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AN EXTENSION OF THE MIZOGUCHI–TAKAHASHI FIXED POINT THEOREM

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Abstract. Our main theorem is an extension of the well–known Mizoguchi–Takahaashi fixed point theorem [N. Mizogochi and W. Takahashi, Fixed point theorems for multi–valued mappings on complete metric spaces, *J. Math. Anal. Appl.* 141 (1989) 177–188].

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let (X,d) be a metric space. $CB(X)$ denotes the collection of all nonempty closed bounded subsets of X. For $A,B \in CB(X)$ and $x \in X$, define

$$D(x,A) := \inf\{d(x,a); a \in A\}$$

and

$$H(A,B) := \max\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(b,A)\}.$$

Notice that H is a metric on $CB(X)$. H is called the Hausdorff metric induced by d.

Definition 1. An element $x \in X$ is said to be a fixed point of a multi–valued mapping $T : X \to CB(X)$, if it is such that $x \in T(x)$.

One can show that $(CB(X),H)$ is a complete metric space, whenever (X,d) is a complete metric space (see for example Lemma 8.1.4, of [11]).

In 1969, Nadler [6] extended the Banach contraction principle [1] to set–valued mappings as follows.

Theorem 1. *Let (X,d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists $r \in [0,1)$ such that $H(Tx,Ty) \le rd(x,y)$ for all $x,y \in X$. Then there exists $z \in X$ such that $z \in T(z)$.*

Many fixed point theorems have been proved by various authors as generalizations to the Nadler's theorem. One such generalization is due to Kaneko in [4] and Nicolae in [7]. Another generalization was proved by Mizoguchi and Takahashi [5] which is also well known as a positive proof of a conjecture posed by Simeon Reich [10].

Nadler's theorem was generalized by Mizoguchi and Takahaashi [5] in the following way.

Theorem 2. *Let (X,d) be a complete metric space and let T be a mapping from (X,d) into $(CB(X),H)$ satisfies*

$$H(Tx,Ty) \leq \alpha(d(x,y))d(x,y)$$

for all $x,y \in X$, where α be a function from $[0,\infty)$ into $[0,1)$ such that $\limsup_{s\to t^+} \alpha(s) < 1$ for all $t \in [0,\infty)$. Then T has a fixed point.

Recently Suzuki [12] proved the Mizoguchi–Takahashi's fixed point theorem by an interesting and short proof.

On the other hand, the Banach contraction principle was generalized by Reich [8, 9] as follows.

Theorem 3. *Let (X,d) be a complete metric space and let T be a mapping from (X,d) into $(CB(X),H)$ satisfies*

$$H(Tx,Ty) \leq \beta[D(x,Tx) + D(y,Ty)]$$

for all $x,y \in X$, where $\beta \in [0,\frac{1}{2})$. Then T has a fixed point.

In 1973, Hardy and Rogers [3] extended the Reich's theorem by the following way.

Theorem 4. *Let (X,d) be a complete metric space and let T be a mapping from X into X such that*

$$d(Tx,Ty) \leq \alpha d(x,y) + \beta[d(x,Tx) + d(y,Ty)] + \gamma[d(x,Ty) + d(y,Tx)]$$

for all $x,y \in X$, where $\alpha,\beta,\gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then T has a fixed point.

Recently, the authors of the present paper [2] extended Theorems 1 and 4, as follows.

Theorem 5. *Let (X,d) be a complete metric space and let T be a mapping from X into $CB(X)$ such that*

$$H(Tx,Ty) \leq \alpha d(x,y) + \beta[D(x,Tx) + D(y,Ty)] + \gamma[D(x,Ty) + D(y,Tx)]$$

for all $x,y \in X$, where $\alpha,\beta,\gamma \geq 0$ and $\alpha + 2\beta + 2\gamma < 1$. Then T has a fixed point.

In this paper, we shall generalize the above results. More precisely, we prove the following theorem, which can be regarded as an extension of all Theorems 1, 2, 3, 4 and 5.

Theorem 6. *Let (X,d) be a complete metric space and let T be mapping from X into $CB(X)$ such that*

$$\begin{aligned} H(Tx,Ty) \leq {} & \alpha(d(x,y))d(x,y) + \beta(d(x,y))[D(x,Tx) + D(y,Ty)] \\ & + \gamma(d(x,y))[D(x,Ty) + D(y,Tx)] \end{aligned} \tag{1.1}$$

for all $x, y \in X$, where α, β, γ are mappings from $[0, \infty)$ into $[0, 1)$ such that $\alpha(t) + 2\beta(t) + 2\gamma(t) < 1$ and

$$\limsup_{s \to t^+} \frac{\alpha(s) + \beta(s) + \gamma(s)}{1 - (\beta(s) + \gamma(s))} < 1$$

for all $t \in [0, \infty)$. Then T has a fixed point.

Moreover, we conclude the following results by using Theorem 6.

Corollary 1. *Let (X, d) be a complete metric space and let T be a mapping from (X, d) into $(CB(X), H)$ satisfies*

$$H(Tx, Ty) \le \beta(d(x, y))[D(x, Tx) + D(y, Ty)]$$

for all $x, y \in X$, where β be a function from $[0, \infty)$ into $[0, \frac{1}{2})$ and $\limsup_{s \to t^+} \beta(s) < \frac{1}{2}$ for all $t \in [0, \infty)$. Then T has a fixed point.

Corollary 2. *Let (X, d) be a complete metric space and let T be a mapping from (X, d) into $(CB(X), H)$ satisfies*

$$H(Tx, Ty) \le \alpha(d(x, y))d(x, y) + \beta(d(x, y))[D(x, Tx) + D(y, Ty)]$$

for all $x, y \in X$, where α, β are function from $[0, \infty)$ into $[0, 1)$ such that $\alpha(t) + 2\beta(t) < 1$ and $\limsup_{s \to t^+} \left(\frac{\alpha(s) + \beta(s)}{1 - \beta(s)} \right) < 1$ for all $t \in [0, \infty)$. Then T has a fixed point.

2. Proof of the main theorem

Proof. Define the function η from $[0, \infty)$ into $[0, 1)$ by $\eta(t) = \frac{\alpha(t) + 1 - 2\beta(t) - 2\gamma(t)}{2}$ for $t \in [0, \infty)$. Then we have the following assertions:

1) $\alpha(t) < \eta(t)$ for all $t \in [0, \infty)$.

2) $\limsup_{s \to t^+} \frac{\eta(s) + \beta(s) + \gamma(s)}{1 - (\beta(s) + \gamma(s))} < 1$ for all $t \in [0, \infty)$.

3) For $x, y \in X$ and $u \in Tx$, there exists $v \in Ty$ such that

$$\begin{aligned} d(v, u) \le \eta(d(x, y))d(x, y) &+ \beta(d(x, y))[D(x, Tx) + D(y, Ty)] \\ &+ \gamma(d(x, y))[D(x, Ty) + D(y, Tx)]. \end{aligned}$$

Putting $u = y$ in (3), we obtain that:

4) For $x \in X$ and $y \in Tx$ there exists $v \in Ty$ such that

$$\begin{aligned} d(v, y) \le \eta(d(x, y))d(x, y) &+ \beta(d(x, y))[D(x, Tx) + D(y, Ty)] \\ &+ \gamma(d(x, y))[D(x, Ty) + D(y, Tx)]. \end{aligned}$$

Hence, we can define a sequence $\{x_n\}$ such that $x_{n+1} \in Tx_n$, $x_{n+1} \ne x_n$ and

$$d(x_{n+2}, x_{n+1}) \le \eta(d(x_{n+1}, x_n))d(x_{n+1}, x_n) + \beta(d(x_{n+1}, x_n))[D(x_n, Tx_n)$$

$$+ D(x_{n+1}, Tx_{n+1})] + \gamma(d(x_{n+1}, x_n)[D(x_n, Tx_{n+1}) + D(x_{n+1}, Tx_n)]$$

for all $n \in \mathbb{N}$. It follows that

$$d(x_{n+2}, x_{n+1}) \le \frac{\eta(d(x_{n+1}, x_n)) + \beta(d(x_{n+1}, x_n)) + \gamma(d(x_{n+1}, x_n))}{1 - (\beta(d(x_{n+1}, x_n)) + \gamma(d(x_{n+1}, x_n)))} d(x_{n+1}, x_n)$$

for all $n \in \mathbb{N}$. On the other hand, we have

$$\frac{\eta(t) + \beta(t) + \gamma(t)}{1 - (\beta(t) + \gamma(t))} < 1$$

for all $t \in [0, \infty)$, so $\{d(x_{n+1}, x_n)\}$ is a non-increasing sequence in $\mathbb{R}$. Hence, $\{d(x_{n+1}, x_n)\}$ converges to some nonnegative integer τ. By assumption,

$$\limsup_{s \to \tau^+} \frac{\eta(s) + \beta(s) + \gamma(s)}{1 - (\beta(s) + \gamma(s))} < 1,$$

so we have

$$\frac{\eta(\tau) + \beta(\tau) + \gamma(\tau)}{1 - (\beta(\tau) + \gamma(\tau))} < 1,$$

and then there exists $r \in [0, 1)$ and $\epsilon > 0$ such that

$$\frac{\eta(s) + \beta(s) + \gamma(s)}{1 - \beta(s) + \gamma(s)} < r$$

for all $s \in [\tau, \tau + \epsilon]$. We can take $\nu \in \mathbb{N}$ such that

$$\tau \le d(x_{n+1}, x_n) \le \tau + \epsilon$$

for all $n \in \mathbb{N}$ with $n \ge \nu$. It follows that

$$\begin{aligned} & d(x_{n+2}, x_{n+1}) \\ \le & \frac{\eta(d(x_{n+1}, x_n)) + \beta(d(x_{n+1}, x_n)) + \gamma(d(x_{n+1}, x_n))}{1 - (\beta(d(x_{n+1}, x_n)) + \gamma(d(x_{n+1}, x_n)))} d(x_{n+1}, x_n) \\ & \le r d(x_{n+1}, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$ with $n \ge \nu$. This implies that

$$\sum_{n=1}^{\infty} d(x_{n+2}, x_{n+1}) \le \sum_{n=1}^{\nu} d(x_{n+1}, x_n) + \sum_{n=1}^{\infty} r^n d(x_{\nu+1}, x_\nu) < \infty.$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, $\{x_n\}$ converges to some point $x^* \in X$. Now, we have

$$\begin{aligned} D(x^*, Tx^*) &\le d(x^*, x_{n+1}) + D(x_{n+1}, Tx^*) \\ &\le d(x^*, x_{n+1}) + H(Tx_n, Tx^*) \\ &\le d(x^*, x_{n+1}) + \alpha(d(x_n, x^*)) d(x_n, x^*) \\ &\quad + \beta(d(x_n, x^*))[D(x_n, Tx_n) + D(x^*, Tx^*)] \end{aligned}$$

$$+\gamma(d(x_n,x^*))[D(x_n,Tx^*)+D(x^*,Tx_n)]$$

for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned} D(x^*,Tx^*) \le d(x^*,x_{n+1}) &+ \alpha(d(x_n,x^*))d(x_n,x^*) \\ &+ \beta(d(x_n,x^*))[d(x_{n+1},x_n)+D(x^*,Tx^*)] \\ &+ \gamma(d(x_n,x^*))[D(x_n,Tx^*)+d(x_{n+1},x^*)] \end{aligned}$$

for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned} & D(x^*,Tx^*) \\ & \le \limsup_{n\to\infty}(\beta(d(x_n,x^*))+\gamma(d(x_n,x^*)))D(x^*,Tx^*) \\ & \le \limsup_{n\to\infty}\left(\frac{\alpha(d(x_n,x^*))+\beta(d(x_n,x^*))+\gamma(d(x_n,x^*))}{1-(\beta(d(x_n,x^*))+\gamma(d(x_n,x^*)))}\right)D(x^*,Tx^*). \end{aligned}$$

On the other hand, we have

$$\limsup_{s\to 0^+}\left(\frac{\alpha(s)+\beta(s)+\gamma(s)}{1-(\beta(s)+\gamma(s))}\right) < 1$$

so $D(x^*,Tx^*)=0$. Since Tx^* is closed, $x^* \in Tx^*$. □

3. An Example

Recently, Suzuki, in [12], has given an example to prove that the Mizoguchi condition is not a contraction. Using the same idea, we give an example in a more general setting.

Example 1. Let l^∞ be the Banach space consisting of all bounded real sequence with supremum norm and let $\{e_n\}$ be the canonical basis of l^∞. Let $\{\tau_n\}$ be a strictly decreasing sequence of positive real numbers that convergent to τ. Put $x_n = \tau_n\, e_n$ and $X_n = \{x_n, x_{n+1}, x_{n+2}, \ldots\}$ for $n \in \mathbb{N}$ and $X = X_1$. It is easy to see that X is a bounded and closed subset of l^∞. Define a mapping T from X into $CB(X)$ by

$$Tx_n = \frac{1}{2}\, X_{n+1} = \{\frac{1}{2}x_{n+1}, \frac{1}{2}x_{n+2}, \ldots\}$$

for $n \in \mathbb{N}$ and functions α, β, γ from $[0,\infty)$ into $[0,1)$ by

$$\alpha(t)=\beta(t)=\gamma(t)=\begin{cases} \frac{\tau_{n+1}}{6\tau_n}, & t=\tau_n \text{ for some } n\in\mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then the following conditions hold:

(a) T satisfies (1.1) for all $x, y \in X$.

(b) $\alpha(t)+2\beta(t)+2\gamma(t) < 1$ for all $t \in [0,\infty)$.

(c) $\limsup_{s\to t^+} \frac{\alpha(s)+\beta(s)+\gamma(s)}{1-\beta(s)-\gamma(s)} < 1$ for all $t \in [0,\infty)$.

In fact, one can show that:

(i)

$$H(Tx_m, Tx_n) = \frac{1}{2}\tau_{n+1},$$

for all $m > n$.

(ii)

$$D(x_n, Tx_n) = D(x_n, \frac{1}{2}X_{n+1}) = \tau_n,$$
$$D(x_n, Tx_m) = D(x_n, \frac{1}{2}X_{m+1}) = \tau_n,$$

for all $m, n \in \mathbb{N}$ with $m > n$.

(iii)

$$\limsup_{t \to s+} \frac{\alpha(t) + \beta(t) + \gamma(t)}{1 - \beta(t) - \gamma(t)} = 0 \ < 1,$$

for all $s \in [0, \infty)$ with $s \neq \tau$, also,

$$\limsup_{t \to \tau+} \frac{\alpha(t) + \beta(t) + \gamma(t)}{1 - \beta(t) - \gamma(t)} = \limsup_{n \to \infty} \frac{\frac{1}{2}\ \frac{\tau_{n+1}}{\tau_n}}{1 - \frac{1}{3}\ \frac{\tau_{n+1}}{\tau_n}} = \frac{3}{4} < 1.$$

Hence, we have (c). Now, let $m, n \in \mathbb{N}$ with $m > n$. Then we have

$$\begin{aligned}
H(Tx_m, Tx_n) &= \frac{1}{2}\ \tau_{n+1} \leq \frac{1}{2}\ \tau_{n+1} + \frac{\tau_{n+1}}{6\tau_n}\ \tau_m \\
&= \frac{1}{2}\ \frac{\tau_{n+1}}{\tau_n}\ \tau_n + \frac{\tau_{n+1}}{6\tau_n}\ \tau_m \\
&= (\alpha(\tau_n) + \beta(\tau_n) + \gamma(\tau_n)\ )\ \tau_n + \beta(\tau_n)\ \tau_m \\
&= \alpha(\tau_n)\ \tau_n + \beta(\tau_n)\ (\tau_n + \tau_m) + \gamma(\tau_n)\ \tau_n \\
&\leq \alpha(d(x_m, x_n))\ d(x_m, x_n) \\
&\quad + \beta(d(x_m, x_n))(D(x_n, Tx_n) + D(x_m, Tx_m)) \\
&\quad + \gamma(d(x_m, x_n))(D(x_n, Tx_m) + D(x_m, Tx_n)).
\end{aligned}$$

This means that (a) holds.

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