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ON ρ-STATISTICAL CONVERGENCE OF ORDER α OF SEQUENCES OF SETS

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Abstract. In this paper we introduce the concepts of Wijsman ρ-statistical convergence of order α and Wijsman strongly ρ-convergence of order α. In addition, some inclusion theorems are also presented.

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1. Introduction

The concept of statistical convergence was introduced by Steinhaus [30] and Fast [18]. Schoenberg [28] established some basic properties of statistical convergence and studied the concept as a summability method. Later on it was further investigated from the sequence space point of view and linked with summability theory by Altınok et al. [1], Aral et al. ([2,3]), Bhardwaj and Dhawan [5], Çakallı et al. ([6–8]), Caserta et al. [9], Çınar et al. [10], Connor [12], Çolak [11], Demirci et al. [13], Et et al. ([14–17]), Fridy [19], Gadjiev and Orhan [20], Işık and Akbaş [21], Salat [26], Savaş and Et [27], Şengül [29] and many others.

Let (X,σ) be a metric space. The distance $d(x,A)$ from a point x to a non-empty subset A of (X,σ) is defined to be

$$d(x,A) = \inf_{y\in A} \sigma(x,y).$$

If $\sup_k d(x,A_k) < \infty$ (for each $x \in X$), then we say that the sequence $\{A_k\}$ is bounded. The set of all bounded sequences of sets denoted $L_\infty(X)$.

The concepts of Wijsman statistical convergence for the sequence were given by Nuray and Rhoades [25] and the concept were generalized by Ulusu et al. ([31],[32]). In this paper we introduce the concepts of Wijsman ρ-statistical convergence of order α and Wijsman strongly ρ-convergence of order α.

2. Results

Definition 1. Let (X,σ) be a metric space and $\alpha \in (0,1]$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman ρ-statistical convergent to A of order α (or WS_ρ^α-convergent to A) if for each $\varepsilon > 0$ and $x \in X$,

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} \left|\{k \leqslant n : |d(x,A_k) - d(x,A)| \geq \varepsilon\}\right| = 0,$$

where $\rho = (\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that

$$\limsup_n \frac{\rho_n}{n} < \infty, \quad \Delta\rho_n = O(1) \quad \text{and} \quad \Delta\rho_n = \rho_{n+1} - \rho_n \tag{2.1}$$

for each positive integer n. In this case, we write $A_k \longrightarrow A(WS_\rho^\alpha)$. The set of all Wijsman ρ-statistical convergent sequences to A of order α will be denoted by WS_ρ^α.

If $\rho_n = n$, for all $n \in \mathbb{N}$, Wijsman ρ-statistical convergence of order α is coincided with Wijsman statistical convergence of order α denoted by $A_k \longrightarrow A\left((WS^\alpha)\right)$. If $\alpha = 1$, Wijsman ρ-statistical convergence of order α is coincided with Wijsman ρ-statistical convergence which were defined by Aral et al. [4]. In this case we write $A_k \to A(WS_\rho)$. If $\rho_n = n$ for all $n \in \mathbb{N}$ and $\alpha = 1$, Wijsman ρ-statistical convergence of order α is coincided with Wijsman statistical convergence which were defined by Nuray and Rhoades [25].

Definition 2. Let (X,σ) be a metric space and $\alpha \in (0,1]$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is said to be Wijsman strongly ρ-summable of order α to A, if for each $\varepsilon > 0$ and $x \in X$,

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} \sum_{k=1}^{n} |d(x,A_k) - d(x,A)| = 0.$$

The set of all Wijsman strongly ρ-summable sequences of order α to A will be denoted by W_ρ^α. In this case we write $A_k \longrightarrow A(W_\rho^\alpha)$. If $\rho_n = n$ for all $n \in \mathbb{N}$, Wijsman strong ρ-summability of order α is reduced to Wijsman strong summability of order α denoted by $A_k \longrightarrow A\left((W^\alpha)\right)$. If $\alpha = 1$, Wijsman strong ρ-summability of order α is reduced to Wijsman strong ρ-summability denoted by $A_k \longrightarrow A\left((W_\rho)\right)$ which were defined by Aral et al. [4]. If $\rho_n = n$ for all $n \in \mathbb{N}$ and $\alpha = 1$, Wijsman strong ρ-summability of order α is reduced to Wijsman strong summability which were defined by Nuray and Rhoades [25].

We give the following results without proof.

Theorem 1. *Let (X,σ) be a metric space and $\alpha \in (0,1]$. If $A_k \longrightarrow A(WS_\rho^\alpha)$, then A is unique.*

Theorem 2. *Let (X,σ) be a metric space, $\alpha \in (0,1]$ and A, A_k (for all $k \in \mathbb{N}$) be non-empty closed subsets of X, then*

(i) If $A_k \longrightarrow A(WS_\rho^\alpha)$ *and* $c \in \mathbb{C}$*, then* $cA_k \longrightarrow cA(WS_\rho^\alpha)$.
(ii) If $A_k \longrightarrow A(WS_\rho^\alpha)$ *and* $B_k \longrightarrow B(WS_\rho^\alpha)$*, then* $(A_k + B_k) \longrightarrow (A+B)(WS_\rho^\alpha)$.
(iii) If $A_k \longrightarrow A(W_\rho^\alpha)$ *and* $c \in \mathbb{C}$*, then* $cA_k \longrightarrow cA(W_\rho^\alpha)$.
(iv) If $A_k \longrightarrow A(W_\rho^\alpha)$ *and* $B_k \longrightarrow B(W_\rho^\alpha)$*, then* $(A_k + B_k) \longrightarrow (A+B)(W_\rho^\alpha)$.

Corollary 1. *In the case* $\rho_n = n$ *(for all* $n \in \mathbb{N}$*) (i)-(iv) hold in the above Theorem 2.*

Theorem 3. *Let* (X,σ) *be a metric space,* $\alpha, \beta \in (0,1]$ *such that* $\alpha \le \beta$ *and* A, A_k *(for all* $k \in \mathbb{N}$*) be non-empty closed subsets of* X*, then* $WS_\rho^\alpha \subset WS_\rho^\beta$ *and the inclusion is strict.*

Proof. The inclusion part of the proof is easy. To prove the inclusion is strict define a sequence $\{A_k\}$ such that

$$A_k := \begin{cases} \{k,k\} & \text{if } k = n^2; \\ \{0,0\} & \text{otherwise.} \end{cases}$$

Then $\{A_k\} \in WS_\rho^\beta$ for $\frac{1}{2} < \beta \leqslant 1$ but $\{A_k\} \notin WS_\rho^\alpha$ for $0 < \alpha \leqslant \frac{1}{2}$. □

Corollary 2. *Let* (X,σ) *be a metric space,* $\alpha \in (0,1]$ *and* A, A_k *(for all* $k \in \mathbb{N}$*) be non-empty closed subsets of* X*. If* $A_k \longrightarrow A(WS_\rho^\alpha)$*, then* $A_k \longrightarrow A(WS_\rho)$.

Theorem 4. *Let* (X,σ) *be a metric space,* $\alpha \in (0,1]$ *and* A, A_k *(for all* $k \in \mathbb{N}$*) be non-empty closed subsets of* X*, then*
(i) $A_k \to A(W_\rho^\alpha) \Rightarrow A_k \to A(WS_\rho^\alpha)$ *and* W_ρ^α *is a proper subset of* WS_ρ^α.
(ii) $\{A_k\} \in L_\infty(X)$ *and* $A_k \to A(WS_\rho) \Rightarrow A_k \to A(W_\rho)$.
(iii) $WS_\rho \cap L_\infty(X) = W_\rho \cap L_\infty(X)$.

Proof.

(i) The inclusion part of the proof is easy. In order to show that the inclusion $W_\rho^\alpha \subseteq WS_\rho^\alpha$ is proper, we define a sequence $\{A_k\}$ as follows:

$$A_k = \begin{cases} \{\sqrt{k}\} & \text{if } k = n^2; \\ \{0\} & \text{otherwise.} \end{cases}$$

Let $X = \mathbb{R}$, $d(x,y) = |x-y|$ $(x,y \in X)$ and $\rho_n = n$ for all $n \in \mathbb{N}$. For $\varepsilon > 0$, $x > 0$ and $\frac{1}{2} < \alpha \leqslant 1$, we have

$$\frac{1}{\rho_n^\alpha} |\{k \leqslant n : |d(x,A_k) - d(x,\{0\})| \ge \varepsilon\}| \le \frac{(\sqrt{n})}{n^\alpha} \to 0, \quad \text{as } n \to \infty$$

and so we get $A_k \to 0\left(WS_\rho^\alpha\right)$.

On the other hand, for $0 < \alpha \leqslant 1$ and $x > 0$,

$$\frac{1}{\rho_n^\alpha} \sum_{k=1}^{n} |d(x,A_k) - d(x,\{0\})| = \frac{\sqrt{n}(\sqrt{n}+1)}{2n^\alpha} \to \infty$$

and for $\alpha = 1$
$$\frac{\sqrt{n}\left(\sqrt{n}+1\right)}{n^{\alpha}} \to 1.$$

So $A_k \nrightarrow 0\left(W_{\rho}^{\alpha}\right)$.

(ii) Omitted.

(iii) Omitted.

□

Theorem 5. *Let (X,σ) be a metric space, $\alpha \in (0,1]$ and A, A_k (for all $k \in \mathbb{N}$) be non-empty closed subsets of X. Then, $WS^{\alpha} \subset WS_{\rho}^{\alpha}$ if $\liminf_n \frac{\rho_n^{\alpha}}{n^{\alpha}} > 0$.*

Proof. Proof follows from the following inequality:

$$\frac{1}{n^{\alpha}}\left|\left\{k \leqslant n : \left|d\left(x,A_k\right)-d\left(x,A\right)\right| \geq \varepsilon\right\}\right| \geqslant \frac{{\rho_n}^{\alpha}}{n^{\alpha}}\frac{1}{\rho_n^{\alpha}}\left|\left\{k \leqslant n : \left|d\left(x,A_k\right)-d\left(x,A\right)\right| \geq \varepsilon\right\}\right|.$$

□

Theorem 6. *Let (X,σ) be a metric space, $\alpha \in (0,1]$ and A, A_k (for all $k \in \mathbb{N}$) be non-empty closed subsets of X. $WS \subset WS_{\rho}^{\alpha}$ if*

$$\liminf_{n\to\infty} \frac{\rho_n^{\alpha}}{n} > 0.$$

Proof. Omitted. □

Theorem 7. *Let (X,σ) be a metric space, $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequences satisfying (2.1) conditions such that $\rho_n \leqslant \tau_n$ for all $n \in \mathbb{N}$ and $0 < \alpha \leqslant \beta \leqslant 1$. If*

$$\lim_{n\to\infty} \frac{\rho_n^{\alpha}}{\tau_n^{\beta}} = a > 0, \tag{2.2}$$

then $WS_{\tau}^{\beta} \subseteq WS_{\rho}^{\alpha}$.

Proof. Omitted. □

Corollary 3. *Let (X,σ) be a metric space, $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequences satisfying (2.1) conditions such that $\rho_n \leqslant \tau_n$ for all $n \in \mathbb{N}$ and $0 < \alpha \leqslant 1$. If (2.2) holds, then*

(i) $WS_{\tau}^{\alpha} \subseteq WS_{\rho}^{\alpha}$.

(ii) $WS_{\tau} \subseteq WS_{\rho}^{\alpha}$.

Theorem 8. *Let (X,σ) be a metric space, $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequences satisfying (2.1) conditions such that $\rho_n \leqslant \tau_n$ for all $n \in \mathbb{N}$ and $0 < \alpha \leqslant \beta \leqslant 1$. If*

$$\limsup_{n\to\infty} \frac{\rho_n^{\alpha}}{\tau_n^{\beta}} < \infty, \tag{2.3}$$

then $WS_{\rho}^{\alpha} \subseteq WS_{\tau}^{\beta}$.

Proof. Let $A_k \to A(WS_\rho^\alpha)$ and for $\varepsilon > 0$. We have

$$\frac{1}{\tau_n^\beta}\left|\{k \leqslant n : |d(x,A_k) - d(x,A)| \geq \varepsilon\}\right| \leqslant \frac{\rho_n^\alpha}{\tau_n^\beta}\frac{1}{\rho_n^\alpha}\left|\{k \leqslant n : |d(x,A_k) - d(x,A)| \geq \varepsilon\}\right|.$$

Using the condition and $A_k \to A(WS_\rho^\alpha)$ we have $WS_\rho^\alpha \subseteq WS_\tau^\beta$. □

Corollary 4. *Let (X,σ) be a metric space, $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequences satisfying (2.1) conditions such that $\rho_n \leqslant \tau_n$ for all $n \in \mathbb{N}$ and $0 < \alpha \leqslant 1$. If (2.3) holds, then*

(i) $WS_\rho^\alpha \subseteq WS_\tau^\alpha$.
(ii) $WS_\rho^\alpha \subseteq WS_\tau$.

3. Results related to Orlicz function

An Orlicz function is a function $M\colon [0,\infty) \to [0,\infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

The study of Orlicz sequence spaces was initiated with a certain specific purpose in Banach space theory. Indeed, Lindberg [23] got interested in Orlicz sequence spaces in connection with finding Banach spaces with symmetric Schauder bases having complementary subspaces isomorphic to c_0 or ℓ_p $(0 \leq p < \infty)$. Subsequently, Lindenstrauss and Tzafriri [24] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0\right\}.$$

The space ℓ_M with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$$

becomes a Banach space, called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = |x|^p$ for $1 \leq p < \infty$. Lindenstrauss and Tzafriri [24] proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to l_p $(1 \leq p < \infty)$. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [22].

It is well known that if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leqslant \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Definition 3. Let (X,σ) be a metric space. Let M be an Orlicz function, $p = (p_k)$ be a sequence of strictly positive real numbers, $\alpha \in (0,1]$, $\rho = (\rho_n)$ be a sequence such as above and for $\lambda > 0$, now we define

$$\left(W_{\rho}^{\alpha},[M,(p)]\right)=\left\{\{A_k\}\in X:\frac{1}{\rho_n^{\alpha}}\sum_{k=1}^{n}\left[M\left(\frac{|d(x,A_k)-d(x,A)|}{\lambda}\right)\right]^{p_k}\to 0,\right.$$
$$\left.\text{for some } A \text{ and for } x\in X\right\}.$$

If $M(x)=x$ and $p_k=1$ for all $k\in\mathbb{N}$ then we shall write $(W_{\rho}^{\alpha},[M,(p)])=W_{\rho}^{\alpha}$ and if $M(x)=x$, then we shall write $(W_{\rho}^{\alpha},[M,(p)])=(W_{\rho}^{\alpha},(p))$. If $p_k=1$ for all $k\in\mathbb{N}$ then we shall write $(W_{\rho}^{\alpha},[M,(p)])=\left(W_{\rho}^{\alpha},[M]\right)$.

In the following theorems, assume that the sequence $p=(p_k)$ is bounded and $0<h=\inf_k p_k\leqslant p_k\leqslant \sup_k p_k=H<\infty$.

Theorem 9. *Let $0<\alpha\leqslant\beta\leqslant 1$, M be an Orlicz function and $\rho=(\rho_n)$ be a sequence such as above, then $(W_{\rho}^{\alpha},[M,(p)])\subset WS_{\rho}^{\beta}$.*

Proof. Let $\{A_k\}\in(W_{\rho}^{\alpha},[M,(p)]$. Let $\varepsilon>0$ be given. As $\rho_n^{\alpha}\leqslant\rho_n^{\beta}$ for each n we can write

$$\begin{aligned}
&\frac{1}{\rho_n^{\alpha}}\sum_{k=1}^{n}\left[M\left(\frac{|d(x,A_k)-d(x,A)|}{\lambda}\right)\right]^{p_k}=\\
&\frac{1}{\rho_n^{\alpha}}\sum_{\substack{k=1\\|d(x,A_k)-d(x,A)|\geqslant\varepsilon}}^{n}\left[M\left(\frac{|d(x,A_k)-d(x,A)|}{\lambda}\right)\right]^{p_k}\\
&+\frac{1}{\rho_n^{\alpha}}\sum_{\substack{k=1\\|d(x,A_k)-d(x,A)|<\varepsilon}}^{n}\left[M\left(\frac{|d(x,A_k)-d(x,A)|}{\lambda}\right)\right]^{p_k}\\
&\geqslant\frac{1}{\rho_n^{\beta}}\sum_{\substack{k=1\\|d(x,A_k)-d(x,A)|\geqslant\varepsilon}}^{n}\left[M\left(\frac{|d(x,A_k)-d(x,A)|}{\lambda}\right)\right]^{p_k}\\
&\quad+\frac{1}{\rho_n^{\alpha}}\sum_{\substack{k=1\\|d(x,A_k)-d(x,A)|<\varepsilon}}^{n}\left[M\left(\frac{|d(x,A_k)-d(x,A)|}{\lambda}\right)\right]^{p_k}\\
&\geqslant\frac{1}{\rho_n^{\beta}}\sum_{\substack{k=1\\|d(x,A_k)-d(x,A)|\geqslant\varepsilon}}^{n}\left[M\left(\frac{\varepsilon}{\lambda}\right)\right]^{p_k}\geqslant\frac{1}{\rho_n^{\beta}}\sum_{\substack{k=1\\|d(x,A_k)-d(x,A)|\geqslant\varepsilon}}^{n}\min\left([M(\varepsilon_1)]^h,[M(\varepsilon_2)]^H\right)\\
&\geqslant\frac{1}{\rho_n^{\beta}}\left|\{k\leqslant n:|d(x,A_k)-d(x,A)|\geq\varepsilon\}\right|\min\left([M(\varepsilon_1)]^h,[M(\varepsilon_1)]^H\right).
\end{aligned}$$

where $\varepsilon_1=\frac{\varepsilon}{\lambda}$. From the above inequality we have $\{A_k\}\in WS_{\rho}^{\beta}$.

□

Corollary 5.

(i) *Let* $0 < \alpha \leqslant 1$*, M be an Orlicz function and* $\rho = (\rho_n)$ *be a sequence as above, then* $(W_\rho^\alpha, [M,(p)]) \subset WS_\rho^\alpha$.

(ii) *Let* $\rho = (\rho_n)$ *be a sequence as above and* $0 < \alpha \leqslant \beta \leqslant 1$*, then* $(W_\rho^\alpha, [M]) \subseteq (WS_\rho^\beta)$.

Theorem 10. *Let M be an Orlicz function,* $\{A_k\}$ *be a sequence in* $L_\infty(X)$ *and* $\rho = (\rho_n)$ *be a sequence. If* $\lim_{n\to\infty} \dfrac{\rho_n}{\rho_n^\alpha} = 1$*, then* $WS_\rho^\alpha \subseteq (W_\rho^\alpha, [M,(p)])$.

Proof. Suppose that $\{A_k\}$ is in $L_\infty(X)$ and $A_k \longrightarrow A(WS_\rho^\alpha)$. As $\{A_k\} \in L_\infty(X)$ there exists $T > 0$ such that $|d(x,A_k) - d(x,A)| \leqslant T$ for all k. For given $\varepsilon > 0$ we have

$$\begin{aligned}
&\frac{1}{\rho_n^\alpha}\sum_{k=1}^{n}\left[M\left(\frac{|d(x,A_k)-d(x,A)|}{\lambda}\right)\right]^{p_k} = \\
&\frac{1}{\rho_n^\alpha}\sum_{\substack{k=1\\ |d(x,A_k)-d(x,A)|\geqslant\varepsilon}}^{n}\left[M\left(\frac{|d(x,A_k)-d(x,A)|}{\lambda}\right)\right]^{p_k} \\
&+\frac{1}{\rho_n^\alpha}\sum_{\substack{k=1\\ |d(x,A_k)-d(x,A)|<\varepsilon}}^{n}\left[M\left(\frac{|d(x,A_k)-d(x,A)|}{\lambda}\right)\right]^{p_k} \\
&\leqslant \frac{1}{\rho_n^\alpha}\sum_{\substack{k=1\\ |d(x,A_k)-d(x,A)|\geqslant\varepsilon}}^{n}\max\left\{\left[M\left(\frac{T}{\lambda}\right)\right]^h, \left[M\left(\frac{T}{\lambda}\right)\right]^H\right\} \\
&\quad+\frac{1}{\rho_n^\alpha}\sum_{\substack{k=1\\ |d(x,A_k)-d(x,A)|<\varepsilon}}^{n}\left[M\left(\frac{\varepsilon}{\lambda}\right)\right]^{p_k} \\
&\leqslant \max\left\{\left[M\left(\frac{T}{\lambda}\right)\right]^h, \left[M\left(\frac{T}{\lambda}\right)\right]^H\right\}\frac{1}{\rho_n^\alpha}\left|\{k\leqslant n : |d(x,A_k)-d(x,A)| \geq \varepsilon\}\right| \\
&\quad+\frac{\rho_n}{\rho_n^\alpha}\max\left\{\left[M\left(\frac{\varepsilon}{\lambda}\right)\right]^h, \left[M\left(\frac{\varepsilon}{\lambda}\right)\right]^H\right\}.
\end{aligned}$$

Therefore $\{A_k\} \in (W_\rho^\alpha, [M,(p)])$. □

We give the following results without proof.

Theorem 11. *Let* $\rho = (\rho_n)$ *be a sequence as above and* $0 < \alpha \leqslant \beta \leqslant 1$*, then* $(W_\rho^\alpha, [M]) \subseteq (W_\rho^\beta, [M])$.

Theorem 12. *Let* $\rho=(\rho_n)$ *and* $\tau=(\tau_n)$ *be two sequences such that* $\rho_n \leqslant \tau_n$ *for all* $n\in\mathbb{N}$, $0<\alpha\leqslant\beta\leqslant 1$. *If* $\lim_{n\to\infty}\dfrac{\rho_n^\alpha}{\tau_n^\beta}=a>0$, *then* $(W_\tau^\beta,[M])\subseteq(W_\rho^\alpha,[M])$.

Corollary 6. *Let* $\rho=(\rho_n)$ *and* $\tau=(\tau_n)$ *be two sequences such that* $\rho_n \leqslant \tau_n$ *for all* $n\in\mathbb{N}$. *If* $\lim_{n\to\infty}\dfrac{\rho_n^\alpha}{\tau_n^\beta}=a>0$ *holds, then* $(W_\tau^\alpha,[M])\subseteq(W_\rho^\alpha,[M])$ *for* $0<\alpha\leqslant 1$.

Theorem 13. *Let* $\rho=(\rho_n)$ *and* $\tau=(\tau_n)$ *be two sequences such that* $\rho_n \leqslant \tau_n$ *for all* $n\in\mathbb{N}$, $0<\alpha\leqslant\beta\leqslant 1$. *If* $\lim_{n\to\infty}\dfrac{\rho_n^\alpha}{\tau_n^\beta}=1$, *then* $(W_\tau^\beta,[M])\subseteq(WS_\rho^\alpha)$.

Theorem 14. *Let* M *be an Orlicz function and* $\inf_k p_k>0$, *the limit of any sequence* $\{A_k\}\in(W_\rho^\alpha,[M,(p)])$ *is unique.*

Proof. Let $\lim_k p_k=s>0$. Suppose that $A_k\to A_1((W_\rho^\alpha,[M,(p)]))$ and $A_k\to A(W_\rho^\alpha,[M,(p)])$. Then there exists $\lambda_1>0$ and $\lambda_2>0$ such that

$$\lim_{n\to\infty}\frac{1}{\rho_n^\alpha}\sum_{k=1}^n\left[M\left(\frac{|d(x,A_k)-d(x,A_1)|}{\lambda_1}\right)\right]^{p_k}=0,$$

and

$$\lim_{n\to\infty}\frac{1}{\rho_n^\alpha}\sum_{k=1}^n\left[M\left(\frac{|d(x,A_k)-d(x,A_2)|}{\lambda_2}\right)\right]^{p_k}=0.$$

Let $\lambda=\max\{2\lambda_1,2\lambda_2\}$. As M is nondecreasing and convex, we have

$$\begin{aligned}
&\frac{1}{\rho_n^\alpha}\sum_{k=1}^n\left[M\left(\frac{|d(x,A_1)-d(x,A_2)|}{\lambda}\right)\right]^{p_k}\\
&\leqslant\frac{D}{\rho_n^\alpha}\sum_{k=1}^n\frac{1}{2^{p_k}}\left(\left[M\left(\frac{|d(x,A_k)-d(x,A_1)|}{\lambda_1}\right)\right]^{p_k}+\left[M\left(\frac{|d(x,A_k)-d(x,A_2)|}{\lambda_2}\right)\right]^{p_k}\right)\\
&\leqslant\frac{D}{\rho_n^\alpha}\sum_{k=1}^n\frac{1}{2^{p_k}}\left(\left[M\left(\frac{|d(x,A_k)-d(x,A_1)|}{\lambda_1}\right)\right]^{p_k}\right.\\
&\quad\left.+\frac{D}{\rho_n^\alpha}\sum_{k=1}^n\frac{1}{2^{p_k}}\left(\left[M\left(\frac{|d(x,A_k)-d(x,A_2)|}{\lambda_2}\right)\right]^{p_k}\right)\right)\to 0
\end{aligned}$$

as $n\to\infty$, where $\sup_k p_k=H$ and $D=\max(1,2^{H-1})$. Therefore we get

$$\lim_{n\to\infty}\frac{1}{\rho_n^\alpha}\sum_{k=1}^n\left[M\left(\frac{|d(x,A_1)-d(x,A_2)|}{\lambda}\right)\right]^{p_k}=0.$$

As $\lim_k p_k=s$, we have

$$\lim_{k\to\infty}\left[M\left(\frac{|d(x,A_1)-d(x,A_2)|}{\lambda}\right)\right]^{p_k}=\left[M\left(\frac{|d(x,A_1)-d(x,A_2)|}{\lambda}\right)\right]^{s}$$

and so $A_1=A_2$. Thus, the limit is unique. □

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