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COMMON TERMS OF TRIBONACCI AND PERRIN SEQUENCES

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Abstract. Assume that T_n is the n^{th} term of Tribonacci sequence and R_m is the m^{th} term of Perrin sequence. In this paper, we solve the equation $T_n = R_m$ completely.

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1. INTRODUCTION

The question of finding common terms of two sequences has an extensive literature in number theory. One of the results of this topic was given by Mignotte who (see [8]) showed that if $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ are two linearly recurrence sequences then, under some weak technical assumptions, the equation $u_n = v_m$ has only finitely many solutions in positive integers m, n. In [9], authors surveyed the equation $Au_x + Bu_y + Cu_z = 0$ for fixed A, B, $C \in \mathbb{Z}$. Szalay [12] gave an algorithm to compute the intersection of the binary sequences with arbitrary initial conditions while Alekseyev [1] described how to compute the intersection of two sequences U_n and V_n that satisfy the recurrence $Z_n = PZ_{n-1} \pm Z_{n-2}$ (Z_n is U_n or V_n) with initial conditions $U_0 = 0, U_1 = 0, V_0 = 2$ and $V_1 = P$. Both of these papers depend on two homogeneous quadratic equations. Marques [6] proved the conjecture about the $k-$generalized Fibonacci sequence $\left\{F^{(k)}\right\}$ (defined by the initial values $0,0,\ldots,0,1$ (k terms) and such that each term afterwards is the sum of the k preceding terms) and showed that the non-trivial solutions of the equation $F_n^{(l)} = F_m^{(k)}$ satisfying $l > k > 1$, $n > l+1$, $m > k+1$ are $(m,n,l,k) = (7,6,3,2)$ and $(12,11,7,3)$. These papers give a motivation us to solve the equation

$$T_n = R_m,$$

where T_n is n^{th} Tribonacci number and R_m is m^{th} Perrin number. The difference of our problem is that the recurrence relations of the sequences are not the same although the orders of the sequences are same. Now, we present the recurrence relations of Tribonacci and Perrin sequences.

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Tribonacci sequence $\{T_n\}$ is defined by the recurrence relation

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n$$

with $T_0 = 0$, $T_1 = T_2 = 1$. The characteristic equation $x^3 - x^2 - x - 1 = 0$ of Tribonacci sequence has one real root α and two conjugate complex roots β and γ. In 1982, Spickerman [11] found the Binet formula of Tribonacci numbers as

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \quad \text{for all } n \geq 0.$$

Afterwards, Dresden and Du [3] presented a Binet-style formula that can be used to produce the k-generalized Fibonacci numbers. The case $k = 3$ corresponds to Binet formula of Tribonacci sequence as follows.

$$T_n = \frac{(\alpha-1)\alpha^{n-1}}{2+4(\alpha-2)} + \frac{(\beta-1)\beta^{n-1}}{2+4(\beta-2)} + \frac{(\gamma-1)\gamma^{n-1}}{2+4(\gamma-2)}, \quad \text{for all } n \geq 0.$$

In fact, these two Binet formulas are same. The difference is the representation. Namely, while each fraction of Dresden and Du's Binet formula contains only one of the roots of the characteristic equation, each fraction in the Binet formula of Spickerman contains all of the roots of the characteristic equation. Moreover, Dresden and Du [3] proved that the terms of Tribonacci sequence can also be written as

$$T_n = c_3\alpha^{n-1} + e_n, \quad \text{with} \quad |e_n| < \frac{1}{\alpha^{1/2}}, \quad \text{for all} \quad n \geq 1 \tag{1.1}$$

where $c_3 = (\alpha-1)/(4\alpha-6) \cong 0.61$.

Perrin sequence $\{R_m\}$ is defined by

$$R_m = R_{m-2} + R_{m-3}$$

with $R_0 = 3,\ R_1 = 0$ and $R_2 = 2$.

Perrin numbers were studied by several authors in the beginning of the nineteenth century (for details, see [10]). The characteristic equation $x^3 - x - 1 = 0$ of the Perrin sequence has distinct roots a, b , and c where $a \approx 1.3247\ldots$ (called a plastic constant) is a unique real root. This constant was first defined in 1924 by Gérard Cordonnier. He described applications to architecture in 1958, he gave a lecture tour that illustrated the use of the plastic constant in many buildings and monuments. The Binet formula of Perrin sequence is

$$R_m = a^m + b^m + c^m, \text{ for all } m \geq 0. \tag{1.2}$$

In this paper, we investigate the common terms of the Tribonacci and Perrin sequences. Our result is following.

Theorem 1. *The solutions exceed* 1 *of the equation* $T_n = R_m$ *are* $T_3 = R_2 = R_4 = 2$*, and* $T_5 = R_7 = 7$.

2. Preliminary Results

Before going further, we give several notations that we will use. Let γ be an algebraic number of degree d over $\mathbb{Q}$ with minimal primitive polynomial, where the leading coefficient a_0 is positive.

The logarithmic height of γ is given by

$$h(\gamma) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log(\max\{|\gamma^i|, 1\})\right). \tag{2.1}$$

Here γ^i are all conjugates of γ number.

Now, we give the upper and lower bounds for the terms of Tribonacci and Perrin numbers.

Lemma 1. *For $n \geq 2$, the inequality*

$$a^{n-1} < R_n < a^{n+1}$$

and

$$\alpha^{n-2} \leq T_n < \alpha^{n-1}$$

hold.

Proof. The proof of the first inequality is in [5] (see Lemma 2.8) and second one is in [2] (the case $k = 3$). □

We use a lower bound for a linear form in logarithms à la Baker and such a bound was given by the following result of Matveev (see [7]).

Lemma 2. *Let $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_t$ be positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, further $b_1, b_2, b_3, \ldots, b_t$ be non-zero integers, and assume that*

$$\Lambda := \gamma_1^{b_1}\gamma_2^{b_2}\gamma_3^{b_3}\ldots\gamma_t^{b_t} - 1 \tag{2.2}$$

is non-zero. Then

$$\Lambda > \exp\left(-1.4 \cdot 30^{t+3} t^{4.5} D^2 (1+\log D)(1+\log B) A_1 A_2 A_3 \ldots A_t\right), \tag{2.3}$$

where

$$B \geq \max\{|b_1|, |b_2|, |b_3|, \ldots, |b_t|\} \tag{2.4}$$

and

$$A_i \geq \max\{Dh(\gamma_i), |\log\gamma_i|, 0.16\} \tag{2.5}$$

for all $i = 1, \ldots, t$.

The following lemma was given by Pethő and Dujella [4] that we will use later.

Lemma 3. *Let γ be an irrational number, M be a positive integer and p/q be a convergent of the continued fraction of γ such that $q > 6M$. Let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\varepsilon := \| \mu q \| - M \| \gamma q \|$, where $\| . \|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality*

$$0 < | m\gamma - n + \mu | < AB^{-m}$$

in positive integers m, n with $m \leq M$ and $m \geq \log(Aq/\varepsilon)/\log B$.

3. Proof of Theorem 1

Firstly, assume that $m \leq 7$. In this case, we see that $T_3 = R_2 = 2$, $T_3 = R_4 = 2$ and $T_5 = R_7 = 7$ are the solutions exceeds 1 in the equation $T_n = R_m$. From now on, we can assume that $m \geq 8$. If we put Binet formula of Perrin sequence and the equation (1.1) in the equation $T_n = R_m$, then we have

$$c_3\alpha^{n-1} - a^m = b^m + c^m - e_n.$$

After dividing the above equation with a^m

$$\frac{c_3\alpha^{n-1}}{a^m} - 1 = \frac{b^m + c^m - e_n}{a^m},$$

then

$$\left| c_3\alpha^{n-1}a^{-m} - 1 \right| = \left| \frac{b^m + c^m - e_n}{a^m} \right| \leq \left| \frac{b^m}{a^m} \right| + \left| \frac{c^m}{a^m} \right| + \left| \frac{e_n}{a^m} \right| < \frac{3}{a^m}$$

follows. Let

$$\Lambda := c_3\alpha^{n-1}a^{-m} - 1. \tag{3.1}$$

To show $\Lambda \neq 0$, we contrary assume $\Lambda = 0$. It yields that $\tau = a^m\alpha^{1-n} = a^m(1/\alpha)^{n-1}$. It is obvious that a and α are algebraic integers. $1/\alpha$ is also algebraic integer since $1/\alpha$ is the root of the monic polynomial $x^3 + x^2 + x - 1$. Since product of algebraic integers is also algebraic integer, then it requires that $\tau = a^m(1/\alpha^{n-1})$ is algebraic integer. But this is a contradiction. Because the minimal polynomial $44x^3 - 44x^2 + 12x - 1$ of c_3 is not monic. So, c_3 is not algebraic integer, but algebraic number. We arrive the claimed fact $\Lambda \neq 0$.

From now on, we suppose that

$$\begin{aligned} \gamma_1 &= c_3, & b_1 &= 1, & D &= 3, \\ \gamma_2 &= \alpha, & b_2 &= n-1, & t &= 3, \\ \gamma_3 &= a, & b_3 &= -m. & & \end{aligned}$$

Since the leading coefficient a_0 is 44 of the minimal polynomial of c_3, then

$$h(c_3) = \frac{1}{3}\{\log 44 + \log(\max\{c_3, 1\})\} = \frac{1}{3}\log 44 \cong 1.26,$$
$$h(\alpha) = \frac{1}{3}\{\log 1 + \log\alpha + \log 1 + \log 1\} = \frac{1}{3}\log\alpha \cong 0.201,$$

$$h(a) = \frac{1}{3}\{\log 1 + \log a + \log 1 + \log 1\} = \frac{1}{3}\log a \cong 0.092.$$

only hold. We choose A_1, A_2 and A_3 as follows.

$$A_1 \geq \max\{3h(c_3), \log c_3, 0, 16\} \cong 3.78, \qquad A_1 = 4,$$
$$A_2 \geq \max\{3h(\alpha),\ \log\alpha,\ 0, 16\} \cong 0.604, \qquad A_2 = 0.61,$$
$$A_3 \geq \max\{3h(a),\ \log a,\ 0, 16\} \cong 0.281, \qquad A_3 = 0.29.$$

Now, we determine the value of B such that

$$B \geq \{1,\ |n-1|,\ |-m|\}.$$

It is obvious that $T_4 > R_4$, $T_5 > R_5$ and $T_6 > R_6$. The recursive relation of the two sequence yields that $T_n > R_n$. If $T_n = R_m$, then $n < m$ holds. So, we choose $B = m$.

If we write all the values into the inequality (2.3), then we have

$$\frac{3}{a^m} > \Lambda > \exp\left(-1.4 \cdot 30^6 3^{4.5} 3^2 (1+\log 3)(1+\log m) \cdot 3.79 \cdot 0.61 \cdot 0.29\right).$$

By taking logarithm it leads to

$$1.81 \cdot 10^{12}(1+\log m) > m\log a.$$

After using the fact $1+\log m < 2\log m$, we can write

$$3.8 \cdot 10^{12}\log m > m\log a.$$

Finally, we obtain

$$m < 1.36 \cdot 10^{13}\log m,$$

which yields that

$$m < 4.6 \cdot 10^{14}.$$

From now on, we will reduce the upper bound on m by using the result of Pethő and Dujella.

Let $M = 4.6 \cdot 10^{14}$ and $\Gamma := \log c_3 + (n-1)\log\alpha - m\log a$. It is known that $\mid x \mid \leq \mid e^x - 1 \mid$ holds for all reals x. So,

$$\mid \Gamma \mid < \mid e^{\Gamma} - 1 \mid < \frac{3}{a^m} < \frac{1}{2}$$

follows. In this case, we have

$$\mid \Gamma \mid < \frac{6}{a^m}.$$

The fact $n < m$ gives that

$$(n-1)\log\alpha - m\log a + \log c_3 < 6a^{-(n-1)}.$$

After dividing both sides with $\log a$,

$$(n-1)\frac{\log\alpha}{\log a} - m + \frac{\log c_3}{\log a} < \frac{6}{\log a}a^{-(n-1)}$$

holds. Let

$$\gamma = \frac{\log \alpha}{\log a} \qquad \mu = \frac{\log c_3}{\log a} \qquad A = \frac{6}{\log a} \qquad B = a,$$

and q_n be the denominator of the n^{th} convergent of the continued fraction of γ. We found

$$q_{24} = 4220424136597687 > 6M$$

and then $\varepsilon := \|\mu q_{24}\| - M\|\gamma q_{24}\| \cong 0.42266\ldots$. We note that the conditions to apply Lemma 3 are fulfilled for $A = \frac{6}{\log a}$ and $B = a$. According to Lemma 3, there is no solution m in the range

$$\left[\frac{\log(Aq_{24}/\varepsilon)}{\log a} + 1, M\right).$$

Thus $m < 142$.

Finally, we use a program written in Mathematica to find the solution of the equation $T_n = R_m$ with $m < 142$, but there is no solution satisfies $m \geq 8$. So, the proof is completed.

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