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ON THE EDGE-TO-VERTEX GEODETIC NUMBER OF A GRAPH

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Abstract. Let $G = (V, E)$ be a connected graph with at least three vertices. For vertices u and v in G, the *distance* $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ *geodesic*. For subsets A and B of V, the distance $d(A, B)$, is defined as $d(A, B) = min\ \{d(x, y) : x \in A, y \in B\}$. A $u - v$ path of length $d(A, B)$ is called an $A - B$ *geodesic* joining the sets $A, B \subseteq V$, where $u \in A$ and $v \in B$. A vertex x is said to *lie on* an $A - B$ geodesic if x is a vertex of an $A - B$ geodesic. A set $S \subseteq E$ is called an edge-to-vertex geodetic set if every vertex of G is either incident with an edge of S or lies on a geodesic joining a pair of edges of S. The edge-to-vertex geodetic number $g_{ev}(G)$ of G is the minimum cardinality of its edge-to-vertex geodetic sets and any edge-to-vertex geodetic set of cardinality $g_{ev}(G)$ is an edge-to-vertex geodetic basis of G. Any edge-to-vertex geodetic basis is also called a g_{ev}-set of G. It is shown that if G is a connected graph of size q and diameter d, then $g_{ev}(G) \leq q - d + 2$. It is proved that, for a tree T with $q \geq 2$, $g_{ev}(T) = q - d + 2$ if and only if T is a caterpillar. For positive integers r, d and $l \geq 2$ with $r \leq d \leq 2r$, there exists a connected graph G with $rad\ G = r$, $diam\ G = d$ and $g_{ev}(G) = l$. Also graphs G for which $g_{ev}(G) = q, q - 1$ or $q - 2$ are characterized.

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Keywords: distance, geodesic, edge-to-vertex geodetic basis, edge-to-vertex geodetic number

1. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q, respectively. We consider connected graphs with at least three vertices. For basic definitions and terminologies we refer to [1, 6]. For vertices u and v in a connected graph G, the *distance* $d(u, v)$ is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ *geodesic*. A vertex v is said to *lie on* an $x - y$ geodesic P if v is a vertex of P including the vertices x and y. A vertex v is an *internal vertex* of an $x - y$ path P if v is a vertex of P and $v \neq x, y$. An edge e of G is an *internal edge* of an $x - y$ path P if e is an edge of P with both its ends internal vertices of P. An edge e is a *pendant edge* if one of its ends is of degree 1. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, $rad\ G$ and the maximum eccentricity is its *diameter*, *diam* G of G. A *double star* is a tree of diameter 3. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete.

The *closed interval* $I[x, y]$ consists of all vertices lying on some $x-y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set*. The geodetic number of a graph was introduced in [7] and further studied in [2],[3] and [4]. It was shown in [7] that determining the geodetic number of a graph is an NP-hard problem. The forcing geodetic number of graph was introduced and studied in [5]. The connected geodetic number of graph was studied in [11]. The upper connected geodetic number and forcing connected geodetic number of a graph were studied in [12].

The edge geodetic number of a graph was studied by in [9]. An *edge geodetic set* of a connected graph G with at least two vertices is a set $S \subseteq V$ such that every edge of G is contained in a geodesic joining some pair of vertices in S. The *edge geodetic number* $g_1(G)$ of G is the minimum order of its edge geodetic sets and any edge geodetic set of order $g_1(G)$ is an *edge geodetic basis* of G.

Consider the graph G given in Figure 1. The sets $S = \{v_3, v_5\}$ and $S_1 = \{v_1, v_2, v_4\}$ are minimum geodetic set and minimum edge geodetic set of G respectively so that $g(G) = 2$ and $g_1(G) = 3$. These concepts have many applications in location the-

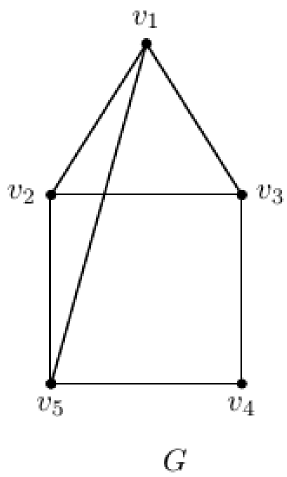

FIGURE 1.

ory and convexity theory. There are interesting applications of these concepts to the problem of designing the route for a shuttle and communication network design. We further extend these concepts to the edge set of G and present several interesting results in [10].

Throughout the following G denotes a connected graph with at least three vertices.

For subsets A and B of V, the *distance* $d(A,B)$ is defined as $d(A,B) = min\{d(x,y) : x \in A, y \in B\}$. A $u-v$ path of length $d(A,B)$ is called an $A-B$ *geodesic* joining the sets A, B, where $u \in A$ and $v \in B$. A vertex x is said to lie on an $A-B$ geodesic if x is a vertex of an $A-B$ geodesic. For $A = \{u,v\}$ and $B = \{z,w\}$ with uv and zw edges, we write an $A-B$ geodesic as $uv-zw$ geodesic and $d(A,B)$ as $d(uv,zw)$.

For the graph G given in Figure 2 with $A = \{v_4, v_5\}$ and $B = \{v_1, v_2, v_7\}$, the paths $P : v_5, v_6, v_7$ and $Q : v_4, v_3, v_2$ are the only two $A-B$ geodesics so that $d(A,B) = 2$. A set $S \subseteq E$ is called an *edge-to-vertex geodetic set* if every vertex of G is either

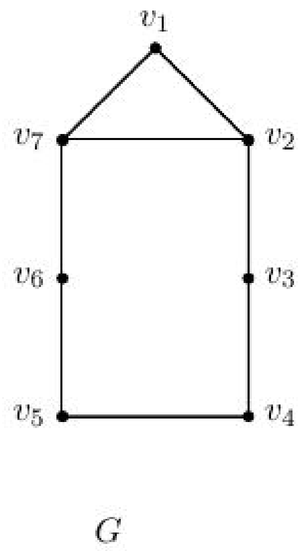

FIGURE 2.

incident with an edge of S or lies on a geodesic joining a pair of edges of S. The *edge-to-vertex geodetic number* $g_{ev}(G)$ of G is the minimum cardinality of its edge-to-vertex geodetic sets and any edge-to-vertex geodetic set of cardinality $g_{ev}(G)$ is an *edge-to-vertex geodetic basis* of G.

For the graph G given in Figure 3, the three $v_1v_6-v_3v_4$ geodesics are $P : v_1, v_2, v_3$; $Q : v_1, v_2, v_4$; and $R : v_6, v_5, v_4$ with each of length 2 so that $d(v_1v_6, v_3v_4) = 2$. Since the vertices v_2 and v_5 lie on the $v_1v_6-v_3v_4$ geodesics P and R respectively, $S = \{v_1v_6, v_3v_4\}$ is an edge-to-vertex geodetic basis of G so that $g_{ev}(G) = 2$. For the graph G given in Figure 2, $S_1 = \{v_1v_2, v_1v_7, v_4v_5\}$ and $S_2 = \{v_1v_2, v_4v_5, v_6v_7\}$ are two g_{ev}-sets of G. Thus there can be more than one g_{ev}-set of G.

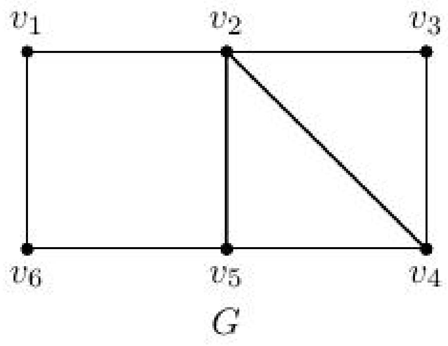

FIGURE 3.

For a connected graph G of size $q \geq 2$, it is clear that $2 \leq g_{ev}(G) \leq q$. Further, these bounds for $g_{ev}(G)$ are sharp. For the star $G = K_{1,q}(q \geq 2)$, it is clear that the set of all edges is the unique edge-to-vertex geodetic set so that $g_{ev}(G) = q$. The set of two end-edges of a path P of length at least 2 is its unique edge-to-vertex geodetic basis so that $g_{ev}(P) = 2$. Thus the star $K_{1,q}$ has the largest possible edge-to-vertex geodetic number q and the paths of length at least 2 have the smallest edge-to-vertex geodetic number 2.

An edge of a connected graph G is called an *extreme edge* of G if one of its ends is an extreme vertex of G. An edge e of a connected graph G is an *edge-to-vertex geodetic edge* in G if e belongs to every edge-to-vertex geodetic basis of G. If G has a unique edge-to-vertex geodetic basis S, then every edge in S is an edge-to-vertex geodetic edge of G.

For the graph G given in Figure 4, $S = \{ux, zv\}$ is the unique edge-to-vertex geodetic basis so that both the edges in S are edge-to-vertex geodetic edges of G. For the graph G given in Figure 5, $S_1 = \{v_1v_2, v_6v_7, v_7v_8\}$, $S_2 = \{v_1v_2, v_5v_6, v_7v_8\}$ and $S_3 = \{v_1v_2, v_5v_8, v_6v_7\}$ are the only g_{ev}-sets of G so that every g_{ev}-set contains the edge v_1v_2. Hence the edge v_1v_2 is the unique edge-to-vertex geodetic edge of G. The following theorems from [10] are used in the sequel.

Theorem 1. *If v is an extreme vertex of a connected graph G, then every edge-to-vertex geodetic set contains at least one extreme edge that is incident with v.*

Theorem 2. *Every pendant edge of a connected graph G belongs to every edge-to-vertex geodetic set of G.*

Theorem 3. *For a non-trivial tree T with k end-vertices, $g_{ev}(T) = k$ and the set of all pendant edges of T is the unique edge-to-vertex geodetic basis of T.*

Theorem 4. *For the complete graph $K_p(p \geq 4)$ with p even, $g_{ev}(K_p) = p/2$.*

Theorem 5. *For the cycle $C_p(p \geq 4)$,* $g_{ev}(C_p) = \begin{cases} 2 & \text{if } p \text{ is even} \\ 3 & \text{if } p \text{ is odd.} \end{cases}$

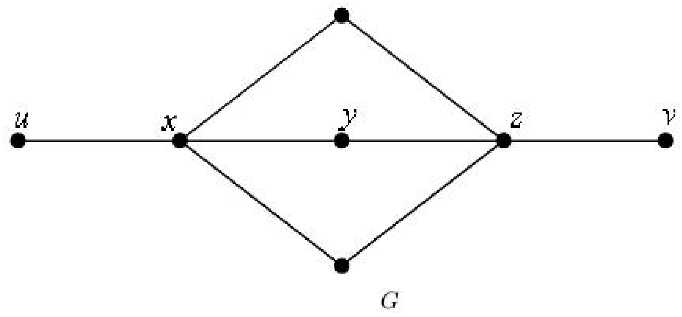

FIGURE 4.

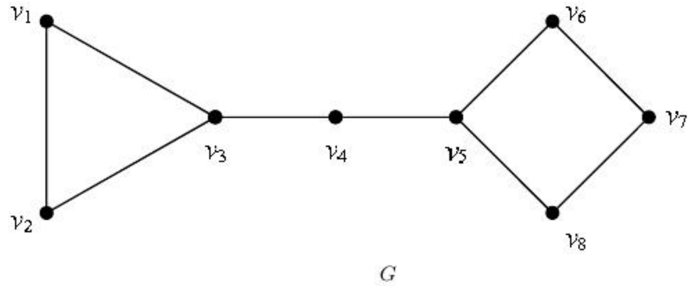

FIGURE 5.

2. THE EDGE-TO-VERTEX GEODETIC NUMBER AND DIAMETER OF A GRAPH

If G is a connected graph of size $q \geq 2$, then $2 \leq g_{ev}(G) \leq q$. An improved upper bound for the edge-to-vertex geodetic number of a graph can be given in terms of its size q and diameter d.

Theorem 6. *For a connected graph G with $q \geq 2$, $g_{ev}(G) \leq q - d + 2$, where d is the diameter of G.*

Proof. Let u and v be vertices of G for which $d(u,v) = d$, where d is the diameter of G and let $P : u = v_0, v_1, v_2, \ldots, v_d = v$ be a $u - v$ path of length d. Let $e_i = v_{i-1}v_i (1 \leq i \leq d)$. Let $S = E(G) - \{v_1v_2, v_2v_3, \ldots, v_{d-2}v_{d-1}\}$. Let x be a vertex of G. If $x = v_i (1 \leq i \leq d-1)$, then x lies on the $e_1 - e_d$ geodesic $P_1 : v_1, v_2, \ldots, v_{d-1}$. If $x \neq v_i (1 \leq i \leq d-1)$, then x is incident with an edge of S. Therefore, S is an edge-to-vertex geodetic set of G. Consequently, $g_{ev}(G) \leq |S| = q - d + 2$. □

Remark 1. The bound in Theorem 6 is sharp. For the star $G = K_{1,q} (q \geq 2)$, $d = 2$ and $g_{ev}(G) = q$, by Theorem 3 so that $g_{ev}(G) = q - d + 2$.

We give below a characterization theorem for trees.

A *caterpillar* is a tree for which the removal of all end-vertices leaves a path.

Theorem 7. *Let $q \geq 2$. For any tree T with diameter d, $g_{ev}(T) = q - d + 2$ if and only if T is a caterpillar.*

Proof. Let $P : v_0, v_1, \ldots, v_{d-1}, v_d$ be a diametral path of length d. Let $e_i = v_{i-1}v_i (1 \leq i \leq d)$ be the edges of the diametral path P. Let k be the number of pendant edges of T and l be the number of internal edges of T other than $e_i (2 \leq i \leq d-1)$. Then $d - 2 + l + k = q$. By Theorem 3, $g_{ev}(T) = k$ and so $g_{ev}(T) = q - d + 2 - l$. Hence $g_{ev}(T) = q - d + 2$ if and only if $l = 0$, if and only if all internal vertices of T lie on the diametral path P, if and only if T is a caterpillar. □

The following theorem gives a realization result.

Theorem 8. *For each triple d, k, q of integers with $2 \leq k \leq q - d + 2$, $d \geq 4$ and $q - d + k + 1 > 0$, there exists a connected graph G of size q with diam $G = d$ and $g_{ev}(G) = k$.*

Proof. Let $2 \leq k = q - d + 2$. Let G be the graph obtained from the path P of length d by adding $q - d$ new vertices to P and joining them to a cut-vertex of P. Then G is a tree of size q and diam $G = d$. By Theorem 3, $g_{ev}(G) = q - d + 2 = k$. Now, let $2 \leq k < q - d + 2$.

Case 1. $q - d - k + 1$ is even. Let $(q - d - k + 1) \geq 2$. Let $n = \frac{q-d-k+1}{2}$. Then $n \geq 1$. Let $P_d : u_0, u_1, \ldots, u_d$ be a path of length d. Add new vertices $v_1, v_2, \ldots, v_{k-2}$ and $w_1, w_2, \ldots, w_n$ and join each $v_i (1 \leq i \leq k-2)$ with u_1 and also join each $w_i (1 \leq i \leq n)$ with u_1 and u_3 in P_d. Now, join w_1 with u_2 and we obtain the graph G in Figure 6. Then G has size q and diameter d. By Theorem 2, all the pendant edges $u_1v_i (1 \leq i \leq k-2)$, u_0u_1 and $u_{d-1}u_d$ lie in every edge-to-vertex geodetic set of G. Let $S = \{u_1v_1, u_1v_2, \ldots, u_1v_{k-2}, u_1u_0, u_{d-1}u_d\}$ be the set of all pendant edges of G. Then it is clear that S is an edge-to-vertex geodetic set of G and so $g_{ev}(G) = k$.

Case 2. $q - d - k + 1$ is odd. Let $q - d - k + 1 \geq 5$. Let $m = \frac{q-d-k}{2}$. Then $m \geq 2$. Let $P_d : u_0, u_1, \ldots, u_d$ be a path of length d. Add new vertices $v_1, v_2, \ldots, v_{k-2}$ and $w_1, w_2, \ldots, w_m$ and join each $v_i (1 \leq i \leq k-2)$ with u_1 and also join each $w_i (1 \leq i \leq m)$ with u_1 and u_3 in P_d. Now join w_1 and w_2 with u_2 and we obtain the graph G in Figure 7. Then G has size q and diameter d. Now, as in Case 1, $S = \{u_1v_1, u_1v_2, \ldots, u_1v_{k-2}, u_0u_1, u_{d-1}u_d\}$ is an edge-to-vertex geodetic set of G so that $g_{ev}(G) = k$. Let $q - d - k + 1 = 1$. Let $P_d : u_0, u_1, \ldots, u_d$ be a path of length d. Add new vertices $v_1, v_2, \ldots, v_{k-2}$ and w_1 and join each $v_i (1 \leq i \leq k-2)$ with u_1 and also join w_1 with u_1 and u_3 in P_d, there by obtaining the graph G in Figure 8. Then the graph is of size q and diameter d. Now, as in Case 1, $S = \{u_1v_1, u_1v_2, \ldots, u_1v_{k-2}, u_0u_1, u_{d-1}u_d\}$ is an edge-to-vertex geodetic set of G so that $g_{ev}(G) = k$.

Now, let $q - d - k + 1 = 3$. Let $P_d : u_0, u_1, \ldots, u_d$ be a path of length d. Add new vertices $v_1, v_2, v_3, \ldots, v_{k-2}, w_1$ and w_2 and join each $v_i (1 \leq i \leq k-2)$ with u_1 and

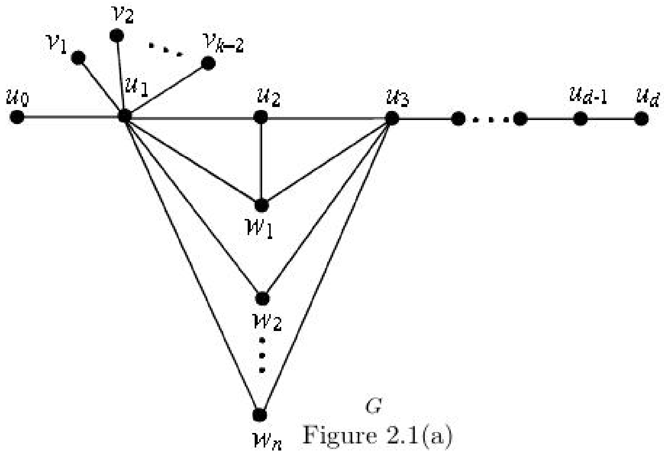

FIGURE 6.

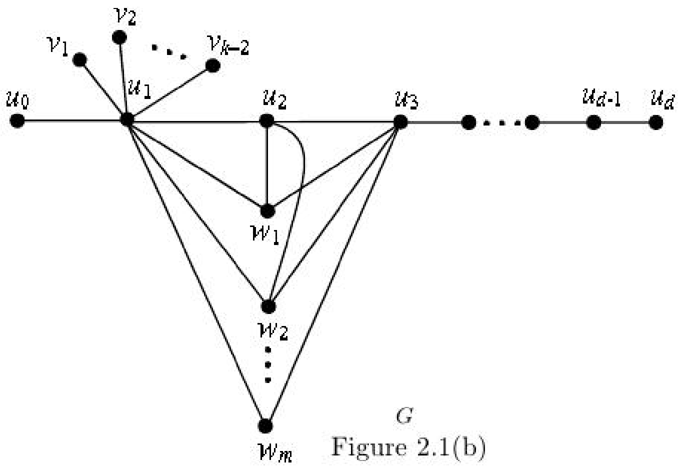

FIGURE 7.

also join w_1 and w_2 with u_1 and u_3 and obtain the graph G in Figure 9. Then G has size q and diameter d. Now, as in Case 1, $S = \{u_1v_1, u_1v_2, \ldots, u_1v_{k-2}, u_0u_1, u_{d-1}u_d\}$ is an edge-to-vertex geodetic set of G so that $g_{ev}(G) = k$. □

For every connected graph, $rad\ G \leq diam\ G \leq 2\ rad\ G$. Ostrand [8] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter, respectively, of some connected graph. Now, Ostrand's theorem can be extended so that the edge-to-vertex geodetic number can also be prescribed.

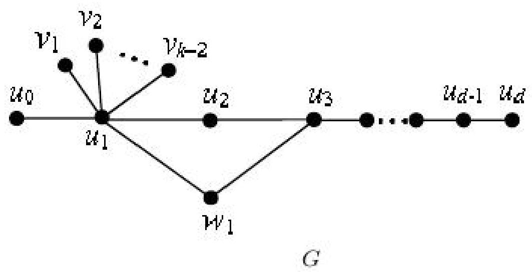

FIGURE 8.

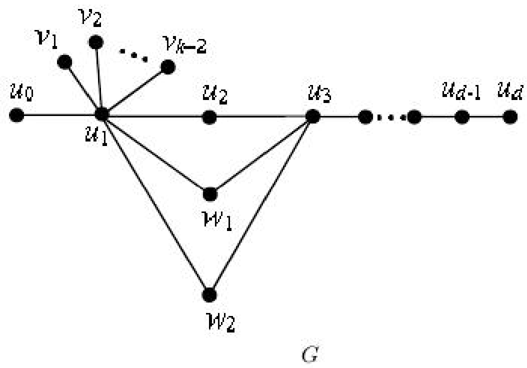

FIGURE 9.

Theorem 9. *For positive integers r, d and $l \geq 2$ with $r \leq d \leq 2r$, there exists a connected graph G with rad $G = r$, diam $G = d$ and $g_{ev}(G) = l$.*

Proof. When $r = 1$, we let $G = K_{2l}$ or $G = K_{1,l}$ according to whether $d = 1$ or $d = 2$ respectively. Then the result follows from Theorem 4 and Theorem 3 respectively. Let $r \geq 2$. If $r = d$ and $l = 2$, let $G = C_{2r}$. Then by Theorem 5, $g_{ev}(G) = 2 = l$. Let $l \geq 3$. Let $C_{2r} : u_1, u_2, \ldots, u_{2r}, u_1$ be the cycle of order $2r$. Let G be the graph obtained by adding the new vertices $x_1, x_2, \ldots, x_{l-1}$ and joining each $x_i (1 \leq i \leq l-1)$ with u_1 and u_2 of C_{2r}. The graph G is shown in Figure 10. It is easily verified that the eccentricity of each vertex of G is r so that rad $G =$ diam $G = r$. Let $S = \{u_1x_1, u_1x_2, \ldots, u_1x_{l-2}, u_2x_{l-1}\}$. It is clear that S is not an edge-to-vertex geodetic set of G. However, $S \cup \{u_{r+1}u_{r+2}\}$ is an edge-to-vertex geodetic set of G. Since $x_1, x_2, \ldots, x_{l-1}$ are the only extreme vertices of G, it follows from Theorem 1 that $g_{ev}(G) = l$.

Let $r < d$. If $l = 2$, then take G to be any path on at least three vertices. Let $l \geq 3$. Let $C_{2r} : v_1, v_2, \ldots, v_{2r}, v_1$ be a cycle of order $2r$ and let $P_{d-r+1} : u_0, u_1, u_2, \ldots, u_{d-r}$ be a path of order $d-r+1$. Let H be the graph obtained from C_{2r} and u_0 in P_{d-r+1} by identifying v_1 in C_{2r} and u_0 in P_{d-r+1}. Now, add $(l-3)$ new vertices

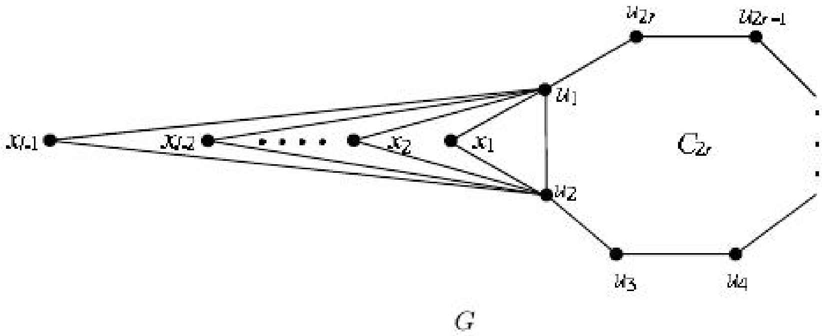

FIGURE 10.

$w_1, w_2, \ldots, w_{l-3}$ to H and join each vertex $w_i (1 \leq i \leq l-3)$ to the vertex u_{d-r-1} and obtain the graph G of Figure 11. Then rad $G = r$ and diam $G = d$. Let $S = \{u_{d-r-1}w_1, u_{d-r-1}w_2, \ldots, u_{d-r-1}w_{l-3}, u_{d-r-1}u_{d-r}\}$ be the set of pendant edges of G. By Theorem 2, S is contained in every edge-to-vertex geodetic set of G. It is clear that S is not an edge-to-vertex geodetic set of G. It is also seen that $S \cup \{e\}$, where $e \in E(G) - S$ is not an edge-to-vertex geodetic set of G. However, the set $S_1 = S \cup \{v_r v_{r+1}, v_{r+1} v_{r+2}\}$ is an edge-to-vertex geodetic set of G so that $g_{ev}(G) = l - 2 + 2 = l$. □

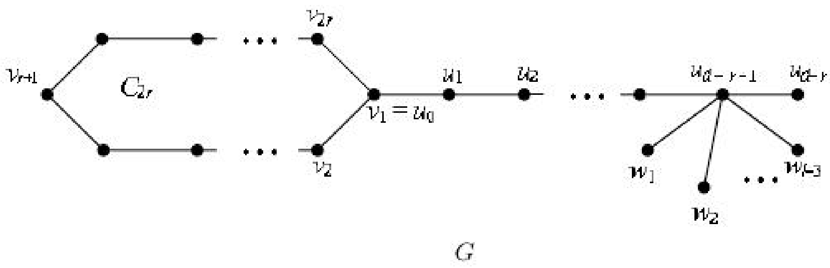

FIGURE 11.

3. GRAPHS G WITH $g_{ev}(G) = q, q-1$ AND $q-2$

In the following we characterize graphs G for which $g_{ev}(G) = q$, $q-1$ or $q-2$. Let G be a graph. A subset $M \subseteq E(G)$ is called a *matching* of G if no pair of edges in M are incident. The maximum size of such M is called the *matching number* of G and is denoted by $\alpha'(G)$. An *edge covering* of G is subset $K \subseteq E(G)$ such that each vertex of G is an end of some edge in K. The number of edges in a minimum edge covering of G, denoted by $\beta'(G)$, is the *edge covering number* of G. The well-known Gallai's theorem states that if $q \geq 1$, then $\alpha'(G) + \beta'(G) = p$. Since every edge covering for G is an edge-to-vertex geodetic set, we have the following.

Lemma A. *For any graph G, $g_{ev}(G) \leq \beta'(G) = p - \alpha'(G)$.*

We will make use of this lemma in the sequel. The proofs of the next two theorems are straightforward.

Theorem 10. *If G is a connected graph such that it is not a star, then $g_{ev}(G) \leq q-1$.*

Theorem 11. *For any connected graph G, $g_{ev}(G) = q$ if and only if G is a star.*

Theorem 12. *Let G be a connected graph which is not a tree. Then $g_{ev}(G) \leq q-2$ $(q \geq 4)$.*

Proof. Since $G \neq C_3$ and it has atleast one cycle, $\alpha'(G) \geq 2$. Thus, by Lemma A, $g_{ev}(G) \leq p - \alpha'(G) \leq q - \alpha\prime(G) \leq q-2$. □

Theorem 13. *For any connected graph G with $q \geq 3$, $g_{ev}(G) = q-1$ if and only if G is either C_3 or a double star.*

Proof. If G is C_3, then $g_{ev}(G) = 2 = q-1$. If G is a double star, then by Theorem 3, $g_{ev}(G) = q-1$. Conversely, let $g_{ev}(G) = q-1$. If G is a tree, then from Lemma A it follows that $\alpha'(G) \leq 2$. If $\alpha'(G) = 1$, then G is a star, which is impossible due to Theorem 11. So $\alpha'(G) = 2$, which implies that G is a double star. If G is not a tree, then $g_{ev}(G) = q-1 \geq p-1$. Again by Lemma A, $\alpha'(G) = 1$, which is the case only when $G = C_3$. Thus the proof is complete. □

Theorem 14. *Let G be a connected graph with $q \geq 4$, which is not a cycle and not a tree and let $C(G)$ be the length of a smallest cycle. Then $g_{ev}(G) \leq q - C(G) + 1$ if $C(G)$ is odd, and $g_{ev}(G) \leq q - C(G) + 2$ if $C(G)$ is even.*

Proof. Let $C(G)$ denote the length of a smallest cycle in G and let C be a cycle of length $C(G)$. We consider two cases.

Case 1. $C(G)$ is odd. First suppose that $C(G) = 3$. Let $C : v_1, v_2, v_3, v_1$ be a cycle of length 3. Since G is not a cycle, there exists a vertex v in G such that v is not on C and v is adjacent to v_1, say. Let $S = E(G) - \{v_1v_2, v_1v_3\}$. Then every vertex of G lies on an edge of S and so S is an edge-to-vertex geodetic set of G set of G. Thus $g_{ev}(G) \leq q-2 = q - C(G) + 1$.

Next suppose that $C(G) \geq 5$. Let $C : v_1, v_2, \ldots, v_k, v_{k+1}, v_{k+2}, \ldots, v_{2k+1}, v_1$ be a cycle of least length $C(G) = 2k+1$. Since G is not a cycle, there exists a vertex v in G such that v is not on C and v is adjacent to v_1, say. We claim that $d(vv_1, v_{k+1}v_{k+2}) = k$. Since $P : v_1, v_2, v_3, \ldots, v_{k+1}$ is a path of length k on C, it follows that $d(vv_1, v_{k+1}v_{k+2}) \leq k$. If $d(vv_1, v_{k+1}v_{k+2}) \leq k-1$, then at least one of $d(v_1, v_i)$ and $d(v, v_i)$ for $i = k+1, k+2$ is less than or equal to $k-1$. First suppose that $d(v_1, v_{k+1}) \leq k-1$. Let Q be a $v_1 - v_{k+1}$ shortest path of length at most $k-1$ different from P. Hence there exists at least one vertex of Q that is not on P and since the length of Q is at most $k-1$, it follows that a cycle of length at most $2k-1$ is formed. This is a contradiction to $C(G) = 2k+1$. Thus $d(v_1, v_{k+1}) = k$. Similarly we can prove that $d(v_1, v_{k+2}) = k$.

Next, suppose that $d(v, v_{k+1}) \leq k-1$. Since $P' : v, v_1, v_2, v_3, \ldots, v_{k+1}$ is a path of length $k+1$, it follows that $d(v, v_{k+1}) \leq k+1$. Then, as above, a cycle of length at most $2k$ is formed and this is a contradiction. Hence $d(v, v_{k+1}) = k$ or $k+1$. Similarly we can prove that $d(v, v_{k+2}) = k$ or $k+1$. Since $d(v_1, v_{k+1}) = d(v_1, v_{k+2}) = k$, it follows that $d(vv_1, v_{k+1}v_{k+2}) = k$.

Now, let $S = (E(G) - E(C)) \bigcup \{v_{k+1}v_{k+2}\}$. It is clear that the vertices $v_2, v_3, \ldots, v_k, v_{k+3}, v_{k+4}, \ldots, v_{2k+1}$ on the cycle C lie on the $vv_1 - v_{k+1}v_{k+2}$ geodesic on the cycle and all the other vertices of G are incident with an edge of S. Thus S is an edge-to-vertex geodetic set of G and so $g_{ev}(G) \leq q - C(G) + 1$.

Case 2. $C(G)$ is even. First suppose that $C(G) = 4$. Let $C : v_1, v_2, v_3, v_4, v_1$ be a cycle of length 4. Since G is not a cycle, there exists a vertex v in G such that v is not on C and v is adjacent to v_1, say. Let $S = E(G) - \{v_1v_2, v_1v_4\}$. Then every vertex of G lies on an edge of S and so S is an edge-to-vertex geodetic set of G. Thus $g_{ev}(G) \leq q - 2 = q - C(G) + 2$.

Next suppose that $C(G) \geq 6$. Let $C : v_1, v_2, \ldots . v_k, v_{k+1}, v_{k+2}, \ldots, v_{2k}, v_1$ be a cycle of least length $C(G) = 2k$. Since G is not a cycle, there exists a vertex v in G such that v is not on C and v is adjacent to v_1, say. We claim that $d(vv_1, v_kv_{k+1}) = d(vv_1, v_{k+1}v_{k+2}) = k-1$. Since $Q : v_1, v_2, v_3, \ldots, v_k$ and $Q' : v_1, v_{2k}, v_{2k-1}, \ldots, v_{k+3}, v_{k+2}$ are paths of length $k-1$ on C, it follows that $d(vv_1, v_kv_{k+1}) = d(vv_1, v_{k+1}v_{k+2}) \leq k-1$. If $d(vv_1, v_kv_{k+1}) \leq k-2$ or $d(vv_1, v_{k+1}v_{k+2}) \leq k-2$, then proceeding as in Case 1, a cycle of length at most $2k-3$ or $2k-2$ or $2k-1$ is formed as the case may be, contradicting that the least length of a cycle is $2k$. Thus $d(vv_1, v_kv_{k+1}) = d(vv_1, v_{k+1}v_{k+2}) = k-1$.

Now, if we let $S = (E(G) - E(C)) \bigcup \{v_kv_{k+1}, v_{k+1}v_{k+2}\}$, then the vertices $v_2, v_3, \ldots, v_{k-1}$ lie on the $vv_1 - v_kv_{k+1}$ geodesic on C, the vertices $v_{k+3}, v_{k+4}, \ldots, v_{2k}$ lie on the $vv_1 - v_{k+1}v_{k+2}$ geodesic on C and all the other vertices of G are incident with an edge of S. Thus S is an edge-to-vertex geodetic set of G and so $g_{ev}(G) \leq q - C(G) + 2$. □

Theorem 15. *If G is a connected graph of size $q \geq 4$ and not a tree such that $g_{ev}(G) = q - 2$, then G is unicyclic.*

Proof. Let G have more than one cycle. Then $q \geq p+1$ and so $p - 1 \leq q - 2 = g_{ev}(G) \leq p - \alpha'(G)$, by Lemma A. Hence $\alpha'(G) = 1$ and so G must be either a star or the cycle C_3, a contradiction. □

Denote by $\Im$ the two classes of graphs given in Figure 12.

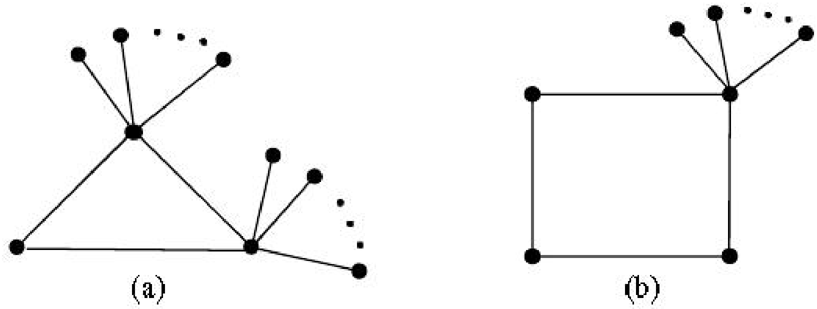

FIGURE 12.

Theorem 16. *For a connected graph G, $g_{ev}(G) = q-2$ ($q \geq 4$) if and only if G is C_4 or C_5 or $K_{1,q-1}+e$ or caterpillar with $d = 4$ or the class of graphs given in family $\Im$ of Figure 12.*

Proof. For $G = C_4$ or C_5, the result follows from Theorem 5. For a caterpillar of diameter 4, the result follows from Theorem 3. For $G = K_{1,q-1}+e$, it follows from Theorem 1 that the set of all end edges of G together with e forms an edge-to-vertex geodetic basis so that $g_{ev}(G) = q-2$. Further, it is easily verified that $g_{ev}(G) = q-2$ for the graphs given in family $\Im$ of Figure 12.

Now, let G be a connected graph such that $g_{ev}(G) = q-2$. By Theorem 15, G is either a tree or unicyclic. If G is a tree, then from Lemma A it follows that $\alpha'(G) \leq 3$. By Theorems 12 and 13, $\alpha' > 2$. So $\alpha' = 3$, which implies that G is a Caterpillar of diameter 4. If G is unicyclic, by Lemma A, $\alpha'(G) \leq 2$. Let C_k be the unique cycle of G. We have $k \leq 5$ since otherwise $\alpha'(G) \geq \alpha'(C_k) \geq 3$. Therefore, we have the following three cases:

Case 1. $k = 5$. Then G cannot have any other vertices since otherwise $\alpha'(G) \geq 3$. Therefore $G = C_5$.

Case 2. $k = 4$. If $G = C_4$, we are done. So, let $G \neq C_4$. Because $\alpha'(G) \leq 2$, only one of the vertices of C_4, say v, is of degree more than 2 and moreover all the neighbors of v are of degree 1. Thus G should be a graph like Figure 12(b).

Case 3. $k = 3$. Since $g_{ev}(C_3) = 2 = q-1$, we have $G \neq C_3$. Let $V(C_3) = \{v_1, v_2, v_3\}$. We note that if $u \in V(G) - V(C_3)$, then deg $u = 1$. Otherwise, there are $u_1, u_2 \in V(G) - V(C_3)$ such that u_1 is adjacent to both u_2 and v_1, say. Then it is easily seen that $E(G) - \{u_1v_1, v_1v_2, v_1v_3\}$ is a edge-to-vertex geodetic set, which implies $g_{ev}(G) \leq q-3$. Further, at least one of the v_is should be of degree 2. Otherwise, $E(G) - E(C_3)$ is a edge-to-vertex geodetic set, which is impossible. Thus G should be either $K_{1,q}+e$ or a graph like Figure 12(a). □

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