



THE CONVERGENCE OF EXPONENTIAL OPERATORS CONNECTED WITH x^3 ON FUNCTIONS OF BOUNDED VARIATION

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Abstract. In the present paper, we estimate the rate of convergence of exponential type operators connected with x^3 on functions of bounded variation.

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1. EXPONENTIAL OPERATORS

For $x \in (0, \infty)$ one of the exponential operators introduced in [11, (3.1.1)] is defined as

$$(Q_n f)(x) = \int_0^\infty \phi_n(x, t) f(t) dt, \quad (1.1)$$

where the kernel is given by

$$\phi_n(x, t) = \sqrt{\frac{n}{2\pi t}} \frac{1}{\sqrt{t}} \exp\left(\frac{n}{x} - \frac{nt}{2x^2} - \frac{n}{2t}\right).$$

These operators satisfy the partial differential equation

$$x^3 \frac{\partial}{\partial x} [(Q_n f)(x)] = n(Q_n \psi_x(t) f)(x),$$

where $\psi_x(t) = t - x$. As per our knowledge these operators have not been studied by researchers in the last four decades due to their complicated behaviour. Very recently in [6] the author estimated moments using the concept of moment generating function and established some of the approximation properties of these operators. In the past several years the convergence estimation of many well-know operators is an active area of research amongst researchers. Many known operators have been appropriately modified and their approximation behaviours have been discussed. In this direction some summation-integral operators were introduced and studied in [4, 10, 15]. For the statistical convergence, we refer the readers to [13], concerning difference between two operators [7], simultaneous approximation [9], Bézier

bases with shape parameter [16] and moment estimations of a generalized class of Szász–Mirakyan–Durrmeyer operators [8].

The convergence rate on functions of bounded variation is also an important area of research in the recent past decades, we mention here some of the work done earlier on different operators viz. Bézier variant of the Baskakov-Kantorovich operators [1], exponential operators connected with $p(x) = 2x^{3/2}$ [2], MKZ operators [3], Baskakov-Durrmeyer type operators [4], Baskakov Bézier operators [5], nonlinear integral operators [12], Bézier variant of the Bleimann–Butzer–Hahn operators [14], Kantorovich variant of the Bleimann, Butzer and Hahn operators [18], Szász–Bézier integral operators [17], general family of operators of Durrmeyer type [15] etc. Also some better bounds to have different basis were established in [19].

We estimate in the present article the rate of convergence for the operators Q_n on functions of bounded variation, by using the optimum upper bound of the basis function.

2. AUXILIARY LEMMAS

Lemma 1. *If we denote $v_{n,m}(x) = (Q_n(\Psi_x^m(t)))(x), m = 0, 1, 2, \dots$, then following [[6, Remark 1]], we have*

$$v_{n,m}(x) = \left[\frac{\partial^m}{\partial F^m} \left\{ \exp \left(\frac{n}{x} \left(1 - \sqrt{\frac{n-2x^2F}{n}} \right) - xF \right) \right\} \right]_{F=0}.$$

Some of the central moments of the operators are given by

$$v_{n,0}(x) = 1, \quad v_{n,1}(x) = 0, \quad v_{n,2}(x) = \frac{x^3}{n}.$$

Also, for each $x \in (0, \infty)$, we get $v_{n,m}(x) = O_x(n^{-(m+1)/2})$, where $[\alpha]$ represents the integral part of α .

Lemma 2. *Let $x \in (0, \infty)$ and the kernel $\phi_n(x, t)$ is defined in (1.1), then we have*

$$\eta_n(x, y) = \int_0^y \phi_n(x, t) dt \leq \frac{x^3}{n(x-y)^2}, \quad 0 < y < x$$

$$1 - \eta_n(x, z) = \int_z^\infty \phi_n(x, t) dt \leq \frac{x^3}{n(z-x)^2}, \quad x < z < \infty.$$

Lemma 3. *For each $x \in (0, \infty)$, we have*

$$\int_0^x \phi_n(x, t) dt \leq \frac{1}{2} + \frac{\sqrt{x}}{2\sqrt{2\pi n}}.$$

Proof. Simple analysis leads us to

$$\int_0^x \phi_n(x, t) dt = \sqrt{\frac{n}{2\pi}} \int_0^x \frac{1}{t\sqrt{t}} \exp \left(\frac{n}{x} - \frac{nt}{2x^2} - \frac{n}{2t} \right) dt$$

$$\begin{aligned}
 &= \sqrt{\frac{n}{2\pi x}} \int_0^1 \frac{2}{t^2} \exp\left[-\frac{n}{2x} \left(t - \frac{1}{t}\right)^2\right] dt \\
 &= \sqrt{\frac{n}{2\pi x}} \left[\int_0^1 \left(1 + \frac{1}{t^2}\right) \exp\left[-\frac{n}{2x} \left(t - \frac{1}{t}\right)^2\right] dt \right. \\
 &\quad \left. - \exp\left(\frac{2n}{x}\right) \int_0^1 \left(1 - \frac{1}{t^2}\right) \exp\left[-\frac{n}{2x} \left(t + \frac{1}{t}\right)^2\right] dt \right] \\
 &= \sqrt{\frac{n}{2\pi x}} \left[\int_{-\infty}^0 \exp\left(-\frac{n}{2x} z^2\right) dz + \exp\left(\frac{2n}{x}\right) \int_2^{\infty} \exp\left(-\frac{n}{2x} z^2\right) dz \right] \\
 &\leq \sqrt{\frac{n}{2\pi x}} \left[\int_{-\infty}^0 \exp\left(-\frac{n}{2x} z^2\right) dz + \exp\left(\frac{2n}{x}\right) \int_2^{\infty} \frac{z}{2} \exp\left(-\frac{n}{2x} z^2\right) dz \right] \\
 &= \sqrt{\frac{n}{2\pi x}} \left[\frac{1}{2} \left(\frac{2x\pi}{n}\right)^{1/2} + \frac{x}{2n} \right] = \frac{1}{2} + \frac{\sqrt{x}}{2\sqrt{2\pi n}}.
 \end{aligned}$$

Thus the desired result follows. □

3. RATE OF CONVERGENCE

Theorem 1. *Let f be a function of bounded variation on each finite subinterval of $(0, \infty)$ satisfying the growth condition $f(t) = O(e^{\gamma t})$ as $t \rightarrow +\infty$. Then, for n large, there holds*

$$\begin{aligned}
 &\left| (Q_n f)(x) - \frac{1}{2}(f(x+) + f(x-)) \right| \\
 &\leq \frac{\sqrt{x}}{2\sqrt{2\pi n}} |f(x+) - f(x-)| + \frac{2x+1}{n} \sum_{k=1}^n v_{x_k^-}^{x_k^+}(f_x) \\
 &\quad + \frac{x e^{\gamma x}}{n} + \frac{\sqrt{x}}{\sqrt{n}} \left(\exp\left(2\gamma x + \frac{x^3(2\gamma)^2}{2n} + \frac{x^5(2\gamma)^3}{2n^2} + \dots\right) \right)^{1/2},
 \end{aligned}$$

where $x_k^- = x - x/\sqrt{k}$, $x_k^+ = x + x/\sqrt{k}$ and

$$f_x(t) = \begin{cases} f(t) - f(x-), & 0 < t < x \\ f(t) - f(x+), & x < t < \infty \\ 0, & t = x. \end{cases}$$

Proof. Let $x \in (0, \infty)$, starting with

$$\begin{aligned}
 &\left| (Q_n f)(x) - \frac{1}{2}(f(x+) + f(x-)) \right| \\
 &\leq \frac{1}{2} |f(x+) - f(x-)| \cdot |(Q_n \text{sign}(t-x))(x)| + |(Q_n f_x)(x)|.
 \end{aligned}$$

Using the constant preservation of the operators i.e $(Q_n e_0)(1) = 1$, we have

$$\begin{aligned} (Q_n \text{sign}(t-x))(x) &= \left(\int_x^\infty - \int_0^x \right) \phi_n(x,t) dt \\ &= 2 \left[\frac{1}{2} - \int_0^x \phi_n(x,t) dt \right]. \end{aligned}$$

Thus by Lemma 3, we have

$$|(Q_n \text{sign}(t-x))(x)| \leq \frac{\sqrt{x}}{\sqrt{2\pi n}}.$$

Next we estimate $(Q_n f_x)(x)$ as follows:

$$\begin{aligned} (Q_n f_x)(x) &= \int_0^\infty \phi_n(x,t) f_x(t) dt \\ &= \left(\int_0^{x_n^-} + \int_{x_n^-}^{x_n^+} + \int_{x_n^+}^\infty \right) \phi_n(x,t) f_x(t) dt \\ &=: e_1 + e_2 + e_3. \end{aligned}$$

First, integrating by parts

$$\begin{aligned} e_1 &= \int_0^{x_n^-} f_x(t) d_t(\eta_n(x,t)) \\ &= f_x(x_n^-) \eta_n(x, x_n^-) - \int_0^{x_n^-} \eta_n(x,t) d_t(f_x(t)). \end{aligned}$$

Since $|f_x(y)| \leq v_y^x(f_x)$, we have

$$|e_1| \leq v_{x_n^-}^x(f_x) \eta_n(x, x_n^-) + \int_0^{x_n^-} \eta_n(x,t) d_t(-v_t^x(f_x)).$$

Applying Lemma 2, and in the next step integrating by parts, we get

$$\begin{aligned} |e_1| &\leq 2v_{x_n^-}^x(f_x) + \frac{x^3}{n} \int_0^{x_n^-} \frac{1}{(x-t)^2} d_t(-v_t^x(f_x)) \\ &= \frac{x^3}{n} \left[\frac{1}{x^2} v_0^x(f_x) + 2 \int_0^{x_n^-} \frac{1}{(x-t)^3} v_t^x(f_x) dt \right] \\ &= \frac{x^3}{n} \left[\frac{1}{x^2} v_0^x(f_x) + \frac{1}{x^2} \sum_{k=1}^n v_{x_k^-}^x(f_x) \right] \\ &\leq \frac{2x}{n} \sum_{k=1}^n v_{x_k^-}^x(f_x). \end{aligned}$$

Next for $t \in [x_n^-, x_n^+]$ and by fact $\int_{x_n^-}^{x_n^+} d_t(\eta_n(x, t)) \leq 1$, we conclude that

$$|e_2| \leq \frac{1}{n} \sum_{k=1}^n v_{x_k}^{x_k^+}(f_x).$$

Finally

$$\begin{aligned} e_3 &= \left(\int_{x_n^+}^{2x} + \int_{2x}^{\infty} \right) \phi_n(x, t) dt \\ &=: e_{31} + e_{32}. \end{aligned}$$

Arguing analogously as in estimate of e_1 , we have

$$|e_{31}| \leq \frac{2x}{n} \sum_{k=1}^n v_{x_k}^{x_k^+}(f_x).$$

Using the growth $f_x(t) = O(e^\gamma), t \rightarrow \infty$, applying Lemma 1 and [6, Lemma 1], we have

$$\begin{aligned} |e_{32}| &\leq \int_{2x}^{\infty} \phi_n(x, t)(e^{\gamma x} + e^\gamma) dt \\ &\leq \frac{e^{\gamma x}}{x^2} \int_0^{\infty} \phi_n(x, t)(t-x)^2 dt + \frac{1}{x} \int_0^{\infty} \phi_n(x, t)e^\gamma |t-x| dt \\ &= \frac{e^{\gamma x}}{x^2} \cdot v_{n,2}(x) + \frac{1}{x} \cdot (v_{n,2}(x) \cdot Q_n(e^{2\gamma}, x))^{1/2} \\ &= \frac{xe^{\gamma x}}{n} + \frac{\sqrt{x}}{\sqrt{n}} \left(\exp \left(2\gamma x + \frac{x^3(2\gamma)^2}{2n} + \frac{x^5(2\gamma)^3}{2n^2} + \dots \right) \right)^{1/2}. \end{aligned}$$

Collecting the estimates of e_1, e_2, e_3 , we get the desired result. □

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