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LOCAL EXISTENCE AND BLOW UP FOR P-LAPLACIAN EQUATION WITH LOGARITHMIC NONLINEARITY

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Abstract. This paper deals with a problem of a wave equation with p-Laplacian and logarithmic nonlinearity term. Firstly, local existence of weak solutions have been obtained by applying Banach fixed theorem. Later, the finite-time blow up of the solutions have been obtained for negative initial energy.

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1. INTRODUCTION

In this work, we consider the following p-Laplacian hyperbolic type equation with logarithmic nonlinearity

$$\begin{cases} u_{tt} - \Delta u - div\left(|\nabla u|^{p-2}\nabla u\right) - \Delta u_t + |u_t|^{k-2}u_t = |u|^{p-2}u\ln|u|, & x\in\Omega,\ t>0, \\ u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x), & x\in\Omega, \\ u(x,t) = 0, \quad x\in\partial\Omega,\ t\geq 0, \end{cases} \tag{1.1}$$

where $p > k > 2$ are real numbers and $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$, the functions u_0, u_1 are given initial data and exponent p satisfies

$$\begin{cases} 2 < p < \infty, & \text{if } n = 1,2, \\ 2 < p < \frac{2(n-1)}{n-2} & \text{if } n \geq 3. \end{cases} \tag{1.2}$$

In absence of p-Laplacian operator $div\left(|\nabla u|^{p-2}\nabla u\right)$, the problem (1.2) can be reduced to the following wave equation with damping and logarithmic source terms:

$$u_{tt} - \Delta u + h(u_t) = |u|^{p-2}u\ln|u|. \tag{1.3}$$

Problems like equation (1.3) is encountered naturally in quantum mechanics, inflation cosmolog, supersymmetric field theories, and a lot of different areas of physics such as, optics, geophysics and nuclear physics [3, 10]. Let us review some works with related to the problem (1.3). Many authors have investigated the local existence, blow up, and asymptotic behaviour of solutions for wave equation with logarithmic nonlinearity, see in this regard [1, 2, 5, 7, 10, 11, 14, 16–18, 21–23, 26] and a long list of references therein.

The p-Laplacian parabolic problems with logarithmic nonlinearity was investigated by [4, 6, 8, 9, 12, 13, 15, 24].

In [24], Le et al. carried out the research on pseudo p-Laplacian evolution equations with logarithmic nonlinearity

$$u_t - div\left(|\nabla u|^{p-2}\nabla u\right) - \Delta u_t = |u|^{p-2}u\ln|u|, \tag{1.4}$$

and established results of existence or nonexistence of global weak solutions when $p > 2$. They proved the large time decay of the global weak solutions. Finally, authors showed that the solution $u(t)$ is not global, that means, it blows up at finite time.

In [4], the authors studied the following equation

$$u_t - div\left(|\nabla u|^{p-2}\nabla u\right) - k\Delta u_t = |u|^{p-2}u\ln|u|, \tag{1.5}$$

where $1 < p < 2$, $\Omega \subset R^n (n \geq 1)$, $k \geq 0$. They showed that the weak solutions of (1.5) are global and can not blow up in finite time when different from the case $p > 2$. They found the sufficient conditions to divide the global boundedness and blowing-up at infinity of the weak solutions.

To best of our knowledge, there are not enough works related to the hyperbolic type equation with p-Laplacian operator and logarithmic nonlinearity. Recently, this type of problems was studied by [20, 25].

In [20] the same author of this paper investigated the following hyperbolic type equation with logarithmic nonlinearity

$$u_{tt} - div\left(|\nabla u|^{p-2}\nabla u\right) - \Delta u + u_t = ku\ln|u|. \tag{1.6}$$

and proved the local existence of solutions for problem (1.6) by using potential well theory combined with Galerkin method.

By the motivation of the above works, we decided to study the local existence and blow up of solution for problem (1.1). The remaining part of this paper is organized as follows: In Section 2, we state some notations and lemmas which will be useful for our main results. In Section 3, the local existence of the solution is given. In Section 4, we established the finite time blow up when the initial energy is negative.

2. PRELIMINARIES

In this section we give some notations and lemmas which will be used throughout this paper. For simplify notations, we adopt the following abbreviations:

$$\|u\|_p = \|u\|_{L^p(\Omega)}, \quad \|u\|_{1,s} = \|u\|_{W_0^{1,p}(\Omega)} = \left(\|u\|_p + \|\nabla u\|_p\right)^{\frac{1}{p}},$$

for $2 < p < \infty$. We use C and C_i $(i = 1, 2, ...)$ to denote various positive constants and they can have different values in different places. In order to state our main results, we define the corresponding energy to problem (1.1) as follows

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{p}\int_\Omega \ln|u|\, u^p dx + \frac{1}{p^2}\|u\|_p^p. \tag{2.1}$$

For proof of Lemma 1, we refer the reader to Pişkin [19]. The proof of Lemma 2 is clear and straightforward, and will be omitted.

Lemma 1. *For any $u \in H_0^1(\Omega)$, we get*

$$\|u\|_q \le C_q \|\nabla u\|_2,$$

for all $1 \le q \le \frac{2n}{n-2}$ if $n \ge 3$; $1 \le q < \infty$ if $n \le 2$, where C_p is the best embedding constant.

Lemma 2. *For each $q > 0$*

$$|r^q \ln r| \le \frac{1}{eq} \text{ for } 0 < r < 1 \text{ and } 0 \le r^{-q}\ln r \le \frac{1}{eq} \text{ for } r \ge 1.$$

Lemma 3. *$E(t)$ is a nonincreasing function, for $t \ge 0$*

$$E'(t) = -\|\nabla u_t\|^2 - \|u_t\|_k^k \le 0. \tag{2.2}$$

Proof. Multiplying equation (1.1) by u_t and integrating on Ω by using Green formula, we have

$$\int_\Omega u_{tt}u_t dx + \int_\Omega \nabla u \nabla u_t dx - \int_\Omega div\left(|\nabla u|^{p-2}\nabla u\right) u_t dx + \int_\Omega \nabla u_t \nabla u_t dx + \int_\Omega |u_t|^{k-1} u_t dx = \int_\Omega u^{p-2} u \ln|u|\, u_t dx,$$

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{p}\int_\Omega \ln|u|\, u^p dx + \frac{1}{p^2}\|u\|_p^p\right) = -\|\nabla u_t\|^2 - \|u_t\|_k^k,$$

$$E'(t) = -\|\nabla u_t\|^2 - \|u_t\|_k^k.$$

□

Lemma 4 ([14])**.** *Let ϑ be a positive number. Then the following inequality holds*

$$\left||s|^{p-2}\log|s|\right| \le A + |s|^{p-2+\vartheta},\ p > 2 \tag{2.3}$$

for $A > 0$.

For reader's straightforwardness, we notice the definition of weak solutions of problem (1.1).

Definition 1. A function $u(t)$ is called a weak solution of problem (1.1) on $\Omega \times [0,T)$, if

$$u \in C\left((0,T);W_0^{1,p}(\Omega)\right) \cap C^1\left((0,T);L^2(\Omega)\right)$$

and

$$u_t \in L^\infty\left((0,T);H_0^1(\Omega)\right)$$

which satisfies

$$\begin{cases} \int\limits_\Omega u_{tt}(x,t)\,w(x)\,dx + \int\limits_\Omega |\nabla u(x,t)|^{p-2}\nabla u(x,t)\nabla w(x)\,dx \\ \quad + \int\limits_\Omega \nabla u(x,t)\nabla w(x)\,dx + \int\limits_\Omega \nabla u_t(x,t)\nabla w(x)\,dx \\ \quad + \int\limits_\Omega |u_t|^{k-2}(x,t)\,|u_t|(x,t)\,w(x)\,dx \\ = \int\limits_\Omega \ln|u(x,t)|\,u^{p-2}(x,t)\,w(x)\,dx, \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x). \end{cases}$$

for $\forall w \in H_0^1(\Omega)$.

3. Existence of Local Solution

In this part, we prove the local existence results by combining contraction mapping principle and Faedo-Galerkin method. Firstly, we state linear problem

$$\begin{cases} v_{tt} - \Delta v - \Delta v_t + |v_t|^{k-2}v_t - div\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{p-2}u\ln|u|, \ (x,t) \in \Omega \times (0,T), \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \qquad x \in \Omega, \\ v = \frac{\partial v}{\partial n} = 0, \qquad x \in \partial\Omega \times R^+, \end{cases} \tag{3.1}$$

in which $T > 0$. We consider the space

$$\mathcal{L} := C\left([0,T];W_0^{1,p}(\Omega)\right) \cap C^1\left([0,T);H_0^1(\Omega)\right)$$

endowed with the norm

$$\|u\|_{\mathcal{L}}^2 = \max_{t\in[0,T]}\left(\|\nabla u(t)\|_{\mathcal{L}}^2 + \|u_t(t)\|_2^2\right).$$

To prove the existence and uniqueness of local solution of problem (1.1), we obtain the following result.

Lemma 5. *Let* $(u_0,u_1) \in W_0^{1,p}(\Omega) \times H_0^1(\Omega)$ *and* $u \in \mathcal{L}$ *for every* $T > 0$. *Then problem (3.1) has a unique weak solution*

$$v \in C\left([0,T);W_0^{1,p}(\Omega)\right), \ v_t \in C\left([0,T);H_0^1(\Omega)\right).$$

Proof. Let $\{w_j\}_{j=1}^{\infty}$ be the orthogonal complete system of eigenfunctions of the Laplace operator in $H_0^1(\Omega)$ with $\|w_j\| = 1$ for all j. According to their multiplicity of

$$-\Delta w_i = \lambda_i w_i, \quad w_j \mid_{\partial\Omega} = 0. \tag{3.2}$$

We denote by λ_j the related eigenvalues repeated. Then, we choose an orthogonal basis $\{w_j\}_{j=1}^{\infty}$ in $H_0^1(\Omega)$ which is orthonormal in $L^2(\Omega)$. Let

$$u_0^m(x,t) = \sum_{j=1}^{m} \left(\int_{\Omega} \nabla u_0 \nabla w_i \right) w_i,$$

and

$$u_1^m(x,t) = \sum_{j=1}^{m} \left(\int_{\Omega} u_1 w_i \right) w_i.$$

So that there exist subsequences $u_0^m \in W_m$, $u_1^m \in W_m$, $u_0^m \to u_0$ in $W_0^{1,p}(\Omega)$ and $u_1^m \to u_1$ in $H_0^1(\Omega)$ as $m \to \infty$. For all $m \geq 1$ we will seek an approximate solution such that

$$v_m(t) = \sum_{i=1}^{m} \gamma_i^m(t) w_i \tag{3.3}$$

which satisfies the following Cauchy problem

$$\begin{cases} \int_{\Omega} \left(v_m''(t) - \Delta v_m - \Delta v_m' + |v_m'|^{k-2} v_m' - div\left(|\nabla u|^{p-2} \nabla u \right) - |u|^{p-2} u \ln|u| \right) \eta dx = 0, \\ v_m(0) = v_0^m, \quad v_m' = v_1^m, \end{cases} \tag{3.4}$$

where $t \geq 0$ and $\eta \in W_m$. Let $\eta = w_i$ for $i = 1, 2 \ldots m$ in (3.4), for the first term, we obtain

$$\int_{\Omega} v_m''(t) w_i dx = \int_{\Omega} \left(\sum_{j=1}^{m} \ddot{\gamma}_j^m(t) w_j \right) w_i dx = \ddot{\gamma}_i^m(t) \int_{\Omega} |w_i|^2 dx = \ddot{\gamma}_i^m(t). \tag{3.5}$$

Similarly we obtain the second term as follows

$$\begin{aligned} \int_{\Omega} -\Delta v_m w_i dx &= -\int_{\Omega} \sum_{j=1}^{m} \gamma_j^m(t) \Delta w_j w_i dx \\ &= \int_{\Omega} \sum_{j=1}^{m} \gamma_j^m(t) \lambda_j w_j w_i dx \\ &= \gamma_i^m(t) \lambda_i \int_{\Omega} |w_i|^2 dx \\ &= \gamma_i^m(t) \lambda_i. \end{aligned} \tag{3.6}$$

For the third term we have

$$\int_{\Omega} -\Delta v_m'(t)\, w_i dx = \int_{\Omega} \left(\sum_{j=1}^{m} -\Delta \dot{\gamma}_j^m(t)\, w_j \right) w_i dx \\ = \nabla \dot{\gamma}_j^m(t). \tag{3.7}$$

For the fourth term, we get

$$\int_{\Omega} \left| v_m'(t) \right|^{k-2} v_m'(t)\, w_i dx = \int_{\Omega} \left(\sum_{j=1}^{m} \left| \dot{\gamma}_j^m(t) \right|^{k-2} \dot{\gamma}_j^m(t)\, w_j \right) w_i dx \\ = \left| \dot{\gamma}_j^m(t) \right|^{k-2} \dot{\gamma}_j^m(t) \int_{\Omega} |w_i|^2 dx \\ = \left| \dot{\gamma}_j^m(t) \right|^{k-2} \dot{\gamma}_j^m(t). \tag{3.8}$$

Then, we insert (3.5)-(3.8) in (3.4). So that (3.4) yields the following Cauchy problem for a linear ordinary differential equation for unknown functions $\gamma_i^m(t)$;

$$\begin{cases} \ddot{\gamma}_i^m(t) + \gamma_i^m(t)\lambda_i + \nabla \dot{\gamma}_j^m(t) + \left| \dot{\gamma}_j^m(t) \right|^{k-2} \dot{\gamma}_j^m(t) = G_i(t), \ i = 1,2,...m, \\ \gamma_i^m(0) = \int_{\Omega} u_0 w_i dx, \ \ \dot{\gamma}_i^m(0) = \int_{\Omega} u_1 w_i dx, \ i = 1,2,...m, \end{cases} \tag{3.9}$$

where

$$G_i(t) = \int_{\Omega} div\left(|\nabla u|^{p-2} \nabla u \right) w_i dx + \int_{\Omega} |u|^{p-2} u \ln|u|\, w_i, \ i = 1,2,...m, t \in [0,T]. \tag{3.10}$$

Then the above problem possesses a unique local solution $\gamma_i^m \in C^2[0,T]$ for all i, which implies a unique v_m defined by (3.3) satisfies (3.4). Replacing η by v_m' in (3.4) and then integrating this over $[0,t] \subset [0,T]$, we obtain

$$\left\| v_m'(t) \right\|^2 + \left\| \nabla v_m(t) \right\|^2 + 2\int_0^t \left\| \nabla v_m'(\tau) \right\|^2 d\tau + 2\int_0^t \left\| v_m'(\tau) \right\|_k^k d\tau \\ = \left\| v_{1m}(t) \right\|^2 + \left\| \nabla v_{0m} \right\|^2 + 2\int_0^t \int_{\Omega} |u(s)|^{p-2} u(s) \ln|u(s)|\, v_m'(s)\, dxds \\ + 2\int_0^t \int_{\Omega} div\left(|\nabla u|^{p-2} \nabla u \right)(s)\, v_m'(s)\, dxds, \tag{3.11}$$

for each $m \geq 1$.

Our aim is to estimate the last two terms in the right-hand side of (3.11). First, we estimate the third term by applying Sobolev's and Young's inequalities, we infer that

$$2\int_0^t\int_\Omega |u(s)|^{p-2}u(s)\ln|u(s)|\,v_m'(s)\,dxds \le 2\int_0^t\int_\Omega \left||u(s)|^{p-1}\ln|u(s)|\right|\left|v_m'(s)\right|dxds$$
$$\le \int_0^t\int_\Omega \left||u(s)|^{p-1}\ln|u(s)|\right|^{\frac{m}{m-1}}dxds+\int_0^t \left\|v_m'(s)\right\|_k^k ds. \tag{3.12}$$

In order to estimate (3.12), we deal with logarithmic term. We define

$$\Omega_1=\{x\in\Omega;|u(x)|<1\} \text{ and } \Omega_2=\{x\in\Omega;|u(x)|\ge 1\},$$

where $\Omega=\Omega_1\cup\Omega_2$. Then we obtain

$$\int_\Omega \left||u(s)|^{p-1}\ln|u(s)|\right|^{\frac{k}{k-1}}dx$$
$$=\int_{\Omega_1}\left||u(s)|^{p-1}\ln|u(s)|\right|^{\frac{k}{k-1}}dx+\int_{\Omega_2}\left||u(s)|^{p-1}\ln|u(s)|\right|^{\frac{k}{k-1}}dx. \tag{3.13}$$

By make use of Lemma 2, we have

$$\int_{\Omega_1}\left||u(s)|^{p-1}\ln|u(s)|\right|^2 dx \le [e(p-1)]^{-\frac{k}{k-1}}|\Omega| = C, \tag{3.14}$$

where

$$\inf_{r\in(0,1)} r^{p-1}\ln r = [e(p-1)]^{-1}.$$

Let

$$\theta=\frac{2n}{n-2}.\frac{k}{k-1}-p+1>0 \text{ for } n\ge 3;\ \text{each positive } \theta \text{ for } n=1,2.$$

By the Sobolev embedding $H_0^1(\Omega)\hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ if $n\ge 3$and to $L^q(\Omega)$ for any $q\ge 1$ if $n=1,2$, recalling $u\in\mathcal{L}:=C\left([0,T];H_0^1(\Omega)\right)$, we obtain

$$\int_{\Omega_2}\left||u(s)|^{p-1}\ln|u(s)|\right|^{\frac{k}{k-1}}dx \le \int_{\Omega_2}\theta^{-\frac{k}{k-1}}\theta^{\frac{k}{k-1}}\left(|u(s)|^{p-1}\ln|u(s)|\right)^{\frac{k}{k-1}}dx$$
$$\le \theta^{-\frac{k}{k-1}}\int_{\Omega_2}\left(|u(s)|^{p-1}\ln|u(s)|^{\theta}\right)^{\frac{k}{k-1}}dx$$
$$\le \theta^{-\frac{k}{k-1}}\int_{\Omega_2}\left(|u(s)|^{p-1+\theta}\right)^{\frac{k}{k-1}}dx$$

$$\begin{aligned}
&\leq \theta^{-\frac{k}{k-1}} \int\limits_{\Omega_2} |u(s)|^{\frac{2n}{n-2}}\, dx \\
&\leq \theta^{-\frac{k}{k-1}} \int\limits_{\Omega=\Omega_1\cup\Omega_2} |u(s)|^{\frac{2n}{n-2}}\, dx \\
&= \theta^{-\frac{k}{k-1}} \|u\|_{\frac{2n}{n-2}}^{\frac{2n}{n-2}} \\
&\leq C \|u\|_{\mathcal{L}}^{\frac{2n}{n-2}} \leq C. \qquad (3.15)
\end{aligned}$$

The proof of the case $n = 1, 2$ is similar.

By using of (3.14), (3.15) and embedding theorem, (3.12) yields

$$2\int_0^t \int_\Omega |u(s)|^{p-2} u(s) \ln |u(s)|\, v_m'(s)\, dxds \leq CT + \int_0^t \left\|v_m'(s)\right\|_k^k ds. \qquad (3.16)$$

By similar calculations we have the second term of (3.11) in the right-hand side as follows

$$\begin{aligned}
2\int_0^t \int_\Omega div\left(|\nabla u|^{p-2} \nabla u\right)(s)\, v_m'(s)\, dxds &\leq 2\int_0^t \left(\frac{1}{p} \|\nabla u(s)\|_p^p \left\|v_m'(s)\right\|\right) ds \\
&\leq \int_0^t \left(\left(\frac{1}{p} \|\nabla(s)\|_p^p\right)^2 + \left\|v_m'(s)\right\|_2^2\right) ds \\
&\leq CT + C\int_0^t \left\|\nabla v_m'(s)\right\|_2^2 ds. \qquad (3.17)
\end{aligned}$$

Adapting (3.17) and (3.16) into (3.11), we have

$$\begin{aligned}
&\left\|v_m'(t)\right\|^2 + \|\nabla v_m(t)\|^2 + \int_0^t \left\|\nabla v_m'(\tau)\right\|^2 d\tau + \int_0^t \left\|v_m'(\tau)\right\|_k^k d\tau, \\
&\qquad \leq \|v_{1m}(t)\|^2 + \|\nabla v_{0m}\|^2 + CT \leq C, \qquad (3.18)
\end{aligned}$$

where C is a positive constant and independent of m. We have from (3.18) that

$$\left\{\begin{array}{l} v_m, \text{ is bounded in } L^\infty\left([0,T]; H_0^1(\Omega)\right), \\ v_m', \text{ is bounded in } L^\infty\left([0,T]; H_0^1(\Omega)\right). \end{array}\right. \qquad (3.19)$$

Thus, up to subsequence, we can pass to the limit in (3.4) satisfies the above regularity. We obtained a weak local solution of the problem (3.4).

To prove the uniqueness, arguing by contradiction, we assume that there are two solutions such that w and v of problem (3.4) which have the same initial data. Subtracting these two equations and testing result by $w_t - v_t$, we can obtain

$$\|w_t - v_t\|^2 + \|\nabla w - \nabla v\|^2 + 2\int_0^T \int_\Omega (\nabla w_\tau - \nabla v_\tau)(\nabla w_\tau - \nabla v_\tau)\,d\tau + 2\int_0^T \int_\Omega \left(|w_\tau|^{k-2} w_\tau - |v_\tau|^{k-2} v_\tau\right)(w_\tau - v_\tau)\,d\tau = 0. \quad (3.20)$$

We define $f : R \to R$, $f(x) = x$ and $g : R \to R$, $g(y) = y$.It follows from the following element inequality

$$\left(|f(x)|^{k-2} f(x) - |g(y)|^{k-2} g(y)\right)(f(x) - g(y)) \geq C\,|f(x) - g(y)|^k \text{ for } k \geq 2,$$

that (3.20) yields

$$\|w_t - v_t\|^2 + \|\nabla w - \nabla v\|^2 + c\left(\int_0^T \|\nabla w_\tau - \nabla v_\tau\|^2 + \int_0^T \|\nabla w_\tau - \nabla v_\tau\|_k^k\right) \leq 0.$$

□

Theorem 1. *There exist* $T > 0$, *such that the problem (1.1) has a unique local weak solution on* $[0, T]$.

Proof. Let $(u_0, u_1) \in W_0^{1,p}(\Omega) \times L^2(\Omega)$ and

$$r^2 = \|u_1\|^2 + \|\nabla u_0\|^2.$$

For $T > 0$, we denote

$$B_r = \{u \in \mathcal{L} : u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ \|u(t)\|_{\mathcal{L}} \leq r\}.$$

By Lemma 5, we define a nonlinear mapping $\Psi : u \to v = \Psi(u)$ where v is the unique solution of problem (3.1). We claim that Ψ is a contraction map satisfying $\Psi(B_r) \subseteq B$ for small $T > 0$.

Indeed, suppose that $u \in B_r$, the corresponding solution $v = \Psi(u)$ satisfies the energy identity for all $t \in [0, T]$ as follows

$$\frac{1}{2}\left(\|v_t\|^2 + \|\nabla v\|^2\right) \leq \frac{1}{2}\left(\|u_1\|^2 + \|\nabla u_0\|^2\right) + \int_0^t \int_\Omega |u(s)|^{p-2} u(s) \ln|u(s)|\, v_t(s)\,dxds + \int_0^t \int_\Omega div\left(|\nabla u|^{p-2} \nabla u\right) v_t(s)\,dxds. \quad (3.21)$$

Next we estimate the last terms in (3.21) as

$$\begin{aligned}\frac{1}{2}\left(\|v_t\|^2+\|\nabla v\|^2\right) &\leq \frac{r^2}{2}+\int_0^t\left(r^{\frac{2n}{n-2}}+\frac{1}{p}r^p\right)\|v_t(s)\|\,ds\\ &=\frac{r^2}{2}+\sqrt{2}\left(r^{\frac{2n}{n-2}}+\frac{1}{p}r^p\right)\int_0^t\left(\frac{1}{2}\|v_t(s)\|^2\right)^{\frac{1}{2}}ds\\ &\leq \frac{r^2}{2}+\sqrt{2}\left(r^{\frac{2n}{n-2}}+\frac{1}{p}r^p\right)\int_0^t\left(\frac{1}{2}\|v_t(s)\|^2+\frac{1}{2}\|\nabla v(s)\|^2\right)^{\frac{1}{2}}ds.\end{aligned} \tag{3.22}$$

By Gronwall inequality, we have

$$\left(\frac{1}{2}\|v_t(s)\|^2+\frac{1}{2}\|\nabla v(s)\|^2\right)^{\frac{1}{2}}\leq \frac{r}{2}+\sqrt{2}\left(r^{\frac{2n}{n-2}}+\frac{1}{p}r^p\right)T. \tag{3.23}$$

If we choose $T>0$ small enough, $\|u(t)\|_{\mathcal{L}}\leq r$, which implies that $\Psi(B_r)\subseteq B$.

Next, we will verify that Ψ is a contraction mapping. Let $v_1,v_2\in X_{r_0,T}$ and $u_1=\Psi v_{1,}\ u_2=\Psi v_{2,}$ be the corresponding solution for problem (3.1). Setting $v_1=\Psi w_{1,}\ v_2=\Psi w_{2,}\ w_{1,}\ w_2\in B_r$, and $v=v_1-v_2$, then, v satisfies

$$\begin{cases}(v_{tt},\eta)+(\nabla v,\nabla\eta)+\int_\Omega(\nabla v_{1t}-\nabla v_{2t})\nabla\eta dx\\ \quad+\int_\Omega(\nabla v_{1t}-\nabla v_{2t})\nabla\eta dx+\int_\Omega\left(|v_{1t}|^{k-2}v_{1t}-|v_{2t}|^{k-2}v_{2t}\right)\eta dx\\ =\int_\Omega\left[div\left(|\nabla w_1|^{p-2}\nabla w_1\right)-div\left(|\nabla w_2|^{p-2}\nabla w_2\right)\right]\eta dx\\ \quad+\int_\Omega\left(|w_1|^{p-2}w_1\ln|w_1|-|w_2|^{p-2}w_2\ln|w_2|\right)\eta dx,\end{cases} \tag{3.24}$$

for any $\eta\in H_0^1(\Omega)$ and a.e. $t\in[0,T]$.

Taking $\eta=v_t=v_{1t}-v_{2t}$ and using Green formula, noticing

$$\begin{aligned}\int_\Omega\left(|v_{1t}|^{k-2}v_{1t}-|v_{2t}|^{k-2}v_{2t}\right)(v_{1t}-v_{2t})\,dx\geq 0,\\ \int_\Omega(\nabla v_{1t}-\nabla v_{2t})(\nabla v_{1t}-\nabla v_{2t})\,dx\geq 0,\end{aligned} \tag{3.25}$$

and integrating both sides of (3.24) over $(0,t)$, we have

$$\left(\|v_t\|^2+\|\nabla v\|^2\right)=2\int_0^t\int_\Omega\left(|w_1|^{p-2}w_1\ln|w_1|-|w_2|^{p-2}w_2\ln|w_2|\right)v_t dxds$$

$$+2\int_0^t\int_\Omega \left[div\left(|\nabla w_1|^{p-2}\nabla w_1\right)-div\left(|\nabla w_2|^{p-2}\nabla w_2\right)\right]v_t dxds. \tag{3.26}$$

Firstly we note that from the Mean Value Theorem there exists $C_p>0$ such that

$$||x|^{p-2}x-|y|^{p-2}y|\leq C_p(|x|^{p-2}+|y|^{p-2})|x-y|,\forall x,y\in R^2. \tag{3.27}$$

Then using Hölder inequality with $\frac{p-2}{2(p-1)}+\frac{1}{2(p-1)}+\frac{1}{2}=1$ and embedding theorems, we have

$$\begin{aligned}&\left|\int_0^t\int_\Omega \left[div\left(|\nabla w_1|^{p-2}\nabla w_1\right)-div\left(|\nabla w_2|^{p-2}\nabla w_2\right)\right]v_t dxds\right|\\
&\leq C\int_0^t\int_\Omega \left(|\nabla w_1|^{p-2}+|\nabla w_2|^{p-2}\right)|\nabla w_1-\nabla w_2|\,|v_t|\,dxds\\
&\leq C\int_0^t \left(\|\nabla w_1\|_{2(p-1)}^{p-2}+\|\nabla w_2\|_{2(p-1)}^{p-2}\right)\|v_t\|_{2(p-1)}\|\nabla w_1-\nabla w_2\|_2^2 dxds\\
&\leq C\int_0^t r^{2(p-2)}\|\nabla w_1-\nabla w_2\|_2^2\|v_t\|_2\\
&\leq C\int_0^t r^{2(p-2)}\|\nabla w_1-\nabla w_2\|_2^2\left(\|v_t\|_2+\|\nabla v\|_2\right),\end{aligned} \tag{3.28}$$

where $C>0$.

Now, we need estimate the logarithmic term in (3.26). If we set

$$G(s)=|s|^{p-2}s\ln|s|,$$

then

$$\begin{aligned}G'(s)&=(p-1)|s|^{p-2}\ln|s|+|s|^{p-2}\\
&=(1+(p-1)\ln|s|)|s|^{p-2}.\end{aligned}$$

Making use of mean value theorem, we get

$$\begin{aligned}&|G(w_1)-G(w_2)|=\left|G'(\vartheta w_1+(1-\vartheta)w_2)(w_1-w_2)\right| \qquad (3.29)\\
&\leq[1+(p-1)\ln|(\vartheta w_1+(1-\vartheta)w_2)|]\,|(\vartheta w_1+(1-\vartheta)w_2)|^{p-2}|w_1-w_2|\end{aligned}$$

where $0<\vartheta<1$.

Then, by recalling Lemma 4 we conclude that

$$|G(w_1)-G(w_2)|\leq|(\vartheta w_1+(1-\vartheta)w_2)|^{p-2}|w_1-w_2|+(p-1)A|w_1-w_2|$$

$$
\begin{aligned}
&+(p-1)\left|w_1-w_2\right|\left|(\vartheta w_1+(1-\vartheta)w_2)\right|^{p-2+\varepsilon}\\
&\leq \left|(w_1+w_2)\right|^{p-2}\left|w_1-w_2\right|+(p-1)A\left|w_1-w_2\right|\\
&+(p-1)\left|w_1-w_2\right|\left|w_1+w_2\right|^{p-2+\varepsilon} \qquad (3.30)
\end{aligned}
$$

As $w_1, w_2 \in B_r$, then it follows from Hölder inequality (for $\frac{1}{p-1}+\frac{p-2}{p-1}=1$) and Sobolev embedding that

$$
\begin{aligned}
\int_\Omega \left[\left|(w_1+w_2)\right|^{p-2}\left|w_1-w_2\right|\right]^2 dx &\leq C\left[\|w_1\|_{2(p-1)}^{2(p-1)}+\|w_2\|_{2(p-1)}^{2(p-1)}\right]^{\frac{p-2}{p-1}}\|w_1-w_2\|_{2(p-1)}^2\\
&\leq C\left[\|\nabla w_1\|_2^{2(p-1)}+\|\nabla w_2\|_2^{2(p-1)}\right]^{\frac{p-2}{p-1}}\|\nabla w_1-\nabla w_2\|_2^2\\
&\leq Cr^{2(p-2)}\|\nabla w_1-\nabla w_2\|_2^2. \qquad (3.31)
\end{aligned}
$$

By similar calculation , we obtain

$$
\begin{aligned}
&\int_\Omega \left[\left|w_1-w_2\right|\left|w_1+w_2\right|^{p-2+\varepsilon}\right]^2 dx\\
&\leq \left(\int_\Omega \left|w_1+w_2\right|^{\frac{2(p-2+\varepsilon)(p-1)}{p-2}}dx\right)^{\frac{p-2}{p-1}}\left(\int_\Omega \left|w_1-w_2\right|^{2(p-1)}dx\right)^{\frac{1}{p-1}}\\
&\leq \left(\int_\Omega \left|w_1+w_2\right|^{2(p-1)+\frac{2\varepsilon(p-1)}{p-2}}dx\right)^{\frac{p-2}{p-1}}\|w_1-w_2\|_{2(p-1)}^2. \qquad (3.32)
\end{aligned}
$$

Because of (1.2) $\left(2p-2<\frac{2n}{n-2}\right)$, we choose $\varepsilon>0$ small enough such that

$$
p_* = 2(p-1)+\frac{2\varepsilon(p-1)}{p-2}<\frac{2n}{n-2}.
$$

Therefore, by using of Sobolev embedding $H_0^1(\Omega)\hookrightarrow L^{2(p-1)}(\Omega)$ and $H_0^1(\Omega)\hookrightarrow L^{p_*}(\Omega)$ (3.32) yields that

$$
\begin{aligned}
\int_\Omega \left[\left|w_1-w_2\right|\left|w_1+w_2\right|^{p-2+\varepsilon}\right]^2 dx &\leq C\left[\|\nabla w_1\|_2^{p_*}+\|\nabla w_2\|_2^{p_*}\right]^{\frac{p-2}{p-1}}\|\nabla w_1-\nabla w_2\|_2^2\\
&\leq Cr^{\frac{p_*(p-2)}{p-1}}\|\nabla w_1-\nabla w_2\|_2^2. \qquad (3.33)
\end{aligned}
$$

Noticing that $\|w_1-w_2\|_2 \leq C\|\nabla w_1-\nabla w_2\|_2$, by using of (3.31) and (3.33), we can infer that

$$
\int_0^t\int_\Omega \left(\left|w_1\right|^{p-2}w_1\ln\left|w_1\right|-\left|w_2\right|^{p-2}w_2\ln\left|w_2\right|\right)v_t\,dxds
$$

$$\leq C\int_0^t \left(1+r^{\frac{p_*(p-2)}{p-1}}+r^{2(p-2)}\right)\|\nabla w_1-\nabla w_2\|_2^2\left(\|v_t\|_2+\|\nabla v\|_2\right)ds. \tag{3.34}$$

Consequently, by inserting (3.34), (3.28) into (3.26), we arrive at

$$\begin{aligned}&\left(\|v_t\|^2+\|\nabla v\|^2\right)\\&\leq C\int_0^t \left(1+r^{\frac{p_*(p-2)}{p-1}}+2r^{2(p-2)}\right)\|\nabla w_1-\nabla w_2\|_2^2\left(\|v_t\|_2+\|\nabla v\|_2\right)ds\\&\leq C\int_0^t \left(1+r^{\frac{p_*(p-2)}{p-1}}+2r^{2(p-2)}\right)\|\nabla w_1-\nabla w_2\|_2^2\left(\|v_t\|^2+\|\nabla v\|^2\right)^{\frac{1}{2}}ds.\end{aligned}$$

By Gronwall inequality, we obtain

$$\left(\|v_t\|^2+\|\nabla v\|^2\right)^{\frac{1}{2}}\leq CT\|\nabla w_1-\nabla w_2\|_2^2.$$

We choose T small enough such that $C\left(1+r^{\frac{p_*(p-2)}{p-1}}+2r^{2(p-2)}\right)T<1$. Thus, we have Ψ is a contraction mapping in B_r. The proof is completed. □

4. Blow Up

In this part, we prove the blow up result of solution for the problem (1.1) with negative initial energy. We give some lemmas which be used in our proof. For proof of Lemmas 6-8, we refer the readers to Kafini and Messaoudi [14].

Lemma 6. *Suppose that (1.2) holds. There exists a positive constant depending on Ω only such that*

$$\left(\int_\Omega u^p\ln|u|\,dx\right)^{\frac{s}{p}}\leq C\left[\int_\Omega u^p\ln|u|\,dx+\|\nabla u\|_2^2\right], \tag{4.1}$$

for any $u\in L^{p+1}(\Omega)$ and $2\leq s\leq p$, provided that $\int_\Omega u^p\ln|u|\,dx\geq 0$.

Lemma 7. *Suppose that (1.2) holds. There exists a positive constant depending on Ω only such that*

$$\|u\|_p^p\leq C\left[\int_\Omega u^p\ln|u|\,dx+\|\nabla u\|_2^2\right], \tag{4.2}$$

for any $u\in L^p(\Omega)$, provided that $\int_\Omega u^p\ln|u|\,dx\geq 0$.

Thus, the result was obtained.

Corollary 1. *Let the assumptions of the Lemma 7 and $k<p$ hold. Using the fact that $\|u\|_k^k \le C\|u\|_p^k \le C\left(\|u\|_k^p\right)^{\frac{k}{p}}$. Then we obtain the following*

$$\|u\|_k^k \le C\left[\left(\int_\Omega u^p \ln|u|\,dx\right)^{\frac{k}{p}} + \|\nabla u\|^{\frac{2k}{p}}\right]. \tag{4.3}$$

Lemma 8. *Suppose that (1.2) holds. There exists a positive constant depending on Ω only such that*

$$\|u\|_p^s \le C\left[\|u\|_p^p + \|\nabla u\|_2^2\right], \tag{4.4}$$

for any $u \in L^p(\Omega)$ and $2 \le s \le p$.

Theorem 2. *Assume that $E(0)<0$. Let the conditions in Lemma 8 hold. Then the solutions of (1.1) blow up in finite time*

$$T^* \le \frac{1-\alpha}{\xi_{\frac{\alpha}{1-\alpha}} L^{\frac{\alpha}{1-\alpha}}(0)}, \tag{4.5}$$

where ξ and α positive constant.

Proof. For this purpose, we denote

$$H(t) = -E(t). \tag{4.6}$$

By using the definition of $H(t)$ and (2.2) we obtain

$$H'(t) = -E'(t) = \|\nabla u_t\|^2 + \|u_t\|_k^k \ge 0. \tag{4.7}$$

Consequently by virtue of (2.1), (4.6) and (4.7) we get

$$0 < H(0) \le H(t) \le \frac{1}{p}\int_\Omega u^p \ln|u|\,dx. \tag{4.8}$$

We set

$$L(t) = H^{1-\alpha}(t) + \varepsilon\int_\Omega u u_t dx + \varepsilon\frac{1}{2}\|\nabla u\|^2, \tag{4.9}$$

for ε small to be chosen later and

$$\frac{2\left(p^2-2k\right)}{(k-1)p^3} < \alpha < \frac{p-k}{(k-1)p}. \tag{4.10}$$

Now, differentiating $L(t)$ with respect to t, we obtain

$$\begin{aligned} L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\int_\Omega |u_t|^2 dx + \varepsilon\int_\Omega u u_{tt} dx + \varepsilon\int_\Omega \nabla u \nabla u_t \\ &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\|u_t\|^2 + \varepsilon\int_\Omega \nabla u \nabla u_t \end{aligned}$$

$$+\varepsilon\int_{\Omega} u\left(\Delta u+div\left(|\nabla u|^{p-2}\nabla u\right)-|u_t|^{k-2}u_t+\Delta u_t+|u|^{p-2}u\ln|u|\right)dx$$
$$=(1-\alpha)H^{-\alpha}(t)H'(t)+\varepsilon\|u_t\|^2-\varepsilon\|\nabla u\|^2-\varepsilon\|\nabla u\|_p^p$$
$$-\varepsilon\int_{\Omega}|u_t|^{k-2}u_t u dx+\varepsilon\int_{\Omega}u^p\ln|u|\,dx. \tag{4.11}$$

Adding and subtracting $\varepsilon p(1-\alpha)H(t)$ for some $0<\alpha<1$ in (4.11), we obtain

$$L'(t)=(1-\alpha)H^{-\alpha}(t)H'(t)+\varepsilon\left(\frac{p(1-\alpha)+2}{2}\right)\|u_t\|^2$$
$$-\varepsilon\left(\frac{2-p(1-\alpha)}{2}\right)\|\nabla u\|^2-\varepsilon\alpha\|\nabla u\|_p^p+\varepsilon\frac{(1-\alpha)}{p}\|u\|_p^p$$
$$+\varepsilon\alpha\int_{\Omega}u^p\ln|u|\,dx-\varepsilon\int_{\Omega}|u_t|^{k-2}u_t u dx+\varepsilon p(1-\alpha)H(t).$$

Exploiting Young's inequality, we get

$$\int_{\Omega}|u_t|^{k-2}u_t u dx\le\frac{\delta^k}{k}\|u\|_k^k+\frac{k-1}{k}\delta^{-\frac{k}{k-1}}\|u_t\|_k^k$$
$$\le\frac{\delta^k}{k}\|u\|_k^k+\frac{k-1}{k}\delta^{-\frac{k}{k-1}}\left(\|u_t\|_k^k+\|\nabla u_t\|^2\right),$$

for any $\delta>0$, (4.11) takes form

$$L'(t)=\left((1-\alpha)H^{-\alpha}(t)-\varepsilon\frac{k-1}{k}\delta^{-\frac{k}{k-1}}\right)H'(t)-\varepsilon\frac{\delta^k}{k}\|u\|_k^k$$
$$+\varepsilon\left(\frac{p(1-\alpha)+2}{2}\right)\|u_t\|^2-\varepsilon\left(\frac{2-p(1-\alpha)}{2}\right)\|\nabla u\|^2$$
$$-\varepsilon\alpha\|\nabla u\|_p^p+\varepsilon p(1-\alpha)H(t)+\varepsilon\frac{(1-\alpha)}{p}\|u\|_p^p+\varepsilon\alpha\int_{\Omega}u^p\ln|u|\,dx. \tag{4.12}$$

Of course (4.12) holds even if δ is time dependent since the integral is taken over the x-variable. Therefore by choosing δ so that $\delta^{-\frac{k}{k-1}}=M_1H^{-\alpha}(t)$, for M_1 to be specified later, and substituting in (4.12), we have

$$L'(t)\ge\left(1-\alpha-\varepsilon\frac{k-1}{k}M_1\right)H^{-\alpha}(t)H'(t)+\varepsilon\frac{(1-\alpha)}{p}\|u\|_p^p$$
$$+\varepsilon\left(\frac{p(1-\alpha)+2}{2}\right)\|u_t\|^2-\varepsilon\alpha\|\nabla u\|_p^p-\varepsilon\left(\frac{2-p(1-\alpha)}{2}\right)\|\nabla u\|^2$$
$$-\varepsilon\frac{(M_1)^{1-k}}{k}H^{\alpha(k-1)}(t)\|u\|_k^k+\varepsilon\alpha\int_{\Omega}u^p\ln|u|\,dx+\varepsilon p(1-\alpha)H(t). \tag{4.13}$$

By using of Corollary 1, embedding theorem and Young's inequality, we obtain

$$\begin{aligned}
H^{\alpha(k-1)}\|u\|_k^k &\le \left(\frac{1}{p}\int_\Omega u^p\ln|u|\,dx\right)^{\alpha(k-1)}\|u\|_k^k \\
&\le C\left(\int_\Omega u^p\ln|u|\,dx\right)^{\alpha(k-1)}\left[\left(\int_\Omega u^p\ln|u|\,dx\right)^{\frac{k}{p}}+\|\nabla u\|^{\frac{2k}{p}}\right] \\
&\le C\left[\left(\int_\Omega u^p\ln|u|\,dx\right)^{\alpha(k-1)+\frac{k}{p}}+\left(\int_\Omega u^p\ln|u|\,dx\right)^{\alpha(k-1)}\|\nabla u\|^{\frac{2k}{p}}\right] \\
&\le C\left[\left(\int_\Omega u^p\ln|u|\,dx\right)^{\alpha(k-1)+\frac{k}{p}}+\left(\int_\Omega u^p\ln|u|\,dx\right)^{\alpha(k-1)}\|\nabla u\|_p^{\frac{2k}{p}}\right] \\
&\le C\left[\left(\int_\Omega u^p\ln|u|\,dx\right)^{\alpha(k-1)+\frac{k}{p}}+\left(\int_\Omega u^p\ln|u|\,dx\right)^{\alpha(k-1)\frac{p^2}{p^2-2k}}+\|\nabla u\|_p^p\right]
\end{aligned} \tag{4.14}$$

Make use of (4.10), we find

$$2<\alpha(k-1)\,p+k\le p \text{ and } 2<\frac{\alpha(k-1)\,p^3}{p^2-2k}\le p.$$

Therefore, Lemma 6 yields that

$$H^{\alpha(k-1)}\|u\|_k^k\le C\left[\int_\Omega u^p\ln|u|\,dx+\|\nabla u\|_p^p\right]. \tag{4.15}$$

Inserting (4.15) into (4.13), we arrive at

$$\begin{aligned}
L'(t)\ge&\left(1-\alpha-\varepsilon\frac{k-1}{k}M_1\right)H^{-\alpha}(t)\,H'(t)+\varepsilon\left(\frac{p(1-\alpha)+2}{2}\right)\|u_t\|^2 \\
&-\varepsilon\left[\alpha+\frac{(M_1)^{1-k}}{k}C\right]\|\nabla u\|_p^p-\varepsilon\left(\frac{2-p(1-\alpha)}{2}\right)\|\nabla u\|^2+\varepsilon\frac{(1-\alpha)}{p}\|u\|_p^p \\
&+\varepsilon\left[\alpha-\frac{(M_1)^{1-k}}{k}C\right]\int_\Omega u^p\ln|u|\,dx+\varepsilon p(1-\alpha)\,H(t).
\end{aligned} \tag{4.16}$$

Since $0 < \frac{2}{p} < 1$, now using the following inequality

$$x^{v} \leq x+1 \leq \left(1+\frac{1}{\beta}\right)(x+\beta), \ \forall x \geq 0, 0 < v < 1, \beta \geq 0 \tag{4.17}$$

especially taking $x = \|\nabla u\|_p^p$, $d = 1+\frac{1}{H(0)}$, $\beta = H(0)$ and by using of the (4.8), we have

$$\|\nabla u\|_2^2 \leq \left(\|\nabla u\|_p^p\right)^{\frac{2}{p}} \leq \left(1+\frac{1}{H(0)}\right)\left(\|\nabla u\|_p^p + H(0)\right) \leq d\left(\|\nabla u\|_p^p + H(t)\right). \tag{4.18}$$

Inserting (4.15)-(4.18) into (4.16) we deduce

$$\begin{aligned} L'(t) \geq & \left(1-\alpha-\varepsilon\frac{k-1}{k}M_1\right)H^{-\alpha}(t)H'(t)+\varepsilon\left(\frac{p(1-\alpha)+2}{2}\right)\|u_t\|^2 \\ & +\varepsilon\left[d\left(\frac{p(1-\alpha)-2}{2}\right)-\alpha-\frac{(M_1)^{1-k}}{k}C\right]\|\nabla u\|_p^p \\ & +\varepsilon\frac{(1-\alpha)}{p}\|u\|_p^p+\varepsilon\left[\alpha-\frac{(M_1)^{1-k}}{k}C\right]\int_{\Omega}u^p\ln|u|\,dx \\ & +\varepsilon\left[p(1-\alpha)+d\left(\frac{p(1-\alpha)-2}{2}\right)\right]H(t), \end{aligned} \tag{4.19}$$

where we used $L^p(\Omega) \hookrightarrow L^2(\Omega), 2 < p$.

At this point, we choose $0 < \alpha < \frac{p-2}{2p}$ enough small that

$$d\left(\frac{p(1-\alpha)-2}{2}\right) > 0$$

and M_1 sufficiently large that

$$d\left(\frac{p(1-\alpha)-2}{2}\right)-\alpha-\frac{(M_1)^{1-k}}{k}C > 0 \ \text{ and } \ \alpha-\frac{(M_1)^{1-k}}{k}C > 0.$$

Once M_1 and α are fixed, we pick $0 < \varepsilon < \frac{1-\alpha}{M_1}$ so that

$$L(0) = H^{1-\alpha}(0)+\varepsilon\int_{\Omega}u_0u_1\,dx > 0. \tag{4.20}$$

Therefore, (4.19) has the form

$$L'(t) \geq \lambda\left[H(t)+\|u_t\|^2+\|\nabla u\|_p^p+\|u\|_p^p+\int_{\Omega}u^p\ln|u|\,dx\right], \tag{4.21}$$

where $\lambda > 0$ is the minimum of the coefficients of $H(t), \|u_t\|^2, \|\nabla u\|_p^p, \|u\|_p^p, \int_\Omega u^p \ln|u|\,dx$.

Consequently we obtain $L(t) > L(0)$, $t \geq 0$. On the other hand, by $(a+b)^p \leq 2^{p-1}(a^p + b^p)$, we have

$$\begin{aligned} L(t)^{\frac{1}{1-\alpha}} &= \left[H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t dx + \varepsilon \frac{1}{2}\|\nabla u\|^2\right]^{\frac{1}{1-\alpha}} \\ &\leq C\left[H(t) + \left(\int_\Omega |u u_t dx|\right)^{\frac{1}{1-\alpha}} + \|\nabla u\|^{\frac{2}{1-\alpha}}\right]. \end{aligned} \tag{4.22}$$

Now make use of the following inequality,

$$x^\nu \leq x + 1 \leq \left(1 + \frac{1}{\beta}\right)(x + \beta),\ \forall x \geq 0,\ 0 < \nu < 1,\ \beta \geq 0$$

for $x = \|\nabla u\|_p^p$, $\nu = \frac{2}{p(1-\alpha)} < 1$, since $\alpha < \frac{p-2}{2p}$, $d = 1 + \frac{1}{H(0)}$, $\beta = H(0)$, we get

$$\|\nabla u\|^{\frac{2}{1-\alpha}} \leq \left(\|\nabla u\|_p^p\right)^{\frac{2}{p(1-\alpha)}} \leq \left(1 + \frac{1}{H(0)}\right)\left(\|\nabla u\|_p^p + H(0)\right) \leq d\left(\|\nabla u\|_p^p + H(t)\right), \tag{4.23}$$

where we used $L^p(\Omega) \hookrightarrow L^2(\Omega)$.

Hölder's inequality gives us

$$\left|\int_\Omega u u_t dx\right| \leq \left(\int_\Omega |u_t|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}} \leq c\left(\int_\Omega |u_t|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}},$$

where c is the positive constant which comes from the embedding $L^p(\Omega) \hookrightarrow L^2(\Omega)$. This inequality implies that there exists a positive constant $C >$ such that

$$\left|\int_\Omega u u_t dx\right|^{1/(1-\alpha)} \leq \left(\int_\Omega |u_t|^2 dx\right)^{\frac{1}{2(1-\alpha)}} \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p(1-\alpha)}}. \tag{4.24}$$

Applying Young's inequality to the right-hand side of the (4.24), we get

$$\left|\int_\Omega u u_t dx\right|^{1/(1-\alpha)} \leq C\left[\left(\int_\Omega |u_t|^2 dx\right)^{\frac{\kappa}{2(1-\alpha)}} + \left(\int_\Omega |u|^p dx\right)^{\frac{\mu}{p(1-\alpha)}}\right], \tag{4.25}$$

for $\frac{1}{\mu} + \frac{1}{\kappa} = 1$.

To be able to use Lemma 8, we take $\kappa = 2/(1-\alpha)$, which gives $\mu = 2(1-\alpha)/(1-2\alpha)$, (4.25) has the form

$$\left|\int_\Omega u u_t dx\right|^{1/(1-\alpha)} \leq C\left[\int_\Omega |u_t|^2 dx + \left(\int_\Omega |u|^p dx\right)^{\frac{2}{p(1-2\alpha)}}\right].$$

By using of Poincare's inequality we get

$$\left|\int_\Omega u u_t dx\right|^{1/(1-\alpha)} \leq C\left[\int_\Omega |u_t|^2 dx + \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{2}{p(1-2\alpha)}}\right].$$

With the re-use of the inequality

$$x^v \leq x+1 \leq \left(1+\frac{1}{\beta}\right)(x+\beta),\ \forall x \geq 0,\ 0<v<1,\ \beta \geq 0,$$

where $x = \|\nabla u\|_p^p$, $d = 1+\frac{1}{H(0)}$, $\beta = H(0)$, $v = \frac{2}{p(1-2\alpha)} < 1$, since $\alpha < \frac{p-2}{2p}$, we obtain

$$\left(\int_\Omega |\nabla u|^p dx\right)^{\frac{2}{p(1-2\alpha)}} \leq \left(1+\frac{1}{H(0)}\right)\left(\|\nabla u\|_p^p + H(0)\right) \leq d\left[\|\nabla u\|_p^p + H(t)\right]. \tag{4.26}$$

Thus, (4.24) becomes

$$\left|\int_\Omega u u_t dx\right|^{1/(1-\alpha)} \leq C\left(H(t) + \|u_t\|^2 + \|\nabla u\|_p^p\right). \tag{4.27}$$

Inserting (4.23) and (4.27) into (4.22), it follows that

$$\begin{aligned} L(t)^{\frac{1}{1-\alpha}} &\leq C\left(H(t) + \|u_t\|^2 + \|\nabla u\|_p^p\right), \\ &\leq \left(H(t) + \|u_t\|^2 + \|\nabla u\|_p^p + \|u\|_p^p + \int_\Omega u^p \ln|u|\, dx\right). \end{aligned} \tag{4.28}$$

By associating (4.28) and (4.21) we arrive at

$$L'(t) \geq \xi L^{\frac{1}{1-\alpha}}(t), \tag{4.29}$$

where ξ is a positive constant which depends only on λ, $C > 0$.

Integration of (4.29) over $(0,t)$ we reach

$$\frac{dL}{dt} \geq \xi L^{\frac{1}{1-\alpha}}(t),$$

$$\int_0^t \frac{dL}{L^{\frac{1}{1-\alpha}}(t)} \geq \int_0^t \xi dt,$$

$$L^{-\frac{\alpha}{1-\alpha}}(t) - L^{-\frac{\alpha}{1-\alpha}}(0) \geq \xi t,$$

$$L^{-\frac{\alpha}{1-\alpha}}(t) \geq L^{-\frac{\alpha}{1-\alpha}}(0) + \xi t,$$

$$L^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{L^{-\frac{\alpha}{1-\alpha}}(0) - \frac{\xi \alpha t}{1-\alpha}}.$$

Therefore the solutions blow up within a time given by the estimate (4.5) above. Consequently we completed our proof. □

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