2-absorbing and $n$-weakly prime submodules

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Abstract. Let $R$ be a commutative ring with identity, and let $n > 1$ be an integer. A proper submodule $N$ of an $R$-module $M$ will be called 2-absorbing [resp. $n$-weakly prime], if $r, s \in R$ and $x \in M$ with $r sx \in N$ [resp. $r sx \in N \setminus (N : M)^{n-1}N$] implies that $rs \in (N : M)$ or $rx \in N$, or $sx \in N$. These concepts are generalizations of the notions of 2-absorbing ideals and weakly prime submodules, which have been studied in [3, 4, 6, 7]. We will study 2-absorbing and $n$-weakly prime submodules in this paper. Among other results, it is proved that if $(N : M)^{n-1}N \neq (N : M)^n N$, then $N$ is 2-absorbing if and only if it is $n$-weakly prime.

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1. INTRODUCTION

Throughout this paper all rings are commutative with identity and all modules are unitary. Also we take $R$ as a commutative ring with identity, $M$ as an $R$-module, and $n > 1$ is a positive integer.

Let $N$ be a submodule of $M$. The ideal $\{ r \in R | r M \subseteq N \}$ is denoted by $(N : M)$.

It is said that a proper submodule $N$ of $M$ is prime if for $r \in R$ and $a \in M$ with $ra \in N$, either $a \in N$ or $r \in (N : M)$. If $N$ is a prime submodule of $M$, then one can easily see that $P = (N : M)$ is a prime ideal of $R$, and we say $N$ is a $P$-prime submodule. Prime submodules have been studied extensively in many papers (see, for example, [2], [4], [3]), so studying its generalization can be helpful in the amplification of this theory.

As a generalization of prime submodules, a proper submodule $N$ of $M$ is called weakly prime, if $r, s \in R$ and $x \in M$ with $r sx \in N$ implies that $rx \in N$ or $sx \in N$ (see [3, 4, 7]).

In this paper, we will introduce and study two generalizations of weakly prime submodules.
2. 2-ABSORBING SUBMODULES

According to [6] an ideal $I$ of a ring $R$ is called 2-absorbing, if $abc \in I$ for $a, b, c \in I$ implies that $ab \in I$ or $bc \in I$ or $ac \in I$.

A generalization of weakly prime submodules, which is also a module version of 2-absorbing ideals, is introduced as follows:

**Definition 1.** A proper submodule $N$ of $M$ will be called 2-absorbing if for $r, s \in R$ and $x \in M$, $rsx \in N$ implies that $rs \in (N : M)$ or $rx \in N$ or $sx \in N$.

**Lemma 1** (Theorem 2.1, Theorem 2.4, and Theorem 2.5 in [6]). Let $I$ be a 2-absorbing ideal of $R$ with $\sqrt{I} = J$. Then

1. $J$ is a 2-absorbing ideal of $R$ with $J^2 \subseteq I \subseteq J = \{ r \in R \mid r^2 \in I \}$.
2. $\{ (I : r) \}_{r \in J \setminus I}$ is a chain of prime ideals.
3. Either $J$ is a prime ideal of $R$, or $J = P_1 \cap P_2$ with $P_1, P_2 \subseteq I$, where $P_1, P_2$ are the only distinct prime ideals of $R$, which are minimal over $I$.

For each $r \in R$ and every submodule $N$ of $M$, we consider $N_r = (N : M) r = \{ x \in M \mid rx \in N \}$.

Part (ii) of the following lemma proves that 2-absorbing submodules are not too far from prime submodules.

**Proposition 1.** Let $N$ be a 2-absorbing submodule of $M$ with $\sqrt{(N : M)} = J$. Then

(i) $(N : M)$ and $J$ are 2-absorbing ideals of $R$. Furthermore
\[ J^2 \subseteq (N : M) \subseteq J = \{ r \in R \mid r^2 \in (N : M) \} \]

(ii) If $(N : M) \neq J$, then for every $r \in J \setminus (N : M)$, $N_r$ is a prime submodule containing $N$ with $J \subseteq (N_r : M)$. Moreover $\{ (N_r : M) \}_{r \in J \setminus (N : M)}$ is a chain of prime ideals.

(iii) Either $J$ is a prime ideal of $R$, or $J = P_1 \cap P_2$, where $P_1, P_2$ are the only distinct minimal prime ideals over $(N : M)$ and $P_1, P_2 \subseteq (N : M)$.

**Proof.** (i) Let $s, t, r \in R$ with $str \in (N : M)$. If $sr, tr \notin (N : M)$, then there exist $x, y \in M \setminus N$ such that $sr x, tr y \notin N$.

Since $st(r(x + y)) \in N$ and $N$ is 2-absorbing, $st \in (N : M)$ or $sr(x + y) \in N$ or $tr(x + y) \in N$. If $sr(x + y) \in N$, then since $sr x \notin N$, we have $sry \notin N$. So as $st(r y) \in N$ and $try \notin N, st \in (N : M)$.

Similarly in case $tr(x + y) \in N$, we get $st \in (N : M)$.

Now since $(N : M)$ is a 2-absorbing ideal, by Lemma 1(1), $J$ is also a 2-absorbing ideal with $J^2 \subseteq (N : M) \subseteq J = \{ r \in R \mid r^2 \in (N : M) \}$.

(ii) To prove that $N_r$ is a prime submodule, let $sx \in N_r$, where $s \in R \setminus (N_r : M)$ and $x \in M$. Then by the definition of $N_r$, $rsx \in N$ and as $N$ is 2-absorbing, $rs \in (N : M)$ or $rx \in N$ or $sx \in N$.

If $rs \in (N : M)$, then $sr M \subseteq N$, that is $s \in (N_r : M)$, which is a contradiction. If $rx \in N$, then $x \in N_r$ by the definition of $N_r$, which completes the proof.
Now suppose $sx \in N$. By part (i), $r^2 \in J^2 \subseteq (N : M)$, so $rM \subseteq N_r$, particularly $rx \in N_r$. Then $(r+s)x \in N_r$, that is $r(r+s)x \in N$, and since $N$ is 2-absorbing, $rx \in N$ or $(r+s)x \in N$ or $r(r+s) \in N$.

If $rx \in N$, then $x \in N_r$, which completes the proof. Also if $(r+s)x \in N$, then from $sx \in N$, again we get $rx \in N$ and so $x \in N_r$.

Now assume $r(r+s) \in (N : M)$. According to part (i), $r^2 \in J^2 \subseteq (N : M)$, hence $rs \in (N : M)$, and so $s \in (N_r : M)$. Whence $N_r$ is a prime submodule of $M$.

One can easily see that $(N : M) : r = (N_r : M)$. By part (i), $rJ \subseteq J^2 \subseteq (N : M)$, so $J \subseteq ((N : M) : r) = (N_r : M)$.

For the proof of the rest of this part note that by part (i), $(N : M)$ is a 2-absorbing ideal. Hence by Lemma 1(2), $\{(N : M) : r\}_{r \in J\setminus(N : M)}$ is a chain of prime ideals and $(N_r : M) = ((N : M) : r)$.

(iii) By part (i), $(N : M)$ is a 2-absorbing ideal, so the proof is clear by Lemma 1(3). \qed

Let $S$ be a multiplicatively closed subset of $R$, and $W$ a submodule of $S^{-1}M$ as $S^{-1}R$-module. We consider $W^c = \{x \in M | \frac{x}{T} \in W\}$.

The proof of the following lemma is easy and we leave it to the reader.

**Lemma 2.** Let $N$ be an 2-absorbing submodule of $M$, and $S$ a multiplicatively closed subset of $R$.

(i) If $S^{-1}N \neq S^{-1}M$, then $S^{-1}N$ is a 2-absorbing submodule of $S^{-1}M$.

(ii) If $W$ is a 2-absorbing submodule of a $S^{-1}R$-module $S^{-1}M$, then $W^c$ is a $2$-absorbing submodule of $M$.

**Lemma 3** (Proposition 1 in [9]). Let $S$ be a multiplicatively closed subset of $R$. If $N$ is a $P$-prime submodule of $M$ such that $(N : M) \cap S = \emptyset$, then $S^{-1}N$ is a prime submodule of $S^{-1}M$ as an $S^{-1}R$-module.

Let $N$ be a 2-absorbing submodule of $M$ with $(N : M) \neq \sqrt{(N : M)}$. Then evidently $(N_r : M) = ((N : M) : r)$, so according to Proposition 1(ii), $\mathcal{W} = \cap \{(N : M) : r \mid r \in \sqrt{(N : M)} \setminus (N : M)\}$ is a prime ideal. In this case we say $\mathcal{W}$ is the prime ideal related to $N$.

**Corollary 1.** Let $N$ be a 2-absorbing submodule of $M$ with $(N : M) \neq \sqrt{(N : M)}$ and $\text{dim } R < \infty$. Suppose $S$ is a multiplicatively closed subset of $R$, and $\mathcal{W}$ is the prime ideal related to $N$.

(i) If $S \cap \mathcal{W} = \emptyset$, then $S^{-1}N$ is a 2-absorbing submodule of $S^{-1}M$.

(ii) $N_{\mathcal{W}}$ is a 2-absorbing submodule of the $R_{\mathcal{W}}$-module $M_{\mathcal{W}}$.

**Proof.** (i) By Lemma 2(i), it is enough to prove that $S^{-1}M \neq S^{-1}N$. According to Proposition 1(ii), $\{(N_r : M) : r \in \sqrt{(N : M)} \setminus (N : M)\}$ is a chain of prime ideals, and since $\text{dim } R < \infty$, this chain has a minimal element, say $(N_{r_0} : M)$. Now since $(N_r : M) = ((N : M) : r)$ for each $r \in \sqrt{(N : M)} \setminus (N : M)$, by our assumption we get
\( S \cap (N_{r_0} : M) = S \cap \mathfrak{P} = \emptyset \). Now according to Proposition 1(ii) and Lemma 3, \( S^{-1}N_{r_0} \) is a prime submodule of \( S^{-1}M \) containing \( S^{-1}N \). Hence \( S^{-1}N \neq S^{-1}M \).

(ii) The proof is clear by part (i).

Lemma 4. Let \( N \) be an \( P \)-primary submodule of \( M \). Then \( N \) is 2-absorbing if and only if \( P^2 \subseteq (N : M) \). In particular for every maximal submodule \( K \) of \( M \), \( (K : M)^2 \) is a 2-absorbing ideal of \( R \).

Proof. If \( N \) is 2-absorbing, then by Proposition 1(i), \( P^2 \subseteq (N : M) \).

For the converse suppose that \( rsx \in N \) for some \( r, s \in R \) and \( x \in M \). If \( rs, sx \notin N \), then since \( N \) is \( P \)-primary, \( r, s \in P \) and so \( rs \in P^2 \subseteq (N : M) \). Therefore \( N \) is 2-absorbing.

Example 1. Let \( \mathfrak{M} \) be a maximal ideal of \( R \).

(a) Evidently, every weakly prime submodule is 2-absorbing. In particular if \( \{ P_i \}_{i \in \mathbb{N}} \) is a chain of prime ideals, then it is easy to see that for the free \( R \)-module \( \oplus_{i \in \mathbb{N}} R \), the submodule \( \oplus_{i \in \mathbb{N}} P_i \) is 2-absorbing.

(b) Let \( F \) be a faithfully flat \( R \)-module. Then \( \mathfrak{M}F \) and \( \mathfrak{M}^2F \) are 2-absorbing submodules, particularly if \( F \) is a free module, or a projective module over an integral domain.

(c) Let \( R \) be a Noetherian domain which is not a field. If \( F \) is a free \( R \)-module, then \( \mathfrak{M}^kF \) is a primary submodule for \( 2 < k \in \mathbb{N} \), but it is not 2-absorbing.

(d) Let \( R \) be a Dedekind domain domain which is not a field. If \( F \) is a free \( R \)-module, then \( \mathfrak{M}^2F \) is a 2-absorbing submodule but it is not weakly prime.

(e) If \( R \) is a unique factorization domain and \( p \) is an irreducible element of \( R \), then for the free \( R \)-module \( R \oplus R \), the submodule \( N = Rp \oplus Rp^2 \) is 2-absorbing, but it is not weakly prime.

Proof. (a) The proof is easy, so it is omitted.

(b) Since \( F \) is faithfully flat, \( \mathfrak{M}F \) and \( \mathfrak{M}^2F \) are proper submodules of \( F \). Clearly \( \sqrt{(N : F)} = \mathfrak{N} \), where \( N = \mathfrak{M}^kF \) for \( k \in \mathbb{N} \). Then \( N \) is a primary submodule, since \( \sqrt{(N : F)} \) is a maximal ideal. Evidently \( \mathfrak{M}^2 \subseteq (\mathfrak{M}^2F : F) \) and \( \mathfrak{M}^2 \subseteq (\mathfrak{M}F : F) \), and by Lemma 4, the submodules \( \mathfrak{M}F \) and \( \mathfrak{M}^2F \) are 2-absorbing.

(c) It is easy to see that in case \( F \) is a free module, \( (IF : F) = I \) for each ideal \( I \) of \( R \). As it was proved in part (b), \( \mathfrak{M}^kF \) is a primary submodule. However, if \( \mathfrak{M}^kF \) is 2-absorbing, then \( \mathfrak{M}^2 \subseteq (\mathfrak{M}^kF : F) = \mathfrak{M}^k \subseteq \mathfrak{M}^2 \) according to Lemma 4. Thus \( \mathfrak{M}^2 = \mathfrak{M}^k \). Now by Nakayama’s lemma, there exists \( r \in R \) such that \( r\mathfrak{M}^2 = 0 \) and \( r - 1 \in \mathfrak{M}^{k-2} \). Then either \( r = 0 \), or \( \mathfrak{M} = 0 \), and both are impossible.

(d) Note that for every weakly prime submodule \( N \) of a module \( M \), the ideal \( (N : M) \) is prime. Although \( (\mathfrak{M}^2F : F) = \mathfrak{M}^2 \) is not a prime ideal, consequently \( \mathfrak{M}^2F \) is not weakly prime.

(e) A straightforward calculation shows that \( N \) is 2-absorbing. But \( N \) is not weakly prime, because \( p, p(1, 1) \in N \), however \( p(1, 1) \notin N \).
Lemma 5 (Lemma 4 in [5]). Let $M$ be a finitely generated $R$-module and $B$ a submodule of $M$. If $(B : M) \subseteq P$, where $P$ is a prime ideal of $R$, then there exists a $P$-prime submodule $N$ of $M$ containing $B$.

Let $P$ be a prime ideal of $R$. For simplification, we denote the submodule $((P^2)pM_P)^c$ of $M$ by $P^{(2)}M$.

The following corollary supplies abundant examples of 2-absorbing submodules.

**Corollary 2.** Let $P$ be a prime ideal of $R$. If one of the following holds, then $P^{(2)}M$ is 2-absorbing.

(i) $(P^2)pM_P \neq M_P$.

(ii) $M$ is finitely generated and $\text{ann}(M) \subseteq P$.

**Proof.** (i) Evidently $(P^2)p \subseteq ((P^2)pM_P : M_P)$, so $P_P \subseteq \sqrt{((P^2)pM_P : M_P)}$, and since $P_P$ is a maximal ideal, $\sqrt{((P^2)pM_P : M_P)} = P_P$. Therefore $(P^2)pM_P$ is a $P_P$-primary submodule of $M_P$. Then clearly $P^{(2)}M$ is a $P$-primary submodule of $M$. Now the proof is given by Lemma 4, as $P^2 \subseteq (P^{(2)}M : M)$.

(ii) By part (i), it is enough to prove that $(P^2)pM_P \neq M_P$.

According to Lemma 5, there exists a $P$-prime submodule $N$ of $M$. Then by Lemma 3, $N_P$ is a $P_P$-primary submodule of $M_P$. Now from $P_PMP \subseteq N_P$, we get $(P^2)pM_P \subseteq N_P$. Consequently $(P^2)pM_P \neq M_P$. \[\square\]

In the following, if $A = \{N \mid N \text{ is a } P \text{-primary and 2-absorbing submodule of } M\} = \emptyset$, then we consider $\bigcap A = M$.

**Corollary 3.** If $P$ is a prime ideal of $R$, then

$$P^{(2)}M = \bigcap \{N \mid N \text{ is a } P \text{-primary and 2-absorbing submodule of } M\}.$$  

**Proof.** Set $A = \{N \mid N \text{ is a } P \text{-primary and 2-absorbing submodule of } M\}$.

If $P^{(2)}M = M$, then $A = \emptyset$, because if $N$ is a $P$-primary and 2-absorbing submodule of $M$, by Lemma 4, $P^2M \subseteq N$. Therefore $M = P^{(2)}M \subseteq (N_P)^c = N$, which is impossible. Hence $A = \emptyset$, and so in this case $\bigcap A = M = P^{(2)}M$.

Now let $P^{(2)}M \neq M$. By Corollary 2(i), $P^{(2)}M$ is 2-absorbing. Also in the proof of Corollary 2(i), we showed that $P^{(2)}M$ is $P$-primary, so $P^{(2)}M \in A$. Consequently $\bigcap A \subseteq P^{(2)}M$.

Now suppose that $N'$ is a $P$-primary and 2-absorbing submodule of $M$. Then Lemma 4 implies that $P^{(2)}M \subseteq (N'_P)^c = N'$. Consequently $P^{(2)}M = \bigcap A$. \[\square\]

A prime ideal $P$ of $R$ is said to be a divided prime ideal if $P \subseteq Rr$ for every $r \in R \setminus P$.

We consider $T(M) = \{m \in M \mid \exists \neq r \in R, \text{ rm} = 0\}$. If $M$ is a nonzero module with $T(M) = 0$, then it is easy to see that $R$ is an integral domain, and in this case we say $M$ is a torsion-free module.
Theorem 1. Let $M$ be a nonzero finitely generated module and $P$ a divided prime ideal. If $T(M) \subseteq P^2 M$, then $P^2 M$ is $2$-absorbing and

$$P^2 M = \bigcap \{N | N \text{ is a } P\text{-primary and } 2\text{-absorbing submodule of } M\},$$

particularly if $M$ is a torsion-free module.

Proof. First we show that $P^2 M$ is a proper submodule of $M$. If $P^2 M = M$, then by Nakayama’s lemma, there exists $a \in R$ such that $1 - a \in P^2$ and $aM = 0$. Since $1 - a \in P$, $a \notin P$ and as $P$ is a divided prime ideal, $1 - a \in P \subseteq Ra$. Thus there exists $t \in R$ with $1 - a = ta$. Therefore $M = (1 - a)M = taM = 0$, which is impossible.

Now by Corollary 3 and Lemma 4, it suffices to show that $P^2 M$ is $P$-primary. Suppose that $rx = s_1t_1y_1 + \cdots + s_nt_ny_n \in P^2 M$, where $s_i,t_i \in P$, $y_i,x \in M$, and $r \in R$. If $r \notin P$, then since $P$ is a divided prime, $P \subseteq Rr$, and hence there exist $r_1, \ldots, r_n \in R$ such that $s_i = rr_i \in P$, for $i = 1, \ldots, n$. Thus for each $i$, $r_i \in P$ and $r(r_1t_1y_1 + \cdots + r_nt ny_n) = rx \in P^2 M$. Hence as $x - (r_1t_1y_1 + \cdots + r_nt ny_n) \in T(M) \subseteq P^2 M$, and $r_1t_1y_1 + \cdots + r_nt ny_n \in P^2 M$, we have $x \in P^2 M$, which completes the proof. □

According to [1] an ideal $I$ of $R$ is called an $n$-almost prime ideal if for $a,b \in R$ with $ab \in I \setminus I^n$, either $a \in I$ or $b \in I$. The case $n = 2$ is called an almost prime ideal and it is due to [8].

Theorem 2. Let $R$ be a Noetherian domain, which is not a field. Then the following are equivalent.

(i) $R$ is Dedekind domain.
(ii) If $I$ is a 2-absorbing ideal of $R$, then $I$ is almost prime or $I = P_1 \cap P_2$ or $I = P^2$, where $P, P_1, P_2$ are prime ideals of $R$.

Proof. (i) $\Rightarrow$ (ii) The proof is given by [6, Theorem 3.14].
(ii) $\Rightarrow$ (i) We prove that every localization of $R$ at any nonzero prime ideal has the property introduced in (ii).

Let $J$ be a 2-absorbing ideal of $R_{\mathfrak{p}}$, where $\mathfrak{p}$ is a nonzero prime ideal of $R$. By Lemma 2, $J^c$ is a 2-absorbing ideal of $R$, and hence by our assumption, $J^c$ is almost prime or $J^c = P_1 \cap P_2$ or $J^c = P^2$, for some prime ideals $P, P_1, P_2$ of $R$.

By [10, Proposition 2.10(ii)], the localization of an almost prime ideal is almost prime if it is a proper ideal. Hence if $J^c$ is an almost prime ideal, then $(J^c)_{\mathfrak{p}} = J \neq R$, and so $J$ is an almost prime ideal of $R_{\mathfrak{p}}$.

If $J^c = P_1 \cap P_2$, then $J = (J^c)_{\mathfrak{p}} = (P_1)_{\mathfrak{p}} \cap (P_2)_{\mathfrak{p}}$, and since $J$ is a proper ideal, at least one of $(P_1)_{\mathfrak{p}}$ or $(P_2)_{\mathfrak{p}}$ is a prime ideal. So in this case either $J$ is a prime ideal or the intersection of two prime ideals.

In case $J^c = P^2$, then $J = (J^c)_{\mathfrak{p}} = (P_{\mathfrak{p}})^2$, and as $J$ is proper, the ideal $(P)_{\mathfrak{p}}$ is prime.
Therefore by considering the localization of $R$, we may suppose that $\mathfrak{M}$ is the only maximal ideal of $R$. If $\mathfrak{M} = \mathfrak{M}^2$, then by Nakayama’s lemma, $\mathfrak{M} = 0$, that is $R$ is a field. Now let $s \in \mathfrak{M} \setminus \mathfrak{M}^2$, and set $I = \mathfrak{M}^2 + Rs$.

First we prove that every ideal $K$ with $\mathfrak{M}^2 \subseteq K$ is almost prime. Hence $R$ is a Dedekind domain.

Evidently $\sqrt{K} = \mathfrak{M}$, and so $K$ is a primary ideal with $\mathfrak{M}^2 \subseteq K$. So by Lemma 4, $K$ is 2-absorbing and the hypothesis in (ii) implies that $K$ is almost prime, or $K = P_1 \cap P_2$ or $K = P^2$, where $P, P_1, P_2$ are prime ideals of $R$. If $K = P^2$, then $\mathfrak{M}^2 \subseteq K = P^2$, and so $\mathfrak{M} = P$. Thus $K = \mathfrak{M}^2$, which is impossible. If $K = P_1 \cap P_2$, then $\mathfrak{M}^2 \subseteq P_1$ and $\mathfrak{M}^2 \subseteq P_2$ and so $P_1 = P_2 = \mathfrak{M}$, that is in this case $K = \mathfrak{M}$, so evidently $K$ is (almost) prime.

By (⋆) in above, $I$ is an almost prime ideal. We will prove that $I^2 = \mathfrak{M}^2$. On the contrary let $a, b \in \mathfrak{M}$ such that $ab \notin I^2$. Thus $ab \in I \setminus I^2$, and since $I$ is almost prime, we have $a \in I$ or $b \in I$ and not both, as $ab \notin I^2$, then suppose $a \in I$ and $b \notin I$. Note that $b^2 \in \mathfrak{M}^2 \subseteq I$. Hence $b(a + b) \in I$. If $b(a + b) \notin I^2$, then $b \in I$ or $a + b \in I$, which is impossible. Hence $b(a + b) \in I^2$, and $ab \notin I^2$, therefore $b^2 \notin I^2$. Then $b^2 \in I \setminus I^2$, and so $b \in I$, which is a contradiction.

Consequently $\mathfrak{M}^2 = I^2 = \mathfrak{M}^4 + \mathfrak{M}^2s + Rs^2 = \mathfrak{M}^2(\mathfrak{M}^2 + Rs) + Rs^2$. Hence by Nakayama’s lemma $\mathfrak{M}^2 = Rs^2 \subseteq Rs$, and as $s \notin \mathfrak{M}^2$, we have $\mathfrak{M}^2 \subseteq Rs$. Thus again by (⋆), $Rs$ is almost prime. By [8, Lemma 2.6], every principal and almost prime ideal is a prime ideal, hence $Rs$ is a prime ideal. Now since $\mathfrak{M}^2 \subseteq Rs$, $\mathfrak{M} = Rs$, that is $\mathfrak{M}$ is a principal ideal. Therefore $R$ is a discrete valuation domain, in case $R$ is local.

Now for the general case, note that every localization of $R$ is a discrete valuation domain, hence $R$ is a Dedekind domain.

3. $n$-weakly prime submodules

Another generalization of weakly prime submodules is introduced in the following. The following definition is also a generalization and a module version of $n$-almost prime ideals which was introduced and studied in [11].

**Definition 2.** Let $n > 1$ be an integer. A proper submodule $N$ of $M$ will be called $n$-weakly prime, if for $r, s \in R$ and $x \in M$, $rsx \in N \setminus (N : M)^{n-1}N$ implies that $rs \in (N : M)$ or $rx \in N$ or $sx \in N$.

If we consider $R$ as an $R$-module, then evidently a proper ideal $I$ of $R$ is $n$-weakly prime if for $a, b, c \in R$, $abc \in I \setminus I^n$ implies that $ab \in I$ or $bc \in I$ or $ac \in I$.

**Remark 1.** For any submodule, we have the following implications:

1. Prime $\implies$ weakly prime $\implies$ 2-absorbing $\implies$ $n$-weakly prime.
2. $n$-weakly prime $\implies$ $(n-1)$-weakly prime, for each $n > 2$.

Evidently the zero submodule is $n$-weakly prime, but it is not necessarily 2-absorbing. The following example introduces non trivial $n$-weakly prime submodules, which are not 2-absorbing.
Example 2. Let $R = \mathbb{K}[X_1, X_2, X_3, X_4] / (X_1^2, X_2^2, X_3^2, X_4^2, X_1X_2, X_3X_4, X_1X_3X_4, X_2X_3X_4)$, where $K$ is a field of characteristic 2 and $X_1, X_2, X_3, X_4$ are independent indeterminates. Consider $M = R \oplus R$ and $I = (\bar{X}_1 \bar{X}_2 + \bar{X}_3 \bar{X}_4)$. Then the two submodules $N = \{(x, x) \mid x \in I\}$ and $N' = I \oplus I$ are $n$-weakly prime, but they are not 2-absorbing.

Proof. Evidently $(R, \mathfrak{M})$ is a local ring with $\mathfrak{M}^2 = 0$, where $\mathfrak{M} = (\bar{X}_1, \bar{X}_2, \bar{X}_3, \bar{X}_4)$. First we prove that $\bar{X}_1 \bar{X}_2 + \bar{X}_3 \bar{X}_4$ is irreducible.

Suppose $fg = \bar{X}_1 \bar{X}_2 + \bar{X}_3 \bar{X}_4 \ (\ast)$, with $f, g$ non unit. Note that $\mathfrak{M}^2 = 0$, then we can consider $f = a_1 \bar{X}_1 + a_2 \bar{X}_2 + a_3 \bar{X}_3 + a_4 \bar{X}_4 \in \mathfrak{M}$, and $g = b_1 \bar{X}_1 + b_2 \bar{X}_2 + b_3 \bar{X}_3 + b_4 \bar{X}_4 \in \mathfrak{M}$, where $a_i, b_i \in K$. From $(\ast)$ we get:

\[
\begin{align*}
(1) \quad & a_1 b_2 + a_2 b_1 = 1 \\
(2) \quad & a_1 b_3 + a_3 b_1 = 0 \\
(3) \quad & a_1 b_4 + a_4 b_1 = 0 \\
(4) \quad & a_2 b_3 + a_3 b_2 = 0 \\
(5) \quad & a_2 b_4 + a_4 b_2 = 0 \\
(6) \quad & a_3 b_4 + a_4 b_3 = 1
\end{align*}
\]

By $(2)$, $\circ = a_1 b_4 (a_1 b_3 + a_3 b_1)$ and by $(3)$, $\circ = a_1 b_3 (a_1 b_4 + a_4 b_1)$ and so $a_1 b_1 (a_3 b_4 - a_4 b_3) = 0$. Since the characteristic of $K$ is 2, $-a_4 b_3 = a_4 b_3$ and so $a_1 b_1 (a_3 b_4 + a_4 b_3) = 0$. Hence by $(6)$, $a_1 b_1 = 0$. Then $a_1 = \circ$ or $b_1 = 0$. The case $a_1 = b_1 = 0$ is impossible, by $(1)$. If $\circ = a_1$ and $\circ \neq b_1$, then $(2)$ and $(3)$ imply that $a_3 = \circ = a_4$ and this is a contradiction by $(6)$.

In case $\circ \neq a_1$ and $\circ = b_1$, then by $(2), (3)$ we get $b_3 = \circ = b_4$, which is a again impossible, according to $(6)$. Consequently $\bar{X}_1 \bar{X}_2 + \bar{X}_3 \bar{X}_4$ is irreducible.

One can easily see that $(N : M) = 0$, and so $(N : M)^n N = 0$. Also it is easy to see that $I \subseteq \mathfrak{M}^2$ and $(N : M) = I$. Then $I^2 \subseteq \mathfrak{M}^2 = 0$, and thus $(N : M)^{n-1} N' = 0$.

To show that $N$ is $n$-weakly prime, let $(\circ, \circ) \neq rs(a, b) \in N$, where $r, s \in R$ and $(a, b) \in M$. We can assume $\circ \neq rsa \in I$. Then for some $h \in R$, $\circ \neq rsa = h(\bar{X}_1 \bar{X}_2 + \bar{X}_3 \bar{X}_4)$. But since $I \mathfrak{M} \subseteq \mathfrak{M}^3 = 0$, $h \in R \setminus \mathfrak{M}$. Thus $h$ is unit and so $rsa h^{-1} = \bar{X}_1 \bar{X}_2 + \bar{X}_3 \bar{X}_4$ and it is irreducible, therefore $r$ or $sa$ is unit. Hence $r$ or $s$ is unit and so $s(a, b) \in r^{-1} N = N$ or $r(a, b) \in s^{-1} N = N$. This show that $N$ is $n$-weakly prime. The same argument proves that $N'$ is $n$-weakly prime.

Now if on the contrary $N$ is a 2–absorbing submodule, then again by Proposition 1(i), $(N : M) = 0$ must be a 2–absorbing ideal and as $0 = \mathfrak{M}^2 \subseteq (N : M)$, we will have $\mathfrak{M}^2 \subseteq (N : M) = 0$, which is impossible. Thus $N$ is not a 2-absorbing submodule.

If $N'$ is a 2–absorbing submodule, then by Proposition 1(i), $(N' : M) = I$ is a 2–absorbing ideal of $R$ and since $0 = \mathfrak{M}^3 \subseteq I$, then $\mathfrak{M}^2 \subseteq I$. Consequently $\bar{X}_1 \bar{X}_2 \in \mathfrak{M}^2 \subseteq I$. Then for some $h' \in R$, $\circ \neq \bar{X}_1 \bar{X}_2 = h'(\bar{X}_1 \bar{X}_2 + \bar{X}_3 \bar{X}_4)$. As $\mathfrak{M}^2 = 0$, $h'$ is unit and since $\bar{X}_1 \bar{X}_2 + \bar{X}_3 \bar{X}_4$ is irreducible, $\bar{X}_1$ or $\bar{X}_2$ is unit, which is impossible.

\[ \square \]

Evidently $(N : M)^{n-1} N \subseteq (N : M)^2 N$, for each submodule $N$ of $M$ for each $n > 2$. We now introduce a simple criteria for an $n$-weakly prime submodule to be 2-absorbing.
Theorem 3. Let $N$ be a submodule of $M$ with $(N : M)^2 N \nsubseteq (N : M)^{n-1} N$. Then $N$ is 2-absorbing if and only if it is $n$-weakly prime.

Proof. Let $N$ be an $n$-weakly prime submodule. Suppose $rsx \in N$, where $r, s \in R$ and $x \in M$. If $rs, sx \notin N$ and $rs \notin (N : M)$, then we prove that $(N : M)^3 N \subseteq (N : M)^{n-1} N$, which is impossible and so $N$ is 2-absorbing.

First we show that the following facts hold:

(i) $rsx \in (N : M)^{n-1} N$.
(ii) $rsN \subseteq (N : M)^{n-1} N$.
(iii) $r(N : M)x, s(N : M)x \subseteq (N : M)^{n-1} N$.
(iv) $(N : M)^2 x \subseteq (N : M)^{n-1} N$.
(v) $r(N : M)N, s(N : M)N \subseteq (N : M)^{n-1} N$.

(i) Since $N$ is $n$-weakly prime and $rs, sx \notin N$ and $rs \notin (N : M)$, then $rsx \in (N : M)^{n-1} N$.

(ii) If $rsN \nsubseteq (N : M)^{n-1} N$, then for some $y \in N$ we have $rsy \notin (N : M)^{n-1} N$. So since $rsx \in (N : M)^{n-1} N$, $rs(x + y) \notin (N : M)^{n-1} N$. Hence $rs(x + y) \notin N \setminus (N : M)^{n-1} N$ and then $r(x + y) \notin N$ or $s(x + y) \notin N$ or $rs \notin (N : M)$. Thus $rx \in N$ or $sx \in N$ or $rs \in (N : M)$, which is impossible. Consequently $rsN \subseteq (N : M)^{n-1} N$.

(iii) Let $r(N : M)x \nsubseteq (N : M)^{n-1} N$. Then there exists $t \in (N : M)$ such that $rtx \in N \setminus (N : M)^{n-1} N$. Clearly $r(s + t)x \in N$. We have $r(s + t)x \notin (N : M)^{n-1} N$, otherwise since $rsx \in (N : M)^{n-1} N$, $rtx \in (N : M)^{n-1} N$, which is a contradiction.

Then $r(s + t)x \notin (N : M)^{n-1} N$ and hence $rx \in N$ or $(s + t)x \in N$ or $r(s + t) \in (N : M)$, which implies $rxx \in N$ or $sx \in N$ or $rs \in (N : M)$, a contradiction to our assumption. Therefore $r(N : M)x \subseteq (N : M)^{n-1} N$. Similarly $s(N : M)x \subseteq (N : M)^{n-1} N$.

(iv) Let $a, b \in (N : M)$. If $abx \notin (N : M)^{n-1} N$, then since $rsx \in N$, $(a + r)(b + s)x \notin (N : M)^{n-1} N$.

If $(a + r)(b + s)x \in (N : M)^{n-1} N$, then $rsx + rbx + asx + abx \in (N : M)^{n-1} N$, and so by parts (i), (iii), $rsx + rbx + asx \in (N : M)^{n-1} N$. Hence $abx \in (N : M)^{n-1} N$, which is impossible. Thus $(a + r)(b + s)x \notin (N : M)^{n-1} N$. Therefore $(a + r)(b + s)x \in N \setminus (N : M)^{n-1} N$ and so $(a + r)x \in N$ or $(b + s)x \in N$ or $(a + r)(b + s) \in (N : M)$, which implies $rxx \in N$ or $sx \in N$ or $rs \in (N : M)$, and this is a contradiction. Then $abx \in (N : M)^{n-1} N$ and so $(N : M)^2 x \subseteq (N : M)^{n-1} N$.

(v) If for some $b \in (N : M)$ and $y \in N$, $rby \notin (N : M)^{n-1} N$, then $r(s + b)(x + y) \in N$. By parts (i),(ii),(iii), $rsx + rsy + rbx \in (N : M)^{n-1} N$ and since $rby \notin (N : M)^{n-1} N$, then $r(s + b)(x + y) \notin (N : M)^{n-1} N$. Hence $r(x + y) \in N$ or $(s + b)(x + y) \in N$ or $r(s + b) \in (N : M)$. Then $rx \in N$ or $sx \in N$ or $rs \in (N : M)$, which is a contradiction. Consequently $r(N : M)N \subseteq (N : M)^{n-1} N$ and similarity $s(N : M)N \subseteq (N : M)^{n-1} N$.

Now we prove the theorem. Let $a, b \in (N : M)$ and $y \in N$. If $aby \notin (N : M)^{n-1} N$, then obviously $(a + r)(b + s)(x + y) \in N$. If $(a + r)(b + s)(x + y) \notin (N : M)^{n-1} N$, then by previous parts $aby = (a + r)(b + s)(x + y) - (abx + asx + asy + rbx +
\[ rby + rsx + rsy \in (N : M)^{n-1}N, \] which is impossible. Thus \((a + r)(b + s)(x + y) \notin (N : M)^{n-1}N\) and so \((a + r)(b + s)(x + y) \in N \setminus (N : M)^{n-1}N\). Hence \((a + r)(x + y) \in N\) or \((b + s)(x + y) \in N\) or \((a + r)(b + s) \in (N : M)\). Therefore \(rsx \in N\) or \(sx \in N\) or \(rs \in (N : M)\), which is impossible. Consequently \((N : M)^2N \subseteq (N : M)^{n-1}N\). □

**Corollary 4.** Let \(n > 3\) and \(M\) be a nonzero torsion-free Noetherian \(R\)-module. Then a submodule is 2-absorbing if and only if it is \(n\)-weakly prime.

**Proof.** Let \(N\) be an \(n\)-weakly prime submodule. By Theorem 3, it is enough to prove that \((N : M)^{n-1}N \neq (N : M)^2N\). On the contrary suppose that \((N : M)^{n-1}N = (N : M)^2N\). Then by Nakayama’s lemma there exists \(a \in (N : M)^{n-3}\) such that \((a - 1)(N : M)^2N = 0\). As \(M\) is torsion-free, we have \(a = 1\), or \((N : M) = 0\) or \(N = 0\).

If \(a = 1\), then \(N = M\), which is impossible. Evidently \(N = 0\) is 2-absorbing. Now suppose \((N : M) = 0\). Assume \(rsx \in N\), where \(r, s \in R\) and \(x \in M\). If \(rsx \neq 0\), then \(rsx \in N \setminus (N : M)^{n-1}N\), and since \(N\) is \(n\)-weakly prime, the proof is clear in this case.

In case \(rsx = 0\), then \(rs = 0 \in (N : M)\), or \(x = 0 \in N\). □

**Proposition 2.** Let \(x \in M\) and \(a \in R\).

(i) If \(ann_M(a) \subseteq aM\), then the submodule \(aM\) is 2-absorbing if and only if it is \(n\)-weakly prime.

(ii) If \(ann_R(x) \subseteq (Rx : M)\), then the submodule \(Rx\) is 2-absorbing if and only if \(Rx\) is \(n\)-weakly prime.

**Proof.** (i) Let \(M\) be an \(n\)-weakly prime submodule and \(r, s \in R\) and \(x \in M\) with \(rsx \in aM\). If \(rsx \notin (aM : M)^{n-1}aM\), then \(rsx \in aM\) or \(sx \in aM\). Therefore assume \(rsx \in (aM : M)^{n-1}aM\). Clearly \(r(s + a)x = rsx + rax \in aM\). If \(r(s + a)x \notin (aM : M)^{n-1}aM\), then \(r(s + a) \in (aM : M)\) or \(rx \in aM\) or \((s + a)x \in aM\). So as \(a \in (aM : M)\), \(rs \in (aM : M)\) or \(rx \in aM\) or \(sx \in aM\).

Now suppose that \(r(s + a)x \in (aM : M)^{n-1}aM\). Then since \(rsx \in (aM : M)^{n-1}M\), for some \(y \in (aM : M)^{n-1}M\), we have \(arx = ay\) and so \(a(rx - y) = 0\). Hence \(rx - y \in ann_M(a) \subseteq aM\) and \(y \in (aM : M)^{n-1}M = (aM : M)^{n-2}(aM : M)M \subseteq aM\). Thus \(rx \in aM\).

(ii) Let \(Rx\) be an \(n\)-weakly prime submodule and \(r, s \in R, y \in M\) with \(rsy \in Rx\). Since \(Rx\) is \(n\)-weakly prime, we may assume \(rsy \in (Rx : M)^{n-1}Rx\). Evidently \(rs(x + y) \in Rx\). If \(rs(x + y) \notin (Rx : M)^{n-1}Rx\), then \(rs \in (Rx : M)\) or \(r(x + y) \in Rx\) or \(s(x + y) \in Rx\). Hence \(rs \in (Rx : M)\) or \(ry \in Rx\) or \(sy \in Rx\).

Now let \(rs(x + y) \in (Rx : M)^{n-1}Rx\). Then as \(rsy \in (Rx : M)^{n-1}Rx\), \(rsx \in (Rx : M)^{n-1}Rx\) and so \(rsx = tx\), for some \(t \in (Rx : M)^{n-1} \subseteq (Rx : M)\). Hence \(rs - t \in ann(x) \subseteq (Rx : M)\) and thus \(rs \in (Rx : M)\). □

**Example 3.** Let \(R\) be a unique factorization domain, \(p\) an irreducible element of \(R\), and \(M = R \oplus R\).
(a) The submodule $N = p^2 M$ is 2-absorbing.
(b) The submodule $N = p^3 M$ is neither 2-absorbing, nor $2$-weakly prime.

**Proof.** (a) Consider $ab(c, d) \in N$, where $a, b, c, d \in R$. Then a straightforward calculation shows that $a(c, d) \in N$ or $b(c, d) \in N$ or $p^2 \mid ab$.

(b) If $N$ is 2-absorbing, then by Proposition 1(i), $(N : M)$ is 2-absorbing and evidently $p^3 \in (N : M)$, therefore $p^2 \in (N : M)$. Then $p^2(1, 0) \in N = p^3 M$. Hence there exists $t \in R$ with $p^2 = p^3 t$. Then $pt = 1$, which is impossible. Therefore $N$ is not 2-absorbing and by Proposition 2(i), $N$ is not 2-weakly prime. 

Recall that the set of zero divisors of $M$, denoted by $Z(M)$ is defined by $Z(M) = \{r \in R | 3 \neq x \in M, rx = 0\}$.

The following result studies the behavior of $n$-weakly prime submodules under localization. Its proof is not difficult and we leave it to the reader.

**Proposition 3.** Let $S$ be a multiplicatively closed subset of $R$.

(i) If $N$ is an $n$-weakly prime submodule of $M$ with $S^{-1}N \neq S^{-1}M$, then $S^{-1}N$ is an $n$-weakly prime submodule of $S^{-1}M$.

(ii) Let $N$ be an $n$-weakly prime submodule of $M$ with $Z(M) \cap S = \emptyset$. Then $S^{-1}N$ is an $n$-weakly prime submodule of $S^{-1}M$ and $(S^{-1}N)^e = N$. Moreover $S^{-1}(N : M) = (S^{-1}N : S^{-1}M)$.

We can introduce the concept of $n$-weak prime as follows:

A proper submodule $N$ of $M$ will be called $n$-weakly prime, if for $r, s \in R$ and $x \in M$, $r sx \in N \setminus (N : M)^{n-1}N$ implies that $rx \in N$ or $sx \in N$.

Then similar to the proof of Theorem 3, Corollary 4 and Proposition 2 we can prove the following results:

1. Let $N$ be a submodule of $M$ with $(N : M)^2 N \not\subseteq (N : M)^{n-1}N$. Then $N$ is weakly prime if and only if it is $n$-weak prime.

2. Let $n > 3$ and $M$ be a nonzero torsion-free Noetherian $R$-module. Then a submodule is weakly prime if and only if it is $n$-weak prime.

3. Let $a \in R$ with $ann_M(a) \subseteq aM$. Then $aM$ is a weakly prime if and only if it is $n$-weak prime.

**References**


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