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ON RINGS OF QUOTIENTS OF SEMIPRIME Γ-RINGS

EMINE KOÇ AND ÖZNUR GÖLBAŞI

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Abstract. In this paper, we investigate the rings of quotients of a semiprime Γ-ring.

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1. INTRODUCTION

Let M be an abelian additive group whose elements are denoted by $a,b,c,\cdots$ and let Γ another abelian additive group whose elements are $\gamma,\beta,\alpha,\cdots$. Suppose that $a\gamma b$ is defined to be an element of M and that $\gamma a\beta$ is defined to be an element of Γ for every a,b,γ and β. If the products satisfy the following three conditions:

(i) $(a+b)\gamma c = a\gamma c + b\gamma c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\gamma(b+c) = a\gamma b + a\gamma c$,

(ii) $(a\gamma b)\beta c = a(\gamma b\beta)c = a\gamma(b\beta c)$,

(iii) $a\alpha b = 0$ for all $a,b \in M$ implies $\alpha = 0$,

then M is called a $\Gamma-$ring in the sense of Nabusawa [7].

Barnes approached Γ-ring in a different way compared to that of Nobusawa and he defined the concept of Γ-rings.

Let M and Γ be additive abelian groups. If there exits a mapping of $M \times \Gamma \times M$ to M (the image of (a,γ,b) $a,b \in M$, $\gamma \in \Gamma$ is denoted by $(a\gamma b)$) satisfying for all $a,b,c \in M$, $\alpha,\beta \in \Gamma$:

(i) $(a+b)\gamma c = a\gamma c + b\gamma c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\gamma(b+c) = a\gamma b + a\gamma c$,

(ii) $(a\gamma b)\beta c = a\gamma(b\beta c)$,

then M is called a $\Gamma-$ring in the sense of Barnes [2].

Throughout the present paper, the symbol $(\Gamma,M)_N$ that stands for M is the $\Gamma-$ring in the sense of Nabusawa, and the symbol $(\Gamma,M)_B$ that stands for M is the $\Gamma-$ring in the sense of Barnes. In [5], we have shown that for all $(\Gamma,M)_B$ there exists Γ' as an additive group such that $(\Gamma',M)_N$. Therefore, if $\Gamma-$ring in the sense of Barnes, then $\Gamma'-$ring in the sense of Nabusawa.

Let M be a $\Gamma-$ring in the sense of Nabusawa. A right (resp. left) ideal of M is an additive subgroup of U such that $U\Gamma M \subseteq U$ (resp. $M\Gamma U \subseteq U$). If U is both a right and left ideal of M, then we say that U is an ideal of M. An ideal P of a $\Gamma-$ring

M is said to be prime if for any ideals A and B of M, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. An ideal P of a Γ- ring M is said to be semiprime if for any ideal U of M, $U\Gamma U \subseteq P$ implies $U \subseteq P$. A Γ-ring M is said to be semiprime if the zero ideal is semiprime. This definition is given as "A Γ-ring M is said to be prime if $a\Gamma b = 0$ with $a, b \in M$, implies $a = 0$ or $b = 0$ and semiprime $a\Gamma a = 0$ with $a \in M$, implies $a = 0$" in [4, Theorem 2.2.23].

Let (Γ_1, M_1) and (Γ_2, M_2) be two gamma rings, $\phi : \Gamma_1 \to \Gamma_2$ and $\theta : M_1 \to M_2$ be two functions. Then an ordered pair (ϕ, θ) of mappings is called a homomorphism of (Γ_1, M_1) into (Γ_2, M_2) if it satisfies the following properties:

(i) $\theta : M_1 \to M_2$ is group homomorphism,

(ii) $\phi : \Gamma_1 \to \Gamma_2$ is group homomorphism,

(iii) $\theta(x\alpha y) = \theta(x)\phi(\alpha)\theta(y)$, for all $x, y \in M$, $\alpha \in \Gamma$,

(iv) $\phi(\alpha x\beta) = \phi(\alpha)\theta(x)\phi(\beta)$, for all $x \in M$, $\alpha, \beta \in \Gamma$.

A homomorphism (ϕ, θ) of a gamma ring (Γ_1, M_1) into a gamma ring (Γ_2, M_2) is called a monomorphism if ϕ and θ are one-one.

Let M be a Γ-ring. A commutative additive group N is called a right gamma M-module (or right ΓM-module) if for all $n, n_1, n_2 \in N$, $m, m_1, m_2 \in M$ and $\alpha, \beta \in \Gamma$,

(i) $n\alpha m \in N$,

(ii) $(n_1 + n_2)\alpha m = n_1\alpha m + n_2\alpha m$,

(iii) $n(\alpha + \beta)m = n\alpha m + n\beta m$,

(iv) $n\alpha(m_1 + m_2) = n\alpha m_1 + n\alpha m_2$.

Let N_1 and N_2 be two right gamma M-modules. Then θ is called a right gamma M-module homomorphism (or right ΓM-module homomorphism) of N_1 into N_2 if it satisfies the following properties:

(i) $\theta : N_1 \to N_2$ is group homomorphism,

(ii)$\theta(x\alpha m) = \theta(x)\alpha m$, for all $x \in N_1$, $m \in M$, $\alpha \in \Gamma$.

A great deal work has been done on Γ-ring in the sense of Barnes and Nabusawa, the results can be compared to those of in the ring theory since 1964. On the other hand, it will be seen that rings of quotients play crucial role in the study of generalized identities in prime and semiprime rings. The study of two-sided rings of quotients was initiated by W. S. Martindale [6] for prime rings and extented for semiprime rings by S. A. Amitsur in [1]. The concept of centroid of a prime Γ-ring was defined and researched in [9], [8], [10] and [11]. In [4], it was defined the rings of quotients of a prime Γ-ring and researched the some properties of it. In this paper, we investigate the rings of quotients of a semiprime Γ-ring. It was first constructed by [12] for rings. We extend these results for semiprime Γ-rings.

Throughout the present paper, M will a Γ-ring in the sense of Nabusawa and the symbol (Γ, M) stands for the $(\Gamma, M)_N$.

2. RESULTS

Definition 1. Let M be a Γ-ring. If there exists $e \in M$ and $\delta \in \Gamma$ such that for all $x \in M$ $e\delta x = x$, then (δ, e) is said to be strong left identity element of (Γ, M). Similarly, if there exists $e \in M$ and $\delta \in \Gamma$ such that for all $x \in M$ $x\delta e = x$, then (δ, e) is said to be strong right identity element of (Γ, M). If (δ, e) is both a right and left strong identity element of (Γ, M), then we say that (δ, e) is an strong identity element of (Γ, M).

Definition 2. Let M be a Γ-ring. A right ideal J of M is said to be dense if given any $0 \neq m_1 \in M, m_2 \in M$ there exists $m \in M$, $\gamma \in \Gamma$ such that $m_1\gamma m \neq 0$ and $m_2\gamma m \in J$. One defines a dense left ideal in an analogous fashion. The collection of all dense right ideal of M will be denoted by $D(\Gamma, M)$.

Let M be a Γ-ring. For a subset S of M,

$$r_\Gamma(S) = \{c \in M \mid S\gamma c = (0),\ \forall\gamma \in \Gamma\}$$

is called the right annihilator of S. A left annihilator $l_\Gamma(S)$ can be defined similarly.

Let N be a $\Gamma M-$ module. For any submodule J of N and any subset $S \subseteq N$ we set

$$(S : J)_\Gamma = \{c \in M \mid S\gamma c \subseteq J,\ \forall\gamma \in \Gamma\}.$$

In particular, for any $a \in M$

$$(a : J)_\Gamma = \{c \in M \mid a\gamma c \in J,\ \forall\gamma \in \Gamma\}.$$

Theorem 1. *Let M be a semiprime Γ-ring, $U, V \in D(\Gamma, M)$ and $f : U \to M$ be a right $\Gamma M-$module homomorphism. Then*

i) $f^{-1}(V) = \{c \in U \mid f(c) \in V\} \in D(\Gamma, M)$.

ii)$(a : U)_\Gamma \in D(\Gamma, M)$ for all $a \in M$.

iii) $U \cap V \in D(\Gamma, M)$.

iv) If W is a right ideal of M and $U \subseteq W$, then $W \in D(\Gamma, M)$.

v) $l_\Gamma(U) = (0) = r_\Gamma(U)$.

vi) If W is a right ideal of M and $(a : W)_\Gamma \in D(\Gamma, M)$ for all $a \in M$, then $W \in D(\Gamma, M)$.

vii) $U\Gamma V \in D(\Gamma, M)$.

Proof. i) Clearly, $f^{-1}(V)$ is a right ideal of M. Let $m_1 \neq 0$, $m_2 \in M$. By the dense right ideal U of M, we get $m_1\gamma' m' \neq 0$ and $m_2\gamma' m' \in U$ for some $m' \in M$, $\gamma' \in \Gamma$. Since $f(m_2\gamma' m') \in M$ and $V \in D(\Gamma, M)$, we see that there exists $m'' \in M$, $\gamma'' \in \Gamma$ such that $(m_1\gamma' m')\gamma'' m'' \neq 0$ and $f(m_2\gamma' m')\gamma'' m'' \in V$. Using f is right $\Gamma M-$module homomorphism, we have

$$f\left(m_2\gamma' m'\right)\gamma'' m'' = f\left(\left(m_2\gamma' m'\right)\gamma'' m''\right) = f\left(m_2\gamma'\left(m'\gamma'' m''\right)\right).$$

Setting $m = m'\gamma'' m''$, we conclude that $m_1\gamma' m \neq 0$ and $f(m_2\gamma' m) \in V$. That is, $m_1\gamma' m \neq 0$ and $m_2\gamma' m \in f^{-1}(V)$ for some $m \in M$, $\gamma' \in \Gamma$. Thus, $f^{-1}(V) \in D(\Gamma, M)$.

ii) Consider the map $\lambda_{a\gamma} : M \to M$, $\lambda_{a\gamma}(m) = a\gamma m$, $\forall m \in M, \forall \gamma \in \Gamma$. One easily checks that $\lambda_{a\gamma}$ is right ΓM−module homomorphism. We have

$$\begin{aligned}\lambda_{a\gamma}^{-1}(U) &= \left\{c \in M \,\middle|\, \lambda_{a\gamma}(c) \in U, \text{ for all } \gamma \in \Gamma\right\} \\ &= \{c \in M \,|a\gamma c \in U, \text{ for all } \gamma \in \Gamma\} \\ &= (a : U)_\Gamma\end{aligned}$$

According to (i), we find that $(a : U)_\Gamma \in D(\Gamma, M)$.

iii) If i is the inclusion map $U \to M$, then

$$i^{-1}(V) = U \cap V.$$

Now apply (i), we conclude that $U \cap V \in D(\Gamma, M)$.

iv) Let W be a right ideal of M, $U \subseteq W$ and $m_1 \neq 0, m_2 \in M$. Since U is a dense right ideal of M, we obtain that $m_1\gamma m \neq 0$ and $m_2\gamma m \in U$ for some $m \in M$, $\gamma \in \Gamma$. Using $U \subseteq W$, we get $m_1\gamma m \neq 0$ and $m_2\gamma m \in W$. That is $W \in D(\Gamma, M)$.

v) We assume that $U\gamma c = (0)$ for some $0 \neq c \in M$, for all $\gamma \in \Gamma$. Setting $m_1 = c = m_2$, we arrive that there exists $\alpha \in \Gamma$, $m \in M$ such that $0 \neq c\alpha m \in U$. Hence

$$(c\alpha m)\gamma(c\alpha m) \in (U\gamma c)\alpha m = (0) \text{ for all } \gamma \in \Gamma,$$

and so

$$(c\alpha m)\,\Gamma\,(c\alpha m) = (0).$$

By the semiprimeness of (Γ, M), we have $c\alpha m = 0$. It contradicts $c\alpha m \neq 0$. Hence $r_\Gamma(U) = (0)$.

Next we suppose $l_\Gamma(U) \neq (0)$. We see that there exists $0 \neq c \in M$ such that $c\gamma U = (0)$ for all $\gamma \in \Gamma$. On the other hand, $c\alpha m \neq 0$ for some $\alpha \in \Gamma$, $m \in M$. Indeed, if $c\alpha m = 0$ for all $\alpha \in \Gamma$, $m \in M$, then $c\Gamma c = (0)$, and so $c = 0$ by the semiprimeness of (Γ, M). But we know that $c \neq 0$.

Setting $m_1 = c\alpha m \neq 0$, $m_2 = m$, we have $(c\alpha m)\gamma' m' \neq 0$ and $m\gamma' m' \in U$ for some $m' \in M$, $\gamma' \in \Gamma$. But

$$(c\alpha m)\gamma' m' = c\alpha\left(m\gamma' m'\right) \in c\Gamma U = (0)$$

and a contradiction is reached. So we must have $l_\Gamma(U) = (0)$.

vi) Let $m_1 \neq 0$, $m_2 \in M$. Using $U \in D(\Gamma, M)$, we see that there exists $m' \in M$, $\gamma' \in \Gamma$ such that $m_1\gamma' m' \neq 0$ and $m_2\gamma' m' \in U$. By the hypothesis, we have $(m_2\gamma' m' : W)_\Gamma \in D(\Gamma, M)$. By (v), we get $l_\Gamma((m_2\gamma' m' : W)_\Gamma) = (0)$. Hence $m_1\gamma' m' \neq 0$, we obtain $m_1\gamma' m' \notin l_\Gamma((m_2\gamma' m' : W)_\Gamma)$. That is $(m_1\gamma' m')\gamma'' m'' \neq 0$ for some $m'' \in (m_2\gamma' m' : W)_\Gamma$, $\gamma'' \in \Gamma$. Therefore,

$$\left(m_1\gamma' m'\right)\gamma'' m'' = m_1\gamma'\left(m'\gamma'' m''\right) \neq 0$$

and $(m_2\gamma' m')\gamma'' m'' = m_2\gamma'(m'\gamma'' m'') \in W$.

Setting $m = m'\gamma'' m''$, we conclude that there exists $m \in M$, $\gamma' \in \Gamma$ such that $m_1\gamma' m \neq 0$ and $m_2\gamma' m \in W$. Thus W is a dense right ideal of M.

vii) Let $m_1 \neq 0, m_2 \in M$. By (ii), we have $(m_2 : U)_\Gamma \in D(\Gamma, M)$. Using (v), we get $l_\Gamma((m_2 : U)_\Gamma) = (0)$. We obtain $m_1 \notin l_\Gamma((m_2 : U)_\Gamma)$. Then we have $m_1\gamma' m' \neq 0$ for some $m' \in (m_2 : U)_\Gamma$, $\gamma' \in \Gamma$. On the other hand, we see that there exists $m'' \in V$, $\gamma'' \in \Gamma$ such that $(m_1\gamma' m')\gamma'' m'' \neq 0$. Indeed, if $(m_1\gamma' m')\gamma'' m'' = 0$, for all $m'' \in V$, $\gamma'' \in \Gamma$, then $m_1\gamma' m' \in l_\Gamma(V)$. Also, we know that $l_\Gamma(V) = (0)$. Hence we get $m_1\gamma' m' = 0$. This is a contradiction.

Setting $m = m'\gamma'' m''$, we obtain that $m_1\gamma' m \neq 0$
and $m_2\gamma' m = m_2\gamma'(m'\gamma'' m'') = (m_2\gamma' m')\gamma'' m'' \in U\Gamma V$. Therefore,
$U\Gamma V \in D(\Gamma, M)$. □

Corollary 1. *Let M be a semiprime Γ-ring and U be a right ideal of M. Then $U \in D(\Gamma, M)$ if and only if $l_\Gamma((a : U)_\Gamma) = (0)$ for all $a \in M$.*

Proof. If $U \in D(\Gamma, M)$, then $(a : U)_\Gamma \in D(\Gamma, M)$ for all $a \in M$ by Theorem 1(ii). By Theorem 1(v), we have $l_\Gamma((a : U)_\Gamma) = (0)$ for all $a \in M$. Conversely, let $m_1 \neq 0, m_2 \in M$. Since $m_1 \neq 0$, we get $m_1 \notin l_\Gamma((m_2 : U)_\Gamma)$. Thus, $m_1\gamma m \neq 0$ for some $m \in (m_2 : U)_\Gamma$, $\gamma \in \Gamma$. That is, $m_1\gamma m \neq 0$ and $m_2\gamma m \in U$ for some $m \in M$, $\gamma \in \Gamma$. We get $U \in D(\Gamma, M)$. This completes the proof. □

We are now in a position to construct the desired gamma ring of quotients of M. Let M be a semiprime Γ-ring. Consider the set

$$\aleph = \{(f;U) \,|\, U \in D(\Gamma, M)\}$$

$$f : U \to M \text{ is a right } \Gamma M\text{-module homomorphism}$$

We define $(f;U) \sim (g;V)$ if there exists $W \subseteq U \cap V$ such that $W \in D(\Gamma, M)$ and $f = g$ on W. By [3], one easily checks that "$\sim$" is an equivalence relation and we let $[f;U]$ denote the equivalence class determined by $(f;U) \in \aleph$. Let Q be the set of all equivalence classes. We define addition on Q as follow:

$$[f;U] + [g;V] = [f+g; U \cap V]$$

First of all we note that by Theorem 1(iii), $U \cap V \in D(\Gamma, M)$. Using similar arguments as in [3], we can prove that Q is a abelian additive group.

In a similar fasion, let (M, Γ) be a semiprime gamma ring. The collection of all dense left ideal of (M, Γ) will be denoted by $D(M, \Gamma)$.

$$\Re = \{(\tau;\Omega) \,|\, \Omega \in D(M, \Gamma),\ \tau : \Omega \to \Gamma \text{ is a left } M\Gamma\text{-module homomorphism}\}$$

We will show that Γ is a left $M\Gamma$-module. Indeed, since (M, Γ) is a gamma ring, Γ is abelian additive group. Also, for all $\gamma, \gamma_1, \gamma_2 \in \Gamma$, $m, m_1, m_2 \in M$ and $\alpha, \alpha_1, \alpha_2 \in \Gamma$,

(i) $\alpha m\gamma \in \Gamma$,
(ii) $\alpha m(\gamma_1 + \gamma_2) = \alpha m\gamma_1 + \alpha m\gamma_2$,
(iii) $\alpha(m_1 + m_2)\gamma = \alpha m_1\gamma + \alpha m_2\gamma$,
(iv) $(\alpha_1 + \alpha_2)m\gamma = \alpha_1 m\gamma + \alpha_2 m\gamma$.

On the other hand, Ω is selected to be left $M\Gamma$-module.

We define $(\tau;\Omega)\approx(\sigma;\Lambda)$ if there exists $\Pi\subseteq\Omega\cap\Lambda$ such that $\Pi\in D(M,\Gamma)$ and $\tau=\sigma$ on Π. "$\approx$" is an equivalence relation. Let $[\tau;\Omega]$ denote the equivalence class determined by $(\tau;\Omega)\in\Re$ and Δ denote the set of all equivalence classes. We define addition of equivalence classes as follow:

$$[\tau;\Omega]+[\sigma;\Lambda]=[\tau+\sigma;\Omega\cap\Lambda]$$

In similar fasion, one easily checks that Δ is abelian additive group.

Let $\tau:\Omega\to\Gamma$ be a left $M\Gamma$-module homomorphism. Define

$$\overset{\Delta}{\tau}:M\Omega M\to M \text{ by } \overset{\Delta}{\tau}(m\gamma n)=m\tau(\gamma)n \text{ for all } m,n\in M,\ \gamma\in\Omega.$$

Now we define multiplication of equivalence classes as follow:

$$[f;U][\tau;\Omega][g;V]=\left[f\overset{\Delta}{\tau}g;(\overset{\Delta}{\tau}g)^{-1}(U)\right]$$

where $[f;U],[g;V]\in Q$ and $[\tau;\Omega]\in\Delta$. We will show that multiplication is well defined. For any $[f_1;U_1],[f_2;U_2],[g_1;V_1],[g_2;V_2]\in Q$ and $[\tau_1;\Omega_1],[\tau_2;\Omega_2]\in\Delta$, we get

$$([f_1;U_1],[\tau_1;\Omega_1],[g_1;V_1])=([f_2;U_2],[\tau_2;\Omega_2],[g_2;V_2]).$$

Then

$$[f_1;U_1]=[f_2;U_2],\ [\tau_1;\Omega_1]=[\tau_2;\Omega_2],\ [g_1;V_1]=[g_2;V_2],$$

and so

$$(f_1;U_1)\sim(f_2;U_2),\ (\tau_1;\Omega_1)\approx(\tau_2;\Omega_2),\ (g_1;V_1)\sim(g_2;V_2).$$

Hence, there exists W_1, $W_2\in D(\Gamma,M)$ such that $W_1\subseteq U_1\cap U_2$, $W_2\subseteq V_1\cap V_2$, $f_1=f_2$ on W_1, $g_1=g_2$ on W_2 and $\Pi\in D(M,\Gamma)$ such that $\Pi\subseteq\Omega_1\cap\Omega_2$, $\tau_1=\tau_2$ on Π.

Set $W=g_1^{-1}(W_1\Pi M)\cap W_2$. Using the same techniques in Theorem 1(vii), we prove that $W_1\Pi M\in D(\Gamma,M)$. By Theorem 1(i), we have $g_1^{-1}(W_1\Pi M)\in D(\Gamma,M)$. Also $g_1^{-1}(W_1\Pi M)\cap W_2\in D(\Gamma,M)$ by Theorem 1(iii).

Now, we will show that $W\subseteq(\overset{\Delta}{\tau_1}g_1)^{-1}(U_1)\cap(\overset{\Delta}{\tau_2}g_2)^{-1}(U_2)$. Let x be any element of $W=g_1^{-1}(W_1\Pi M)\cap W_2$. That is, $g_1(x)\in W_1\Pi M$ and $x\in W_2$. Taking $g_1(x)$ by $w_1\gamma m$, where $w_1\in W_1$, $\gamma\in\Pi$, $m\in M$, we get

$$\overset{\Delta}{\tau_1}g_1(x)=\overset{\Delta}{\tau_1}(g_1(x))=\overset{\Delta}{\tau_1}(w_1\gamma m)=w_1\tau_1(\gamma)m.$$

On the other hand, $g_1(x)=g_2(x)$ for $x\in W_2$, there by obtain

$$\overset{\Delta}{\tau_2}g_2(x)=\overset{\Delta}{\tau_2}(g_2(x))=\overset{\Delta}{\tau_2}(g_1(x))=\overset{\Delta}{\tau_2}(w_1\gamma m)=w_1\tau_2(\gamma)m.$$

For $w_1\in W_1\subseteq U_1\cap U_2$, we conclude that

$$\overset{\Delta}{\tau_1}g_1(x)=w_1\tau_1(\gamma)m\in U_1 \text{ and } \overset{\Delta}{\tau_2}g_2(x)=w_1\tau_2(\gamma)m\in U_2.$$

Hence, we find that $x \in (\overset{\Delta}{\tau_1} g_1)^{-1}(U_1)$ and $x \in (\overset{\Delta}{\tau_2} g_2)^{-1}(U_2)$, i.e., $x \in (\overset{\Delta}{\tau_1} g_1)^{-1}(U_1) \cap (\overset{\Delta}{\tau_2} g_2)^{-1}(U_2)$.

Moreover, we prove that $f_1 \overset{\Delta}{\tau_1} g_1 = f_2 \overset{\Delta}{\tau_2} g_2$ on W. Let $x \in W$. We know that $g_1(x) \in W_1 \Pi M$. Replacing $g_1(x)$ by $w_1 \gamma m$, where $w_1 \in W_1$, $\gamma \in \Pi$, $m \in M$, we have

$$(f_1 \overset{\Delta}{\tau_1} g_1)(x) = f_1(\overset{\Delta}{\tau_1}(g_1(x))) = f_1(\overset{\Delta}{\tau_1}(w_1 \gamma m)) = f_1(w_1 \tau_1(\gamma) m).$$

Using $w_1 \in W_1 \subseteq U_1$ and f_1 is a right ΓM-module homomorphism, we get

$$f_1(w_1 \tau_1(\gamma) m) = f_1(w_1) \tau_1(\gamma) m.$$

Since $f_1 = f_2$ on W_1 and $\tau_1 = \tau_2$ on Π, we obtain that

$$f_1(w_1)\tau_1(\gamma) m = f_2(w_1)\tau_2(\gamma) m = f_2(w_1 \tau_2(\gamma) m) = f_2(\overset{\Delta}{\tau_2}(w_1 \gamma m)).$$

Appliying $x \in W_2$ and $g_1 = g_2$ on W_2, we conclude that

$$(f_2 \overset{\Delta}{\tau_2})(w_1 \gamma m) = f_2(\overset{\Delta}{\tau_2}(g_1(x))) = (f_2 \overset{\Delta}{\tau_2} g_2)(x).$$

Hence we find that $f_1 \overset{\Delta}{\tau_1} g_1 = f_2 \overset{\Delta}{\tau_2} g_2$ on W, and also

Therefore,

$$\left[f_1 \overset{\Delta}{\tau_1} g_1; (\overset{\Delta}{\tau_1} g_1)^{-1}(U_1) \right] = \left[f_2 \overset{\Delta}{\tau_2} g_2; (\overset{\Delta}{\tau_2} g_2)^{-1}(U_2) \right].$$

As a result multiplication is well defined. Also, for all $[f;U], [g;V] \in Q$ and $[\tau;\Omega] \in \Delta$, we get $[f;U][\tau;\Omega][g;V] \in Q$. Indeed, we defined that $[f;U][\tau;\Omega][g;V] = \left[f \overset{\Delta}{\tau} g; (\overset{\Delta}{\tau} g)^{-1}(U) \right]$ by Theorem 1(i), we have $(\overset{\Delta}{\tau} g)^{-1}(U) \in D(\Gamma, M)$. It is clear that $f \overset{\Delta}{\tau} g$ is a right ΓM-module homomorphism. Thus, $[f;U][\tau;\Omega][g;V] \in Q$ for all $[f;U], [g;V] \in Q$ and $[\tau;\Omega] \in \Delta$.

Now we will show that (Δ, Q) is a gamma ring.

a) i) For all $[f;U], [g;V], [h;K] \in Q$ and $[\tau;\Omega] \in \Delta$,

$$\begin{aligned}
([f;U] + [g;V])[\tau;\Omega][h;K] &= [f+g; U \cap V][\tau;\Omega][h;K] \\
&= \left[(f+g) \overset{\Delta}{\tau} h; (\overset{\Delta}{\tau} h)^{-1}(U \cap V) \right] \\
&= \left[f \overset{\Delta}{\tau} h + g \overset{\Delta}{\tau} h; (\overset{\Delta}{\tau} h)^{-1}(U) \cap (\overset{\Delta}{\tau} h)^{-1}(V) \right] \\
&= \left[f \overset{\Delta}{\tau} h; (\overset{\Delta}{\tau} h)^{-1}(U) \right] + \left[g \overset{\Delta}{\tau} h; (\overset{\Delta}{\tau} h)^{-1}(V) \right] \\
&= [f;U][\tau;\Omega][h;K] + [g;V][\tau;\Omega][h;K].
\end{aligned}$$

ii) For all $[f;U],[g;V]\in Q$ and $[\tau;\Omega],[\sigma;\Lambda]\in\Delta$,

$$[f;U]([\tau;\Omega]+[\sigma;\Lambda])[g;V]=[f;U][\tau+\sigma;\Omega\cap\Lambda][g;V] \tag{2.1}$$
$$=\left[f(\tau\overset{\Delta}{+}\sigma)g;((\tau\overset{\Delta}{+}\sigma)g)^{-1}(U)\right],$$

and

$$[f;U][\tau;\Omega][g;V]+[f;U][\sigma;\Lambda][g;V] \tag{2.2}$$
$$=\left[f\overset{\Delta}{\tau}g;(\overset{\Delta}{\tau}g)^{-1}(U)\right]+\left[f\overset{\Delta}{\sigma}g;(\overset{\Delta}{\sigma}g)^{-1}(U)\right] \tag{2.3}$$
$$=\left[f\overset{\Delta}{\tau}g+f\overset{\Delta}{\sigma}g;(\overset{\Delta}{\tau}g)^{-1}(U)\cap(\overset{\Delta}{\sigma}g)^{-1}(U)\right].$$

We will show that (2.1) and (2.3) are equivalent. Let $W=(\overset{\Delta}{\tau}g)^{-1}(U)\cap(\overset{\Delta}{\sigma}g)^{-1}(U)$ and $x\in W$. Then $x\in(\overset{\Delta}{\tau}g)^{-1}(U)$ and $x\in(\overset{\Delta}{\sigma}g)^{-1}(U)$, i.e., $(\overset{\Delta}{\tau}g)(x)\in U$ and $(\overset{\Delta}{\sigma}g)(x)\in U$. Thus $(\overset{\Delta}{\tau}g)(x)+(\overset{\Delta}{\sigma}g)(x)\in U$ and so $(\overset{\Delta}{\tau}+\overset{\Delta}{\sigma})g(x)\in U$. That is, $x\in((\tau\overset{\Delta}{+}\sigma)g)^{-1}(U)$. Hence

$$W\subseteq((\tau\overset{\Delta}{+}\sigma)g)^{-1}(U)\cap(\overset{\Delta}{\tau}g)^{-1}(U)\cap(\overset{\Delta}{\sigma}g)^{-1}(U).$$

Moreover, for all $w\in W$

$$(f(\tau\overset{\Delta}{+}\sigma)g)(w)=(f\overset{\Delta}{\tau}g)(w)+(f\overset{\Delta}{\sigma}g)(w)=(f\overset{\Delta}{\tau}g+f\overset{\Delta}{\sigma}g)(w).$$

iii) For all $[f;U],[g;V],[h;K]\in Q$ and $[\tau;\Omega]\in\Delta$,

$$[f;U][\tau;\Omega]([g;V]+[h;K])=[f;U][\tau;\Omega][g+h;V\cap K]$$
$$=\left[f\overset{\Delta}{\tau}(g+h);(\overset{\Delta}{\tau}(g+h))^{-1}(U)\right]$$
$$=\left[f\overset{\Delta}{\tau}g+f\overset{\Delta}{\tau}h;(\overset{\Delta}{\tau}g)^{-1}(U)\cap(\overset{\Delta}{\tau}h)^{-1}(U)\right]$$
$$=\left[f\overset{\Delta}{\tau}g;(\overset{\Delta}{\tau}g)^{-1}(U)\right]+\left[f\overset{\Delta}{\tau}h;(\overset{\Delta}{\tau}h)^{-1}(U)\right]$$
$$=[f;U][\tau;\Omega][g;V]+[f;U][\tau;\Omega][h;K].$$

Selected $W=(\overset{\Delta}{\tau}g)^{-1}(U)\cap(\overset{\Delta}{\tau}h)^{-1}(U)$, above equality is proved in anagous way.

b) i) For all $[f;U],[g;V],[h;K]\in Q$ and $[\tau;\Omega],[\sigma;\Lambda]\in\Delta$,

$$([f;U][\tau;\Omega][g;V])[\sigma;\Lambda][h;K]=\left[f\overset{\Delta}{\tau}g;(\overset{\Delta}{\tau}g)^{-1}(U)\right][\sigma;\Lambda][h;K] \tag{2.4}$$

$$= \left[(f \overset{\Delta}{\tau} g) \overset{\Delta}{\sigma} h; (\overset{\Delta}{\sigma} h)^{-1} (\overset{\Delta}{\tau} g)^{-1} (U) \right].$$

$$[f;U][\tau;\Omega]([g;V][\sigma;\Lambda][h;K]) = [f;U][\tau;\Omega]\left[g \overset{\Delta}{\sigma} h; (\overset{\Delta}{\sigma} h)^{-1} (V) \right] \tag{2.5}$$
$$= \left[f \overset{\Delta}{\tau} (g \overset{\Delta}{\sigma} h); (\overset{\Delta}{\tau} (g \overset{\Delta}{\sigma} h))^{-1} (U) \right]$$

We will prove that (2.4) and (2.5) are equivalent. Let $W = (\overset{\Delta}{\sigma} h)^{-1} (\overset{\Delta}{\tau} g)^{-1} (U)$ and $x \in W$. Then $(\overset{\Delta}{\sigma} h)(x) \in (\overset{\Delta}{\tau} g)^{-1} (U)$ and so $(\overset{\Delta}{\tau} g)(\overset{\Delta}{\sigma} h)(x) \in U$. Finally, $x \in (\overset{\Delta}{\tau} (g \overset{\Delta}{\sigma} h))^{-1} (U)$. Thus $W \subseteq (\overset{\Delta}{\sigma} h)^{-1} (\overset{\Delta}{\tau} g)^{-1} (U) \cap (\overset{\Delta}{\tau} (g \overset{\Delta}{\sigma} h))^{-1} (U)$, and also

$$((f \overset{\Delta}{\tau} g) \overset{\Delta}{\sigma} h)(w) = (f \overset{\Delta}{\tau} (g \overset{\Delta}{\sigma} h))(w), \text{ for all } w \in W.$$

ii) For all $[f;U],[g;V],[h;K] \in Q$ and $[\tau;\Omega],[\sigma;\Lambda] \in \Delta$,

$$([f;U][\tau;\Omega][g;V])[\sigma;\Lambda][h;K] = \left[f \overset{\Delta}{\tau} g; (\overset{\Delta}{\tau} g)^{-1} (U) \right][\sigma;\Lambda][h;K]$$
$$= \left[(f \overset{\Delta}{\tau} g) \overset{\Delta}{\sigma} h; (\overset{\Delta}{\sigma} h)^{-1} (\overset{\Delta}{\tau} g)^{-1} (U) \right]$$
$$= \left[f (\overset{\Delta}{\tau} g \overset{\Delta}{\sigma}) h; (\overset{\Delta}{\tau} g \overset{\Delta}{\sigma} h)^{-1} (U) \right]$$
$$= [f;U][\overset{\Delta}{\tau} g \overset{\Delta}{\sigma}; (g \overset{\Delta}{\sigma})^{-1} (\Omega)][h;K]$$
$$= [f;U]([\tau;\Omega][g;V][\sigma;\Lambda])[h;K].$$

Similar to the above can be shown.

c) Let $[f;U][\tau;\Omega][g;V] = [0;M]$ for all $[f;U],[g;V] \in Q$ and $[\tau;\Omega] \in \Delta$. For $[I_M;M] \in Q$, where $I_M : M \to M$, is an identity right ΓM-module homomorphism, we get

$$[I_M;M][\tau;\Omega][I_M;M] = [0;M],$$

and so

$$\left[I_M \overset{\Delta}{\tau} I_M; (\overset{\Delta}{\tau} I_M)^{-1} (M) \right] = [0;M].$$

Let $W = M\Omega M$. We first prove that $M\Omega M \subseteq M \cap (\overset{\Delta}{\tau} I_M)^{-1} (M)$. For any $x = m\gamma n \in M\Omega M$, where $m,n \in M$, $\gamma \in \Omega$, we have

$$(\overset{\Delta}{\tau} I_M)(x) = (\overset{\Delta}{\tau} I_M)(m\gamma n) = \overset{\Delta}{\tau} (I_M (m\gamma n)) = \overset{\Delta}{\tau} (m\gamma n) = m\tau(\gamma)n.$$

Thus we get $(\overset{\Delta}{\tau} I_M)(x) \in M$, and so $x \in (\overset{\Delta}{\tau} I_M)^{-1} (M)$. On the other hand, we know that $x \in M$. Hence $x \in M \cap (\overset{\Delta}{\tau} I_M)^{-1} (M)$.

Now, for all $w = m\gamma n$, where $m,n \in M$, $\gamma \in \Omega$,

$$I_M \overset{\Delta}{\tau} I_M (w) = I_M \overset{\Delta}{\tau} I_M (m\gamma n) = 0\,(m\gamma n).$$

That is

$$m\tau(\gamma)n = 0, \text{ for all } m,n \in M \text{ and } \gamma \in \Omega.$$

Since (Γ, M) is gamma ring, we have $\tau(\gamma) = 0$, for all $\gamma \in \Omega$, and so, $\tau = 0$. That is, $[\tau;\Omega] = [0;\Gamma]$.

So, we see that (Δ, Q) is a gamma ring and we call it the maximal right gamma ring of quotients of (Γ, M).

Theorem 2. *Let (M,Γ) be a semiprime gamma ring. The collection of all dense left ideal of (M,Γ) will be denoted by $D(M,\Gamma)$, $\Omega, \Lambda \in D(M,\Gamma)$ and $\tau : \Omega \to \Gamma$ be a left $M\Gamma-$module homomorphism. Then*

i) $\tau^{-1}(\Lambda) = \{\gamma \in \Omega \,|\, \tau(\gamma) \in \Lambda\} \in D(M,\Gamma)$.

ii)$(\alpha : \Omega)_M \in D(M,\Gamma)$ for all $\alpha \in \Gamma$.

iii) $\Omega \cap \Lambda \in D(M,\Gamma)$.

iv) If Π is a left ideal of Γ and $\Omega \subseteq \Pi$, then $\Pi \in D(M,\Gamma)$.

v) $l_M(\Omega) = (0) = r_M(\Omega)$.

vi) If Π is a left ideal of Γ and $(\alpha : \Pi)_M \in D(M,\Gamma)$ for all $\alpha \in \Gamma$, then $\Pi \in D(M,\Gamma)$.

vii) $\Omega M \Lambda \in D(M,\Gamma)$.

Proof. The procedures in Theorem 1 can be exactly applied in set $D(M,\Gamma)$ and the same results are obtained. □

One can construct the maximal left gamma ring of quotient of (M,Γ) using the following operations. Now we can define multiplication of equivalence classes as follow:

$$[\tau;\Omega][f;U][\sigma;\Lambda] = \left[\tau \overset{\Delta}{f} \sigma; (\overset{\Delta}{f}\sigma)^{-1}(\Omega)\right]$$

where $\overset{\Delta}{f} : \Gamma U \Gamma \to \Gamma$, $\overset{\Delta}{f}(\gamma m \beta) = \gamma f(m)\beta$ and $[f;U] \in Q$ and $[\tau;\Omega], [\sigma;\Lambda] \in \Delta$.

We will prove that multiplication is well defined. For any $[\tau_1;\Omega_1], [\tau_2;\Omega_2], [\sigma_1;\Lambda_1], [\sigma_2;\Lambda_2] \in \Delta$ and $[f_1;U_1], [f_2;U_2] \in Q$, we get

$$([\tau_1;\Omega_1], [f_1;U_1], [\sigma_1;\Lambda_1]) = ([\tau_2;\Omega_2], [f_2;U_2], [\sigma_2;\Lambda_2]).$$

Then

$$[\tau_1;\Omega_1] = [\tau_2;\Omega_2], \; [f_1;U_1] = [f_2;U_2], \; [\sigma_1;\Lambda_1], [\sigma_2;\Lambda_2]$$

and so

$$(\tau_1;\Omega_1) \approx (\tau_2;\Omega_2), \; (f_1;U_1) \sim (f_2;U_2), \; (\sigma_1;\Lambda_1) \approx (\sigma_2;\Lambda_2).$$

Therefore, there exists $\Pi_1,\ \Pi_2 \in D(M,\Gamma)$ such that $\Pi_1 \subseteq \Omega_1 \cap \Omega_2,\ \Pi_2 \subseteq \Lambda_1 \cap \Lambda_2, \tau_1 = \tau_2$ on $\Pi_1,\ \sigma_1 = \sigma_2$ on Π_2 and $W \in D(\Gamma, M)$ such that $W \subseteq U_1 \cap U_2$, $f_1 = f_2$ on W.

Let $\Pi = \sigma_1^{-1}(\Gamma W \Pi_1) \cap \Pi_2$. Using Theorem 2(vii), we show that $\Gamma W \Pi_1 \in D(M,\Gamma)$. By Theorem 2(i), we see that $\sigma_1^{-1}(\Gamma W \Pi_1) \in D(M,\Gamma)$. Hence $\sigma_1^{-1}(\Gamma W \Pi_1) \cap \Pi_2 \in D(M,\Gamma)$ by Theorem 2(iii).

Now, we will show that $\Pi \subseteq (\overset{\Delta}{f_1}\sigma_1)^{-1}(\Omega_1) \cap (\overset{\Delta}{f_2}\sigma_2)^{-1}(\Omega_2)$. Let α be any element of $\Pi = \sigma_1^{-1}(\Gamma W \Pi_1) \cap \Pi_2$. That is $\sigma_1(\alpha) \in \Gamma W \Pi_1$ and $\alpha \in \Pi_2$. Replacing $\sigma_1(\alpha)$ by $\gamma w \beta_1$, where $\beta_1 \in \Pi_1$, $w \in W$, $\gamma \in \Gamma$, yields that

$$(\overset{\Delta}{f_1}\sigma_1)(\alpha) = \overset{\Delta}{f_1}(\sigma_1(\alpha)) = \overset{\Delta}{f_1}(\gamma w \beta_1) = \gamma f_1(w)\beta_1.$$

On the other hand, $\sigma_1(\alpha) = \sigma_2(\alpha)$ for $\alpha \in \Pi_2$, we get

$$(\overset{\Delta}{f_2}\sigma_2)(\alpha) = \overset{\Delta}{f_2}(\sigma_2(\alpha)) = \overset{\Delta}{f_2}(\sigma_1(\alpha)) = \overset{\Delta}{f_2}(\gamma w \beta_1) = \gamma f_2(w)\beta_1.$$

For $\beta_1 \in \Pi_1 \subseteq \Omega_1 \cap \Omega_2$, we see that

$$(\overset{\Delta}{f_1}\sigma_1)(\alpha) = \gamma f_1(w)\beta_1 \in \Omega_1 \text{ and } (\overset{\Delta}{f_2}\sigma_2)(\alpha) = \gamma f_2(w)\beta_1 \in \Omega_2.$$

Hence, we arrive that $\alpha \in (\overset{\Delta}{f_1}\sigma_1)^{-1}(\Omega_1)$ and $\alpha \in (\overset{\Delta}{f_2}\sigma_2)^{-1}(\Omega_2)$, i.e., $\alpha \in (\overset{\Delta}{f_1}\sigma_1)^{-1}(\Omega_1) \cap (\overset{\Delta}{f_2}\sigma_2)^{-1}(\Omega_2)$.

Moreover, we show that $\tau_1 \overset{\Delta}{f_1}\sigma_1 = \tau_2 \overset{\Delta}{f_2}\sigma_2$ on Π. Let $\alpha \in \Pi$. We obtain that $\sigma_1(\alpha) \in \Gamma W \Pi_1$. Substituting $\gamma w \beta_1$ for $\sigma_1(\alpha)$, where $\beta_1 \in \Pi_1$, $w \in W$, $\gamma \in \Gamma$, we have

$$(\tau_1 \overset{\Delta}{f_1}\sigma_1)(\alpha) = \tau_1(\overset{\Delta}{f_1}(\sigma_1(\alpha))) = \tau_1(\overset{\Delta}{f_1}(\gamma w \beta_1)) = \tau_1(\gamma f_1(w)\beta_1).$$

Using $\beta_1 \in \Pi_1 \subseteq \Omega_1$ and τ_1 is a left $M\Gamma$-module homomorphism, we arrive that

$$\tau_1(\gamma f_1(w)\beta_1) = \gamma f_1(w)\tau_1(\beta_1).$$

Since $\tau_1 = \tau_2$ on Π_1 and $f_1 = f_2$ on W, we find that

$$\gamma f_1(w)\tau_1(\beta_1) = \gamma f_2(w)\tau_2(\beta_1) = \tau_2(\gamma f_2(w)\beta_1) = \tau_2(\overset{\Delta}{f_2}(\gamma w \beta_1)).$$

Using $\alpha \in \Pi_2$ and $\sigma_1 = \sigma_2$ on Π_2, we see that

$$(\tau_2 \overset{\Delta}{f_2})(\gamma w \beta_1) = \tau_2(\overset{\Delta}{f_2}(\sigma_1(\alpha))) = (\tau_2 \overset{\Delta}{f_2}\sigma_2)(\alpha).$$

Thus, we conclude that $\tau_1 \overset{\Delta}{f_1}\sigma_1 = \tau_2 \overset{\Delta}{f_2}\sigma_2$ on Π, and so

$$\left[\tau_1 \overset{\Delta}{f_1}\sigma_1; (\overset{\Delta}{f_1}\sigma_1)^{-1}(\Omega_1)\right] = \left[\tau_2 \overset{\Delta}{f_2}\sigma_2; (\overset{\Delta}{f_2}\sigma_2)^{-1}(\Omega_2)\right].$$

Consequently, multiplication is well defined. It is clearly that $[\tau;\Omega][f;U][\sigma;\Lambda]\in\Delta$ for all $[f;U]\in Q$ and $[\tau;\Omega],[\sigma;\Lambda]\in\Delta$.

Now we will show that (Q,Δ) is a gamma ring.

a) i) For all $[f;U]\in Q$ and $[\tau;\Omega],[\sigma;\Lambda],[\delta;\Sigma]\in\Delta$,

$$\begin{aligned}([\tau;\Omega]+[\sigma;\Lambda])[f;U][\delta;\Sigma] &= [\tau+\sigma;\Omega\cap\Lambda][f;U][\delta;\Sigma]\\ &= \left[(\tau+\sigma)\overset{\Delta}{f}\delta;(\overset{\Delta}{f}\delta)^{-1}(\Omega\cap\Lambda)\right]\\ &= \left[\tau\overset{\Delta}{f}\delta+\sigma\overset{\Delta}{f}\delta;(\overset{\Delta}{f}\delta)^{-1}(\Omega)\cap(\overset{\Delta}{f}\delta)^{-1}(\Lambda)\right]\\ &= \left[\tau\overset{\Delta}{f}\delta;(\overset{\Delta}{f}\delta)^{-1}(\Omega)\right]+\left[\sigma\overset{\Delta}{f}\delta;(\overset{\Delta}{f}\delta)^{-1}(\Lambda)\right]\\ &= [\tau;\Omega][f;U][\delta;\Sigma]+[\sigma;\Lambda][f;U][\delta;\Sigma].\end{aligned}$$

ii) For all $[f;U],[g;V]\in Q$ and $[\tau;\Omega],[\sigma;\Lambda]\in\Delta$,

$$\begin{aligned}[\tau;\Omega]([f;U]+[g;V])[\sigma;\Lambda] &= [\tau;\Omega][f+g;U\cap V][\sigma;\Lambda] \qquad (2.6)\\ &= \left[\tau(f\overset{\Delta}{+}g)\sigma;((f\overset{\Delta}{+}g)\sigma)^{-1}(\Omega)\right],\end{aligned}$$

and

$$[\tau;\Omega][f;U][\sigma;\Lambda]+[\tau;\Omega][g;V][\sigma;\Lambda] \qquad (2.7)$$

$$= \left[\tau\overset{\Delta}{f}\sigma;(\overset{\Delta}{f}\sigma)^{-1}(\Omega)\right]+\left[\tau\overset{\Delta}{g}\sigma;(\overset{\Delta}{g}\sigma)^{-1}(\Omega)\right] \qquad (2.8)$$

$$= \left[\tau\overset{\Delta}{f}\sigma+\tau\overset{\Delta}{g}\sigma;(\overset{\Delta}{f}\sigma)^{-1}(\Omega)\cap(\overset{\Delta}{g}\sigma)^{-1}(\Omega)\right].$$

We will show that (2.6) and (2.8) are equivalent. Let $\Pi=(\overset{\Delta}{f}\sigma)^{-1}(\Omega)\cap(\overset{\Delta}{g}\sigma)^{-1}(\Omega)$ and $\alpha\in\Pi$. Then $\alpha\in(\overset{\Delta}{f}\sigma)^{-1}(\Omega)$ and $\alpha\in(\overset{\Delta}{g}\sigma)^{-1}(\Omega)$, i.e., $(\overset{\Delta}{f}\sigma)(\alpha)\in\Omega$ and $(\overset{\Delta}{g}\sigma)(\alpha)\in\Omega$. Thus $(\overset{\Delta}{f}\sigma)(\alpha)+(\overset{\Delta}{g}\sigma)(\alpha)\in\Omega$, and so $(\overset{\Delta}{f}+\overset{\Delta}{g})\sigma(\alpha)\in\Omega$. That is $\alpha\in((f\overset{\Delta}{+}g)\sigma)^{-1}(\Omega)$. Then

$$\Pi\subseteq((f\overset{\Delta}{+}g)\sigma)^{-1}(\Omega)\cap(\overset{\Delta}{f}\sigma)^{-1}(\Omega)\cap(\overset{\Delta}{g}\sigma)^{-1}(\Omega).$$

Moreover, for all $\alpha\in\Pi$

$$(\tau(f\overset{\Delta}{+}g)\sigma)(\alpha)=(\tau\overset{\Delta}{f}\sigma)(\alpha)+(\tau\overset{\Delta}{g}\sigma)(\alpha)=(\tau\overset{\Delta}{f}\sigma+\tau\overset{\Delta}{g}\sigma)(\alpha).$$

iii) For all $[f;U]\in Q$ and $[\tau;\Omega],[\sigma;\Lambda],[\delta;\Sigma]\in\Delta$,

$$[\tau;\Omega][f;U]([\sigma;\Lambda]+[\delta;\Sigma])=[\tau;\Omega][f;U][\sigma+\delta;\Lambda\cap\Sigma]$$

$$
\begin{aligned}
&= \left[\tau \overset{\Delta}{f}(\sigma+\delta); (\overset{\Delta}{f}(\sigma+\delta))^{-1}(\Omega)\right] \\
&= \left[\tau \overset{\Delta}{f}\sigma + \tau \overset{\Delta}{f}\delta; (\overset{\Delta}{f}\sigma)^{-1}(\Omega) \cap (\overset{\Delta}{f}\delta))^{-1}(\Omega)\right] \\
&= \left[\tau \overset{\Delta}{f}\sigma; (\overset{\Delta}{f}\sigma)^{-1}(\Omega)\right] + \left[\tau \overset{\Delta}{f}\delta; (\overset{\Delta}{f}\delta))^{-1}(\Omega)\right] \\
&= [\tau;\Omega][f;U][\sigma;\Lambda] + [\tau;\Omega][f;U][\delta;\Sigma].
\end{aligned}
$$

Set $\Pi = (\overset{\Delta}{f}\sigma)^{-1}(\Omega) \cap (\overset{\Delta}{f}\delta))^{-1}(\Omega)$, application of similar arguments yields the above equality.

b) i) For all $[f;U], [g;V] \in Q$ and $[\tau;\Omega], [\sigma;\Lambda], [\delta;\Sigma] \in \Delta$,

$$
\begin{aligned}
([\tau;\Omega][f;U][\sigma;\Lambda])[g;V][\delta;\Sigma] &= \left[\tau \overset{\Delta}{f}\sigma; (\overset{\Delta}{f}\sigma)^{-1}(\Omega)\right][g;V][\delta;\Sigma] \\
&= \left[(\tau \overset{\Delta}{f}\sigma)\overset{\Delta}{g}\delta; (\overset{\Delta}{g}\delta)^{-1}(\overset{\Delta}{f}\sigma)^{-1}(\Omega)\right].
\end{aligned} \tag{2.9}
$$

$$
\begin{aligned}
[\tau;\Omega][f;U]([\sigma;\Lambda][g;V][\delta;\Sigma]) &= [\tau;\Omega][f;U]\left[\sigma \overset{\Delta}{g}\delta; (\overset{\Delta}{g}\delta)^{-1}(\Lambda)\right] \\
&= \left[\tau \overset{\Delta}{f}(\sigma \overset{\Delta}{g}\delta); (\overset{\Delta}{f}(\sigma \overset{\Delta}{g}\delta))^{-1}(\Omega)\right]
\end{aligned} \tag{2.10}
$$

We will prove that (2.9) and (2.10) are equivalent. Let $\Pi = (\overset{\Delta}{g}\delta)^{-1}(\overset{\Delta}{f}\sigma)^{-1}(\Omega)$ and $\alpha \in \Pi$. Then $(\overset{\Delta}{g}\delta)(\alpha) \in (\overset{\Delta}{f}\sigma)^{-1}(\Omega)$, and so $(\overset{\Delta}{f}\sigma)(\overset{\Delta}{g}\delta)(\alpha) \in \Omega$. Hence, $\alpha \in (\overset{\Delta}{f}(\sigma \overset{\Delta}{g}\delta))^{-1}(\Omega)$. Thus $\Pi \subseteq (\overset{\Delta}{g}\delta)^{-1}(\overset{\Delta}{f}\sigma)^{-1}(\Omega) \cap (\overset{\Delta}{f}(\sigma \overset{\Delta}{g}\delta))^{-1}(\Omega)$, and also

$$((\tau \overset{\Delta}{f}\sigma)\overset{\Delta}{g}\delta)(\alpha) = (\tau \overset{\Delta}{f}(\sigma \overset{\Delta}{g}\delta))(\alpha), \text{ for all } \alpha \in \Pi.$$

ii) For all $[f;U], [g;V] \in Q$ and $[\tau;\Omega], [\sigma;\Lambda], [\delta;\Sigma] \in \Delta$,

$$
\begin{aligned}
([\tau;\Omega][f;U][\sigma;\Lambda])[g;V][\delta;\Sigma] &= \left[\tau \overset{\Delta}{f}\sigma; (\overset{\Delta}{f}\sigma)^{-1}(\Omega)\right][g;V][\delta;\Sigma] \\
&= \left[(\tau \overset{\Delta}{f}\sigma)\overset{\Delta}{g}\delta; (\overset{\Delta}{g}\delta)^{-1}(\overset{\Delta}{f}\sigma)^{-1}(\Omega)\right] \\
&= \left[\tau(\overset{\Delta}{f}\sigma\overset{\Delta}{g})\delta; (\overset{\Delta}{f}\sigma\overset{\Delta}{g}\delta)^{-1}(\Omega)\right] \\
&= [\tau;\Omega][\overset{\Delta}{f}\sigma\overset{\Delta}{g}; (\sigma\overset{\Delta}{g})^{-1}(U)][\delta;\Sigma] \\
&= [\tau;\Omega]([f;U][\sigma;\Lambda][g;V])[\delta;\Sigma].
\end{aligned}
$$

Using similar arguments as above, we can prove.

c) Let $[\tau;\Omega][f;U][\sigma;\Lambda]=[0;\Gamma]$ for all $[f;U]\in Q$ and $[\tau;\Omega],[\sigma;\Lambda]\in\Delta$. For $[I_\Gamma;\Gamma]\in\Delta$, where $I_\Gamma:\Gamma\to\Gamma$ is an identity left $M\Gamma$-module homomorphism, we see that

$$[I_\Gamma;\Gamma][f;U][I_\Gamma;\Gamma]=[0;\Gamma],$$

and so

$$\left[I_\Gamma \overset{\Delta}{f} I_\Gamma;(\overset{\Delta}{f} I_\Gamma)^{-1}(\Gamma)\right]=[0;\Gamma].$$

Set $\Pi=\Gamma U\Gamma$. We first prove that $\Gamma U\Gamma\subseteq\Gamma\cap(\overset{\Delta}{f} I_\Gamma)^{-1}(\Gamma)$. For any $\gamma=\alpha u\beta\in\Gamma U\Gamma$ where $u\in U$, $\alpha,\beta\in\Gamma$, we arrive at

$$(\overset{\Delta}{f} I_\Gamma)(\gamma)=(\overset{\Delta}{f} I_\Gamma)(\alpha u\beta)=\overset{\Delta}{f}(I_\Gamma(\alpha u\beta))=\overset{\Delta}{f}(\alpha u\beta)=\alpha f(u)\beta.$$

We have $(\overset{\Delta}{f} I_\Gamma)(\gamma)\in\Gamma$, and so $\gamma\in(\overset{\Delta}{f} I_\Gamma)^{-1}(\Gamma)$. On the other hand, we get $\gamma\in\Gamma$. As a result $\gamma\in\Gamma\cap(\overset{\Delta}{f} I_\Gamma)^{-1}(\Gamma)$.

Now, for all $\gamma=\alpha u\beta\in\Gamma U\Gamma$ where $u\in U$, $\alpha,\beta\in\Gamma$,

$$I_\Gamma \overset{\Delta}{f} I_\Gamma(\gamma)=I_\Gamma \overset{\Delta}{f} I_\Gamma(\alpha u\beta)=0(\alpha u\beta).$$

That is

$$\alpha f(u)\beta=0,\ \text{for all } u\in U,\alpha,\beta\in\Gamma.$$

Since (M,Γ) is gamma ring, we get $f(u)=0$, for all $u\in U$, and so $f=0$. That is $[f;U]=[0;M]$.

We arrive that (Q,Δ) is a gamma ring and we call it the maximal left gamma ring of quotient of (M,Γ).

We proceed by showing that (Δ,Q) is characterized by certain reasonable properties that any gamma ring of quotient should have.

Remark 1. Let M be a Γ-ring, $\varepsilon\in\Gamma, e\in M$. If (ε,e) is the strong right identity element of (Γ,M), then (e,ε) is the strong left identity element of (M,Γ).

Proof. Assume that (ε,e) is the strong right identity element of (Γ,M). Thus we get $x\varepsilon e=x$ for all $x\in M$. For any $\gamma\in\Gamma$, $x,y\in M$, we have

$$x(\varepsilon e\gamma-\gamma)y=x(\varepsilon e\gamma)y-x\gamma y=(x\varepsilon e)\gamma y-x\gamma y=x\gamma y-x\gamma y=0$$

and so

$$M(\varepsilon e\gamma-\gamma)M=(0).$$

Since (Γ,M) is gamma ring, we get

$$\varepsilon e\gamma=\gamma \text{ for all } \gamma\in\Gamma.$$

□

Using similar arguments as above, we can prove the followings:

Remark 2. Let M be a Γ-ring, $\varepsilon \in \Gamma, e \in M$. If (ε, e) is the strong left identity element of (Γ, M), then (e, ε) is the strong right identity element of (M, Γ).

Let (ε, e) is the strong right identity element of (Γ, M). For a fixed element a in M, consider a mapping $\lambda_{a\varepsilon} : M \to M$ defined by $\lambda_{a\varepsilon}(x) = a\varepsilon x$ for all $x \in M$. It is easy to prove that the mapping $\lambda_{a\varepsilon}$ is a right $\Gamma M-$module homomorphism. Moreover, for all $m \in M$,

$$\lambda_{(a+b)\varepsilon}(m) = (a+b)\varepsilon m = a\varepsilon m + b\varepsilon m = \lambda_{a\varepsilon}(m) + \lambda_{b\varepsilon}(m) = (\lambda_{a\varepsilon} + \lambda_{b\varepsilon})(m),$$

and so

$$\lambda_{(a+b)\varepsilon} = (\lambda_{a\varepsilon} + \lambda_{b\varepsilon}). \tag{2.11}$$

Now consider a mapping $\mu_{e\beta} : \Gamma \to \Gamma$ defined by $\mu_{e\beta}(\alpha) = \alpha e\beta$ for all $\alpha \in \Gamma$ is a left $M\Gamma-$module homomorphism. Indeed, we shown that Γ is a left $M\Gamma-$module.

i) For all $\alpha, \gamma \in \Gamma$,

$$\mu_{e\beta}(\alpha+\gamma) = (\alpha+\gamma)e\beta = \alpha e\beta + \gamma e\beta.$$

ii) For all $\alpha, \gamma \in \Gamma$ and $m \in M$,

$$\mu_{e\beta}(\alpha m\gamma) = (\alpha m\gamma)e\beta = \alpha m(\gamma e\beta) = \alpha m\mu_{e\beta}(\gamma).$$

Thus, $\mu_{e\beta}$ is a left $M\Gamma-$module homomorphism.

Using arguments as above, we can prove the followings:

$$\mu_{e(\beta+\gamma)} = \mu_{e\beta} + \mu_{e\gamma}. \tag{2.12}$$

Let

$$\wp = \{\lambda_{a\varepsilon} | a \in M\} \text{ and } \mho = \{\mu_{e\beta} | \beta \in \Gamma\}.$$

These sets are additive groups and defining the mappings

$$\wp \times \mho \times \wp \to \wp, \ (\lambda_{x\varepsilon}, \mu_{e\gamma}, \lambda_{y\varepsilon}) \mapsto \lambda_{x\varepsilon e\gamma y\varepsilon} = \lambda_{x\gamma y\varepsilon} \tag{2.13}$$

and

$$\mho \times \wp \times \mho \to \mho, \ (\mu_{e\gamma}, \lambda_{x\varepsilon}, \mu_{e\beta}) \mapsto \mu_{e\gamma x\varepsilon e\beta} = \mu_{e\gamma x\beta}. \tag{2.14}$$

It can be shown that $(\mho, \wp)$ is a gamma ring.

In the following theorem, $D(\Gamma, M)$ will denote collection of all dense ideal of M.

Theorem 3. *Let (Γ, M) be a semiprime gamma ring with strong right identity element. Then (Γ, M) is a subring of (Δ, Q).*

Proof. Consider the mappings

$$\phi_\varepsilon : M \to Q, \ a \mapsto [\lambda_{a\varepsilon}; M] \text{ and } \psi_e : \Gamma \mapsto \Delta, \ \beta \to [\mu_{e\beta}; \Gamma].$$

One readily checks that ϕ_ε is well defined. Using (2.11) equation, we have

$$\phi_\varepsilon(a+b) = [\lambda_{(a+b)\varepsilon}; M] = [\lambda_{a\varepsilon}; M] + [\lambda_{b\varepsilon}; M] = \phi_\varepsilon(a) + \phi_\varepsilon(b),$$

for all $a, b \in M$.

Hence, ϕ_ε is a group homomorphism. Now, we suppose that $\phi_\varepsilon(a) = \phi_\varepsilon(b)$. Then $[\lambda_{a\varepsilon}; M] = [\lambda_{b\varepsilon}; M]$, and so $(\lambda_{a\varepsilon}; M) \sim (\lambda_{b\varepsilon}; M)$. Hence there exists $W \subseteq M \cap M$ such that $W \in D(\Gamma, M)$ and $\lambda_{a\varepsilon} = \lambda_{b\varepsilon}$ on W, i.e., $a\varepsilon w = b\varepsilon w$, for all $w \in W$. That is $(a-b)\varepsilon W = (0)$. Since W is a ideal of M, we have $(a-b)\varepsilon M\Gamma W = (0)$, and so $(a-b)\varepsilon M \in l_\Gamma(W)$. By Theorem 1(v), we get $(a-b)\varepsilon M = (0)$. In particular, taking m by e, we get $(a-b)\varepsilon e = 0$, and so $a = b$. Thus ϕ_ε is one-one. Hence ϕ_ε is a group monomorphism.

On the other hand, once easly checks that ψ_e is a group monomorphism.

For all $\alpha, \beta \in \Gamma$, $m, n \in M$

$$\phi_\varepsilon(m\beta n) = \left[\lambda_{(m\beta n)\varepsilon}; M\right] = \left[\lambda_{m\varepsilon e\beta n\varepsilon}; M\right], \tag{2.15}$$

and

$$\phi_\varepsilon(m)\psi_e(\beta)\phi_\varepsilon(n) = [\lambda_{m\varepsilon}; M]\left[\mu_{e\beta}; \Gamma\right][\lambda_{n\varepsilon}; M] \tag{2.16}$$

$$= [\lambda_{m\varepsilon}\overset{\Delta}{\mu}_{e\beta}\lambda_{n\varepsilon}; (\overset{\Delta}{\mu}_{e\beta}\lambda_{n\varepsilon})^{-1}(M)].$$

We will show that (2.15) and (2) are equivalent. Set $W = M\Gamma M$ and $w = x\gamma y \in W$, where $\gamma \in \Gamma$, $x, y \in M$.

$$(\overset{\Delta}{\mu}_{e\beta}\lambda_{n\varepsilon})(w) = (\overset{\Delta}{\mu}_{e\beta}\lambda_{n\varepsilon})(x\gamma y) = \overset{\Delta}{\mu}_{e\beta}(n\varepsilon x\gamma y) = n\varepsilon x\mu_{e\beta}(\gamma)y = n\varepsilon x\gamma e\beta y.$$

Thus $(\overset{\Delta}{\mu}_{e\beta}\lambda_{n\varepsilon})(w) \in M$, and so $w \in (\overset{\Delta}{\mu}_{e\beta}\lambda_{n\varepsilon})^{-1}(M)$. Hence we obtain that $W \subseteq (\overset{\Delta}{\mu}_{e\beta}\lambda_{n\varepsilon})^{-1}(M) \cap M$.

On the other hand, using (2.13), we conclude that

$$\lambda_{m\varepsilon}\overset{\Delta}{\mu}_{e\beta}\lambda_{n\varepsilon} = \lambda_{m\varepsilon e\beta n\varepsilon} = \lambda_{(m\beta n)\varepsilon}.$$

That is $\lambda_{m\varepsilon}\overset{\Delta}{\mu}_{e\beta}\lambda_{n\varepsilon} = \lambda_{(m\beta n)\varepsilon}$ on W. In that case, we get

$$\phi_\varepsilon(m\beta n) = \phi_\varepsilon(m)\psi_e(\beta)\phi_\varepsilon(n).$$

Using similar arguments as above, we can shown that

$$\psi_e(\alpha m\gamma) = \psi_e(\alpha)\phi_\varepsilon(m)\psi_e(\gamma).$$

Hence $(\psi_e, \phi_\varepsilon)$ is a gamma ring monomorphism, and so (Γ, M) is a subring of (Δ, Q). □

Theorem 4. *Let (Γ, M) be a semiprime gamma ring with strong left identity element. (Δ, Q) satisfies the following properties:*

i) If $U \in D(\Gamma, M)$ and $f : U \to M$ is a right $\Gamma M-$module homomorphism, then there exists an element $q \in Q$ such that $f(u) = q\varepsilon u$ for all $u \in U$.

ii) For all $q \in Q$ there exists $U \in D(\Gamma, M)$ such that $q\varepsilon U \subseteq M$.

iii) For all $q \in Q$ and $U \in D(\Gamma, M)$, $q\varepsilon U = (0)$ if and only if $q = 0$.

Proof. i) Let $q \in Q$, $u \in U$, $f : U \to M$ be a right $\Gamma M-$module homomorphism and $q = [f; U]$. Since M can be embedded in Q, we can write $u = [\lambda_{u\varepsilon}; M]$ such that $\lambda_{u\varepsilon} : M \to M$, $x \mapsto u\varepsilon x$ for all $u \in U$. Then we have

$$q\varepsilon u = [f; U][\mu_{e\varepsilon}; \Gamma][\lambda_{u\varepsilon}; M] = [f \overset{\Delta}{\mu}_{e\varepsilon}\lambda_{u\varepsilon}; (\overset{\Delta}{\mu}_{e\varepsilon}\lambda_{u\varepsilon})^{-1}(U)] \tag{2.17}$$

and

$$f(u) = \left[\lambda_{f(u)\varepsilon}; M\right]. \tag{2.18}$$

We will show that (2.17) and (2.18) are equivalent. Indeed, for $W = U\Gamma M$ and $x = y\gamma m \in W$, where $y \in U$, $\gamma \in \Gamma$, $m \in M$, we have

$$\begin{aligned}(\overset{\Delta}{\mu}_{e\varepsilon}\lambda_{u\varepsilon})(x) &= (\overset{\Delta}{\mu}_{e\varepsilon}\lambda_{u\varepsilon})(y\gamma m) = \overset{\Delta}{\mu}_{e\varepsilon}(u\varepsilon y\gamma m)\\ &= u\varepsilon y\mu_{e\varepsilon}(\gamma)m = u\varepsilon y\gamma e\varepsilon m = u\varepsilon y\gamma m \in U.\end{aligned}$$

In this way, we get $x = y\gamma m \in (\overset{\Delta}{\mu}_{e\varepsilon}\lambda_{u\varepsilon})^{-1}(U)$, and so $W \subseteq (\overset{\Delta}{\mu}_{e\varepsilon}\lambda_{u\varepsilon})^{-1}(U) \cap M$. Moreover, for all $x \in W$, we get

$$\begin{aligned}(f\overset{\Delta}{\mu}_{e\varepsilon}\lambda_{u\varepsilon})(x) &= (f\overset{\Delta}{\mu}_{e\varepsilon}\lambda_{u\varepsilon})(y\gamma m) = (f\overset{\Delta}{\mu}_{e\varepsilon})(u\varepsilon y\gamma m)\\ &= f(u\varepsilon y\mu_{e\varepsilon}(\gamma)m) = f(u\varepsilon y\gamma e\varepsilon m)\\ &= f(u\varepsilon y\gamma m) = f(u)\varepsilon y\gamma m\\ &= \lambda_{f(u)\varepsilon}(y\gamma m) = \lambda_{f(u)\varepsilon}(x).\end{aligned}$$

That is $f\overset{\Delta}{\mu}_{e\varepsilon}\lambda_{u\varepsilon} = \lambda_{f(u)\varepsilon}$ on W.

Therefore, there exists an element $q \in Q$ such that $f(u) = q\varepsilon u$ for all $u \in U$.

ii) Let $q \in Q$. Then there exists $U \in D(\Gamma, M)$ such that $q = [f; U]$ and $f : U \to M$ be a right $\Gamma M-$module homomorphism. According to (i), we have $f(u) = q\varepsilon u$ for all $u \in U$. Thus $q\varepsilon U \subseteq M$. Consequently, there exists $U \in D(\Gamma, M)$ such that $q\varepsilon U \subseteq M$.

iii) Let $q \in Q$ and $U \in D(\Gamma, M)$, $q\varepsilon U = (0)$. Then $q = [f; U]$. If $q\varepsilon U = (0)$, then $q\varepsilon u = 0$, for all $u \in U$. By (i), $f(u) = q\varepsilon u = 0$, for all $u \in U$. Thus $f(u) = 0$, for all $u \in U$, and so $q = [0; U] = [0; M] = 0_Q$. Conversely, let $q = 0$. Then $q = [f; U] = [0; M]$, and so $(f; U) \sim (0; M)$. That is $f(u) = 0$ for all $u \in U \subseteq M$. According to (i), $f(u) = q\varepsilon u = 0$ for all $u \in U$, so $q\varepsilon U = (0)$. This completes the proof. □

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Authors' addresses

Emine Koç
Cumhuriyet University, Faculty of Science, Department of Mathematics, 58140 Sivas, TURKEY
E-mail address: `eminekoc@cumhuriyet.edu.tr`

Öznur Gölbaşı
Cumhuriyet University, Faculty of Science, Department of Mathematics, 58140 Sivas, TURKEY
E-mail address: `ogolbasi@cumhuriyet.edu.tr`