\alpha}}. \end{aligned} \tag{2.3}$$

We claim that $b_n^{1/\alpha} \Delta z_n \to \infty$ as $n \to \infty$. If this is not the case, then $b_n^{1/\alpha} \Delta z_n \to 2d < \infty$ as $n \to \infty$ From the definition of z_n and using the fact that $\{\frac{z_n}{C_n}\}$ is decreasing, we have

$$y_n \geq z_n \left(1 - p_n \frac{C_{\sigma(n)}}{C_n}\right) = E_n z_n.$$

Summing equation (1.1) from n to ∞ and using the last inequality, we obtain

$$\Delta(b_n(\Delta z_n)^\alpha) \geq \frac{1}{a_n} \sum_{s=n}^{\infty} q_s E_{s+1}^\alpha z_{s+1}^\alpha.$$

Now $b_n^{1/\alpha} \Delta z_n \to 2d$ as $n \to \infty$ implies $b_n^{1/\alpha} \Delta z_n > d$ for n large enough, which in turn implies $z_n > d B_n$. Combining the last two inequalities and summing once more, we obtain

$$(2d)^\alpha \geq d^\alpha \sum_{n=N_1}^{\infty} \frac{1}{a_n} \sum_{s=n}^{\infty} q_s E_{s+1}^\alpha B_{s+1}^\alpha,$$

which contradicts (2.1). Thus, $b_n^{1/\alpha} \Delta z_n \to \infty$ as $n \to \infty$ as we claimed.

Hence, in view of (2.3) and (H_4), we obtain

$$z_n \leq b_n^{1/\alpha} \Delta z_n B_n, \quad n \geq N.$$

From the last inequality, we see that

$$\Delta\left(\frac{z_n}{B_n}\right) = \frac{B_n \Delta z_n - \frac{z_n}{b_n^{1/\alpha}}}{B_n B_{n+1}} \geq 0$$

eventually, and we conclude that $\{\frac{z_n}{B_n}\}$ is increasing. Therefore (iii) holds and this completes the proof of the lemma. □

Lemma 3. *Assume that* (2.1) *holds and let* $\{y_n\}$ *be a positive solution of equation* (1.1) *with the corresponding sequence* $\{z_n\} \in S_2$ *for all* $n \geq N$. *Then*

$$E_n z_n \leq y_n \leq z_n \quad \text{for } n \geq N. \tag{2.4}$$

Proof. From the definition of z_n, we have $z_n \geq y_n$ for all $n \geq N$. Furthermore,

$$y_n \geq z_n - p_n z_{\sigma(n)} \geq E_n z_n$$

where we have used the fact that $\{\frac{z_n}{C_n}\}$ is decreasing for all $n \geq N$. This completes the proof of the lemma. □

Next, we obtain sufficient condition for all nonoscillatory solutions of (1.1) to be of the Kneser type.

Theorem 1. *If condition* (2.1) *holds and*

$$\limsup_{n\to\infty} \left\{ \frac{1}{A_{n+1}} \sum_{s=N}^{n} q_s E_{s+1}^{\alpha} C_{s+1}^{\alpha} A_{s+1} + \frac{C_{n+1}^{\alpha}}{B_{n+1}^{\alpha}} \sum_{s=n+1}^{\infty} q_s E_{s+1}^{\alpha} B_{s+1}^{\alpha} \right\} > 1, \tag{2.5}$$

then $S_2 = \varnothing$.

Proof. Let $\{y_n\}$ be a positive solution of equation (1.1), with the corresponding sequence $\{z_n\} \in S_2$ for all $n \geq N$. Using (2.4) in equation (1.1), we obtain

$$\Delta(a_n \Delta(b_n(\Delta z_n)^{\alpha})) + q_n E_{n+1}^{\alpha} z_{n+1}^{\alpha} \leq 0, \quad n \geq N. \tag{2.6}$$

Summing inequality (2.6) from n to ∞, we have

$$\Delta(b_n(\Delta z_n)^{\alpha}) \geq \sum_{s=N}^{n-1} \frac{1}{a_s} \left(\sum_{t=s}^{\infty} q_t E_{t+1}^{\alpha} z_{t+1}^{\alpha} \right). \tag{2.7}$$

From (2.7), it follows that

$$\begin{aligned} b_n(\Delta z_n)^{\alpha} &\geq \sum_{s=N}^{n-1} \frac{1}{a_s} \left(\sum_{t=s}^{\infty} q_t E_{t+1}^{\alpha} z_{t+1}^{\alpha} \right) \\ &= \sum_{s=N}^{n-1} \frac{1}{a_s} \sum_{t=s}^{n-1} q_t E_{t+1}^{\alpha} z_{t+1}^{\alpha} + \sum_{s=N}^{n-1} \frac{1}{a_s} \sum_{t=n}^{\infty} q_t E_{t+1}^{\alpha} z_{t+1}^{\alpha} \\ &= \sum_{s=N}^{n-1} q_s E_{s+1}^{\alpha} A_{s+1} z_{s+1}^{\alpha} + A_n \sum_{s=n}^{\infty} q_s E_{s+1}^{\alpha} z_{s+1}^{\alpha}. \end{aligned}$$

Using (2.2), in the last inequality, we obtain

$$\frac{A_{n+1} z_{n+1}^{\alpha}}{C_{n+1}^{\alpha}} \geq \sum_{s=N}^{n} q_s A_{s+1} E_{s+1}^{\alpha} z_{s+1}^{\alpha} + A_{n+1} \sum_{s=n+1}^{\infty} q_s E_{s+1}^{\alpha} z_{s+1}^{\alpha}. \tag{2.8}$$

In view of the monotonicity properties (i) and (iii) in Lemma 2, (2.8) yields

$$1 \geq \frac{1}{A_{n+1}} \sum_{s=N}^{n} q_s A_{s+1} E_{s+1}^{\alpha} C_{s+1}^{\alpha} + \frac{C_{n+1}^{\alpha}}{B_{n+1}^{\alpha}} \sum_{s=n+1}^{\infty} q_s E_{s+1}^{\alpha} B_{s+1}^{\alpha}.$$

Taking the lim sup as $n \to \infty$ on both sides of the last inequality leads to a contradiction of (2.5). This completes the proof of the theorem. □

Next, we wish to obtain estimates for the Kneser solutions. We begin with a lemma.

Lemma 4. *Let $\{y_n\}$ be a positive solution of equation (1.1) with the corresponding sequence $\{z_n\} \in S_0$ for $n \geq N$. Then:*

(i) *$(1-p)z_n \leq y_n \leq z_n$ for all $n \geq N$;*
(ii) *$\{z_n \phi_n\}$ is increasing for all $n \geq N$.*

Proof. Assume that $\{y_n\}$ is a positive solution of equation (1.1) with the corresponding sequence $\{z_n\} \in S_0$. Then z_n is positive, $z_n \geq y_n$, and

$$y_n = z_n - p_n y_{\sigma(n)} \geq (1-p)z_n, \quad n \geq N \geq n_0,$$

so (i) is proved.

It easy to see that $z_n \in S_0$ implies

$$\lim_{n\to\infty} b_n(\Delta z_n)^{\alpha} = 0;$$

otherwise we would eventually have $\Delta z_n > 0$ contradicting $z_n \in S_0$. Similarly,

$$\lim_{n\to\infty} a_n \Delta(b_n(\Delta z_n)^{\alpha}) = 0.$$

A summation of equation (1.1) then yields

$$a_n \Delta(b_n(\Delta z_n)^{\alpha}) = \sum_{s=n}^{\infty} q_s y_{s+1}^{\alpha} \leq \sum_{s=n}^{\infty} q_s z_{s+1}^{\alpha} \leq z_{n+1}^{\alpha} \sum_{s=n}^{\infty} q_s.$$

Summing once more, we obtain

$$b_n(\Delta z_n)^{\alpha} \geq -z_{n+1}^{\alpha} \sum_{s=n}^{\infty} \frac{1}{a_s} \sum_{t=s}^{\infty} q_t,$$

or

$$\Delta z_n \geq -z_{n+1} Q_n.$$

Hence,

$$\Delta(z_n \phi_n) = \phi_n \Delta z_n + z_{n+1} \Delta \phi_n \geq z_{n+1}(\Delta \phi_n - \phi_n Q_n) = 0$$

since $\{\phi_n\}$ is a solution of the difference equation $(\Delta \phi_n - Q_n \phi_n) = 0$. Therefore, $\{z_n \phi_n\}$ is increasing and this completes the proof of the lemma. □

Theorem 2. *Assume that conditions (2.1) and (2.5) hold. If $\{y_n\}$ is a positive solution of equation (1.1), then there are positive constants C_1 and C_2 such that*

$$C_1\psi_1(n) \le y_n \le C_2\psi_2(n), \tag{2.9}$$

where

$$\psi_1(n) = \frac{1}{\phi_n} \quad \text{and} \quad \psi_2(n) = \prod_{s=N}^{n-1}(1-R_s).$$

Proof. Assume that $\{y_n\}$ is a positive solution of equation (1.1). Then, by Theorem 1, $\{y_n\}$ is a Kneser type solution. From Lemma 4, we have that $\{z_n\phi_n\}$ is increasing for all $n \ge N$. Therefore,

$$y_n \ge \frac{(1-p)}{\phi_n} z_n\phi_n \ge \frac{(1-p)}{\phi_n} z_N\phi_N = C_1\psi_1(n).$$

On the other hand, summing equation (1.1) from n to ∞ and applying Lemma 4 (i), we have

$$a_n\Delta(b_n(\Delta z_n)^\alpha) = \sum_{s=n}^{\infty} q_s y_{s+1}^\alpha \ge (1-p)^\alpha \sum_{s=n}^{\infty} q_s z_{s+1}^\alpha \ge (1-p)^\alpha \phi_{n+1}^\alpha z_{n+1}^\alpha \sum_{s=n}^{\infty} \frac{q_s}{\phi_{s+1}^\alpha}.$$

Again summing the last inequality and applying Lemma 4 (ii) gives

$$-b_n(\Delta z_n)^\alpha \ge (1-p)^\alpha \phi_n^\alpha z_n^\alpha \sum_{s=n}^{\infty} \frac{1}{a_s} \sum_{t=s}^{\infty} \frac{q_t}{\phi_{t+1}^\alpha}$$

or

$$\frac{\Delta z_n}{z_n} \le -R_n, \; n \ge N.$$

Summing the last inequality from N to $n-1$, we obtain

$$y_n \le z_n \le C_2\psi_2(n).$$

This proves the theorem. □

From Theorem 2 we deduce the following result for the half-linear difference equation (1.2).

Corollary 1. *Assume that*

$$\sum_{n=N}^{\infty}\left(\sum_{s=n}^{\infty} s^\alpha q_s\right) = \infty \tag{2.10}$$

and

$$\limsup_{n\to\infty}\left\{\frac{1}{n+1}\sum_{s=N}^{n}(s+1)^\alpha(s+2)^\alpha q_s + (n+2)^\alpha \sum_{s=n+1}^{\infty}(s+1)^\alpha q_s\right\} > 2^\alpha. \tag{2.11}$$

Then every positive solution $\{y_n\}$ of equation (1.2) is decreasing and satisfies estimates of the form (2.9) with $p = 0$ in R_n.

3. Examples

In this section, we present two examples to illustrate the significance of our main results.

Example 1. Consider the third order difference equation

$$\Delta^3 y_n + \frac{10}{n(n+1)(n+2)} y_{n+1} = 0, \quad n \geq 1. \tag{3.1}$$

A simple calculation shows that conditions (2.10) and (2.11) are satisfied, so by Corollary 1, positive nonoscillatory solutions of equation (3.1) are of the Kneser type and a positive solution $\{y_n\}$ satisfies the estimates

$$\frac{C_1}{n} \leq y_n \leq \frac{C_2}{n}$$

for some positive constants C_1 and C_2. Therefore, equation (3.1) has property (A).

Example 2. Consider the third order neutral difference equation

$$\Delta\left(\frac{1}{n+2}\Delta^2\left(y_n + \frac{1}{2}y_{2n}\right)\right) + \frac{3\beta}{n(n+1)(n+2)(n+3)} y_{n+1} = 0, \quad n \geq 1. \tag{3.2}$$

Again calculations show that conditions (2.1) and (2.5) hold if $\beta > 6$. Therefore, all positive nonoscillatory solutions of equation (3.2) are of the Kneser type and Theorem 2 provides estimates for a positive solution $\{y_n\}$ to be of the form

$$\frac{C_1}{n^\beta} \leq y_n \leq \frac{C_2}{n}$$

for some positive constants C_1 and C_2. We also see that (3.2) has property (A).

4. Conclusions

In this paper, we give conditions for all positive solutions of a third order half-linear neutral difference equation to be of the Kneser type and we derive upper and lower bounds for them. We also wish to point out that the results obtained in Lemmas 2 and 3 are new and complement those in [4, 5, 7, 8, 11–19].

References

[1] R. P. Agarwal, *Difference Equations and Inequalities, Theory, Methods and Applications, Second Edition.* New York: Dekker, 2000.

[2] R. P. Agarwal, M. Bohner, S. R. Grace, and D. O‘Regan, *Discrete Oscillation Theory Applications.* New York: Hindawi, 2005.

[3] R. P. Agarwal, R. P. Gala, and M. A. Ragusa, “A regularity criterion in weak spaces to Boussinesq equations.” *Mathematics*, vol. 8, no. 920, 2020.

[4] R. P. Agarwal and S. R. Grace, “Oscillation of certain third order difference equations.” *Comput. Math. Appl.*, vol. 42, pp. 379–384, 2001.

[5] R. P. Agarwal, S. R. Grace, and D. O‘Regan, “On the oscillation of certain third order difference equations.” *Adv. Difference Equ.*, vol. 2005, pp. 345–367, 2005.

[6] B. Baculíková and J. Džurina, "Remark on properties of Kneser solutions for third-order neutral differential equations." *Appl. Math. Letters.*, vol. 63, pp. 1–5, 2017, doi: 10.1016/j.aml.2016.07.005.

[7] Z. Došlá and A. Kobza, "On third order linear difference equations involving quasi-differences." *Adv. Difference Equ.*, vol. 2006, no. 65652, pp. 1–13, 2006, doi: 10.1155/ADE/2006/65652.

[8] J. R. Graef and E. Thandapani, "Oscillatory and asymptotic behavior of solutions of third order delay difference equations." *Funkcial. Ekvac.*, vol. 42, pp. 355–369, 1999.

[9] C. Kome and Y. Yazlik, "The m-extension of Fibonacci and Lucas p-difference sequences." *Filomat*, vol. 33, pp. 6187–6194, 2019.

[10] M. A. Ragusa and A. Tachikawa, "Regularity for minimizers for functionals of double phase with variable exponents." *Adv. Nonlinear Anal.*, vol. 9, pp. 710–728, 2020, doi: 10.1515/anona-2020-0022.

[11] S. H. Saker, "Oscillation and asymptotic behavior of third order nonlinear neutral delay difference equations." *Dynam. Systems Appl.*, vol. 15, pp. 549–567, 2006.

[12] S. H. Saker, "New oscillation criteria for third order nonlinear neutral difference equations." *Math. Slovaca*, vol. 61, pp. 579–600, 2011, doi: 10.2478/s12175-011-0030-5.

[13] S. H. Saker, J. O. Alzabut, and A. Mukheimer, "On the oscillatory behavior for a certain class of third order nonlinear delay difference equations." *Electron. J. Qual. Theory Differ. Equ.*, vol. 2010, no. 67, pp. 1–16, 2010, doi: 10.14232/ejqtde.2010.1.67.

[14] B. Smith, "Oscillatory and asymptotic behavior in certain third order difference equations." *Rocky Mountain J. Math.*, vol. 17, pp. 597–606, 1987, doi: 10.1216/RMJ-1987-17-3-597.

[15] A. A. Sobilo and A. Drozdowicz, "Asymptotic behavior of solutions of third order nonlinear difference equations of neutral type." *Math. Bohem.*, vol. 133, pp. 247–258, 2008.

[16] E. Thandapani and K. Mahalingam, "Oscillatory properties of third order neutral delay difference equations." *Demonstratio Math.*, vol. 35, pp. 325–337, 2002.

[17] E. Thandapani and S. Selvarangam, "Oscillation results for third order half-linear neutral difference equations," *Bull. Math. Anal. Appl.*, vol. 4, pp. 91–102, 2012.

[18] E. Thandapani and S. Selvarangam, "Oscillation of third order half-linear neutral difference equations." *Math. Bohem.*, vol. 138, pp. 87–104, 2013.

[19] E. Thandapani, M. Vijaya, and T. Li, "On the oscillation of third order half-linear neutral type difference equations." *Electron. J. Qual. Theory Differ. Equ.*, vol. 2011, no. 76, pp. 1–13, 2011, doi: 10.14232/ejqtde.2011.1.76.

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