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NONEXISTENCE OF GLOBAL SOLUTIONS OF A DELAYED WAVE EQUATION WITH VARIABLE-EXPONENTS

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Abstract. This work deals with a Petrovsky equation with delay term and variable exponents. Firstly, we establish the local existence result by the Faedo-Galerkin method. Later, we prove the blow-up of solutions in a finite time. Our results are more general than the earlier results.

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1. Introduction

In this work, we study the following Petrovsky equation with variable exponents and delay term

$$\begin{cases} u_{tt}+\Delta^2 u-\Delta u_t+\mu_1 u_t(x,t)\left|u_t\right|^{m(x)-2}(x,t) & \\ +\mu_2 u_t(x,t-\tau)\left|u_t\right|^{m(x)-2}(x,t-\tau)=bu\left|u\right|^{p(x)-2} & \text{in } \Omega\times R^+, \\ u(x,t)=\frac{\partial u(x,t)}{\partial v}=0 & \text{in } \partial\Omega\times[0,\infty), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x) & \text{in } \Omega, \\ u_t(x,t-\tau)=f_0(x,t-\tau) & \text{in } \Omega\times(0,\tau), \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n, $n\geq 1$. Here, $\tau>0$ is a time delay term, $b\geq 0$ is a constant, μ_1 is a positive constant and μ_2 is a real number. The functions u_0, u_1, f_0 are the initial data to be specified later.

The variable exponents $p(\cdot)$ and $m(\cdot)$ are given as measurable functions on $\overline{\Omega}$ satisfy:

$$\begin{cases} 2\leq m^-\leq m(x)\leq m^+\leq m^* \\ 2\leq p^-\leq p(x)\leq p^+\leq p^* \end{cases} \tag{1.2}$$

where

$$m^-=ess\inf_{x\in\Omega} m(x),\qquad m^+=ess\sup_{x\in\Omega} m(x),$$
$$p^-=ess\inf_{x\in\Omega} p(x),\qquad p^+=ess\sup_{x\in\Omega} p(x),$$

and

$$m^*, p^* = \frac{2(n-2)}{n-4} \qquad \text{if } n > 4.$$

The problems with variable exponents arises in many branches in sciences such as electrorheological fluids, nonlinear elasticity theory and image processing [3, 4, 20–23]. Time delay often appears in many practical problems like thermal, biological, chemical, physical and economic phenomena [7].

There has been published much work concerning the wave equations with variable exponents or time delay. Our goal is to consider the Petrovsky equation both with the delay term ($\mu_2 u_t(x,t-\tau)$) and variable exponents which make the problem more interesting than from those concerned in the literature.

Li et al. [11] considered the Petrovsky equation with strong damping term as follows

$$u_{tt} + \Delta^2 u - \Delta u_t + |u_t|^{p-2} u_t = |u|^{q-2} u. \tag{1.3}$$

The authors established the blow up of solutions, existence and decay of the problem (1.3). Then, Polat and Pişkin [19] proved the global existence and decay of solutions of (1.3).

In [12], Messaoudi studied the Petrovsky equation as follows

$$u_{tt} + \Delta^2 u + g(u_t) = \beta |u|^{r-1} u, \tag{1.4}$$

where $g(u_t) = \alpha |u_t|^{p-1} u_t$ and he investigated the blow-up result in finite time for $r > p$. In [25], for when $g(u_t) = \alpha |u_t|^{p-1} u_t$, Tsai and Wu looked into that the solution is global for equation (1.4). Moreover, they established the blow-up result in finite time for the nonnegative initial energy.

Messaoudi and Kafini [6] looked into the nonlinear wave equation with variable exponents and delay term as follows

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) |u_t|^{m(x)-2}(x,t) + \mu_2 u_t(x,t-\tau) |u_t|^{m(x)-2}(x,t-\tau) = bu|u|^{p(x)-2}.$$

They proved the global nonexistence and decay estimates of the equation (1).

In recent years, some other authors investigate hyperbolic type equation with variable exponents (see [8, 13, 16, 18, 24]).

There is no research, to our best knowledge, about Petrovsky equation with delay term and variable exponents, hence, our paper is generalization of the previous ones. In this paper, our main goal is to study the local existence and blow-up result of Petrovsky equation (1.1) with variable exponents and delay term.

The plan of this paper is as follows. Firstly, in Section 2, the definition of the variable exponent Sobolev and Lebesgue spaces are introduced. In Section 3, we obtain the local existence result. Finally, in Section 4, we prove the blow-up result for negative initial energy.

2. Preliminaries

In this part, we state the results related to Lebesgue $L^{p(\cdot)}(\Omega)$ and Sobolev $W^{1,p(\cdot)}(\Omega)$ spaces with variable exponents (see [1, 2, 4, 5, 10, 17]).

Let $p\colon \Omega \to [1,\infty)$ be a measurable function. We define the variable exponent Lebesgue space with a variable exponent $p(\cdot)$ by

$$L^{p(\cdot)}(\Omega) = \left\{ u\colon \Omega \to R;\ \text{measurable in } \Omega\ :\ \int_{\Omega} |u|^{p(\cdot)}\, dx < \infty \right\},$$

with a Luxemburg-type norm

$$\|u\|_{p(\cdot)} = \inf\left\{ \lambda > 0 : \int_{\Omega} \left|\frac{u}{\lambda}\right|^{p(x)} dx \leq 1 \right\}.$$

Equipped with this norm, $L^{p(\cdot)}(\Omega)$ is a Banach space (see [4]).

Next, we define the variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ as following:

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) :\ \nabla u \text{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}.$$

Variable exponent Sobolev space with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$$

is a Banach space. $W_0^{1,p(\cdot)}(\Omega)$ is the space which is defined as the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. For $u \in W_0^{1,p(\cdot)}(\Omega)$, we can define an equivalent norm

$$\|u\|_{1,p(\cdot)} = \|\nabla u\|_{p(\cdot)}.$$

The dual of $W_0^{1,p(\cdot)}(\Omega)$ is defined as $W_0^{-1,p'(\cdot)}(\Omega)$, similar to Sobolev spaces, where

$$\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1.$$

We also assume that:

$$|p(x) - p(y)| \leq -\frac{A}{\log|x-y|} \quad \text{and} \quad |m(x) - m(y)| \leq -\frac{B}{\log|x-y|} \tag{2.1}$$

for all $x, y \in \Omega$, $A, B > 0$ and $0 < \delta < 1$ with $|x-y| < \delta$ (log-Hölder condition).

Lemma 1 ([1] Poincare inequality). *Let Ω be a bounded domain of R^n and suppose that $p(\cdot)$ satisfies (2.1). Then,*

$$\|u\|_{p(\cdot)} \leq c\,\|\nabla u\|_{p(\cdot)} \qquad \text{for all } u \in W_0^{1,p(\cdot)}(\Omega),$$

where $c = c(p^-, p^+, |\Omega|) > 0$.

Lemma 2 ([1]). *If $p\colon \overline{\Omega} \to [1,\infty)$ is continuous,*

$$2 \leq p^- \leq p(x) \leq p^+ \leq \frac{2n}{n-2}, \qquad n \geq 3,$$

satisfies, then the embedding $H_0^1(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ *is continuous.*

Lemma 3 ([1]). *If* $p^+ < \infty$ *and* $p\colon \Omega \to [1,\infty)$ *is a measurable function, then* $C_0^\infty(\Omega)$ *is dense in* $L^{p(\cdot)}(\Omega)$.

Lemma 4 ([1] Hölder' inequality). *Let* $p, q, s \geq 1$ *be measurable functions defined on* Ω *and*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \qquad \text{for a.e. } y \in \Omega,$$

satisfies. If $f \in L^{p(\cdot)}(\Omega)$ *and* $g \in L^{q(\cdot)}(\Omega)$*, then* $fg \in L^{s(\cdot)}(\Omega)$ *and*

$$\|fg\|_{s(\cdot)} \leq 2\,\|f\|_{p(\cdot)}\,\|g\|_{q(\cdot)}\,.$$

Lemma 5 ([6, Lemma 2.5] unit ball property). *Let* $p \geq 1$ *be a measurable function on* Ω. *Then* $\|f\|_{p(\cdot)} \leq 1$ *if and only if* $\rho_{p(\cdot)}(f) \leq 1$*, where*

$$\rho_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}\,dx.$$

Lemma 6 ([1]). *If* $p \geq 1$ *is a measurable function on* Ω*, then*

$$\min\left\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right\} \leq \rho_{p(\cdot)}(u) \leq \max\left\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right\},$$

for any $u \in L^{p(\cdot)}(\Omega)$ *and for a.e.* $x \in \Omega$.

Remark 1. Let c be various positive constants which may be different from line to line. Then, we use the embedding

$$H_0^2(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^p(\Omega)$$

which satisfies

$$\|u\|_p \leq c\,\|\nabla u\| \leq c\,\|\Delta u\|\,,$$

where $2 \leq p < \infty$ $(n = 1,2)$, $2 \leq p \leq \frac{2n}{n-2}$ $(n \geq 3)$.

Moreover

$$\|u\|_q \leq C\,\|\Delta u\|\,,$$

$$q = \begin{cases} \infty & \text{if } n < 4, \\ \text{any number in } [1,\infty) & \text{if } n = 4, \\ \frac{2(n-2)}{n-4} & \text{if } n > 4. \end{cases}$$

3. LOCAL EXISTENCE

In this part, our goal is to prove the local existence result for our main problem (1.1) by using Faedo-Galerkin method. We use similar arguments as in [14, 16] to get the result. Firstly, we give the lemma which we need:

Lemma 7 ([9, Lemma 3.1]). *(Lemma 3.1 in [9]) Let $x \in \Omega$ and $p(\cdot)$ satisfies*

$$2 \leq p^- \leq p(x) \leq p^+ \leq \infty,$$

then, $h(s) = b|s|^{p(x)-2} s$ is differentiable function and $|h'(s)| = b|p(x)-1||s|^{p(x)-2}$.

Suppose that μ_1 and μ_2 satisfy

$$|\mu_2| < \frac{m^-}{m^+}\mu_1, \tag{3.1}$$

where $m^- = es\sin f_{x\in\Omega} m(x)$, $m^+ = es\sup f_{x\in\Omega} m(x)$. Assume that ζ is a positive constant such that

$$\tau\left(m^+ - 1\right)\mu_2 < \zeta < \tau\left(m^-\mu_1 - |\mu_2|\right). \tag{3.2}$$

Now, similar to [15], we introduce, the new variable

$$z(x,\rho,t) = u_t(x,t-\tau\rho), \qquad x \in \Omega, \rho \in (0,1), t > 0.$$

Hence, problem (1.1) takes the form

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta u_t + \mu_1 |u_t(x,t)|^{m(x)-2} u_t(x,t) & \\ \quad + \mu_2 |z(x,1,t)|^{m(x)-2} z(x,1,t) = bu|u|^{p(x)-2} & \text{in } \Omega \times R^+, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 & \text{in } \Omega \times (0,1) \times (0,\infty), \\ u(x,t) = \frac{\partial u(x,t)}{\partial v} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in } \Omega, \\ z(x,\rho,0) = f_0(x,-\tau\rho) & \text{in } \Omega \times (0,1), \\ z(x,0,t) = u_t(x,t) & \text{in } \Omega \times (0,\infty). \end{cases} \tag{3.3}$$

Theorem 1. *Assume that (3.1) holds and $m(\cdot)$ satisfies (1.2), (2.1) and $p(\cdot)$ satisfies (2.1) and*

$$2 \leq p^- \leq p(x) \leq p^+ \leq \frac{2(n-2)}{n-4} \qquad \text{if } n > 4. \tag{3.4}$$

Assume further that $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$, $f_0 \in L^{m(\cdot)}(\Omega \times (0,1))$ and $T > 0$. Then, the problem (3.3) has a unique local solution

$$\begin{aligned} & u \in C\left([0,T]; H_0^2(\Omega)\right), \\ & u_t \in C\left([0,T]; L^2(\Omega)\right) \cap L^{m(\cdot)}(\Omega \times (0,T)), \\ & z \in L^{m(\cdot)}(\Omega \times (0,1)). \end{aligned}$$

Proof. **Existence:** Let $v \in L^{\infty}\left((0,T);H_0^2(\Omega)\right)$. Since

$$2\left(p^{-}-1\right) \leq 2\left(p^{+}-1\right) \leq \frac{2n}{n-4},$$

then

$$\|h(v)\|^2 \leq |b|^2 \left\{ \int_{\Omega} |v|^{2(p^{-}-1)} dx + \int_{\Omega} |v|^{2(p^{+}-1)} dx \right\} < \infty.$$

Hence, we have

$$h(v) \in L^{\infty}\left((0,T);L^2(\Omega)\right) \subset L^2(\Omega \times (0,T)).$$

Thus, for each $v \in L^{\infty}\left((0,T);H_0^2(\Omega)\right)$, there exists a unique solution

$$\begin{aligned} u &\in L^{\infty}\left((0,T);H_0^2(\Omega)\right), \\ u_t &\in L^{\infty}\left((0,T);L^2(\Omega)\right) \cap L^{m(\cdot)}(\Omega \times (0,T)), \\ z &\in L^{m(\cdot)}(\Omega \times (0,1)) \end{aligned}$$

satisfying the following problem

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta u_t + \mu_1 |u_t(x,t)|^{m(x)-2} u_t(x,t) & \\ \quad + \mu_2 |z(x,1,t)|^{m(x)-2} z(x,1,t) = h(v) & \text{in } \Omega \times (0,T), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 & \text{in } \Omega \times (0,1) \times (0,T), \\ u(x,t) = \frac{\partial u(x,t)}{\partial v} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in } \Omega, \\ z(x,0,t) = u_t(x,t) & \text{in } \Omega \times (0,T), \\ z(x,\rho,0) = f_0(x,-\tau\rho) & \text{in } \Omega \times (0,1). \end{cases} \tag{3.5}$$

Define the following space that the sequence (u^k) is Cauchy in

$$X := C\left([0,T];H_0^2(\Omega)\right) \cap C^1\left([0,T];L^2(\Omega)\right),$$

equipped with the norm

$$\|u\|_X^2 = \max_{0 \leq t \leq T} \left\{ \|u_t\|^2 + \|\Delta u\|^2 \right\}.$$

We define the nonlinear mapping $K\colon X \to X$ by $K(v) = u$, here, u is the unique solution of (3.5). Now, we shall show that there exist $T > 0$, such that

(i) $K\colon X \to X$,
(ii) K is a contraction mapping in X.

To show (i), we multiply the first equation in (3.5) by u_t and integrate over $\Omega\times(0,t)$, to obtain

$$\begin{aligned}&\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\Delta u\|^2+\int_0^t\|\nabla u_t\|^2ds+\mu_1\int_0^t\int_\Omega|u_t(s)|^{m(x)}dxds\\&+\mu_2\int_0^t\int_\Omega|z(x,1,s)|^{m(x)-2}z(x,1,s)u_t(s)dxds\\&=\frac{1}{2}\int_\Omega u_1^2dx+\frac{1}{2}\int_\Omega|\Delta u_0|^2dx+b\int_0^t\int_\Omega|v|^{p(x)-2}vu_t(s)dxds.\end{aligned}\tag{3.6}$$

We multiply the second equation in (3.5) by $\frac{\zeta}{\tau}z^{m(x)-1}$, and integrate over $\Omega\times(0,1)\times(0,t)$, to have

$$\begin{aligned}&\int_0^1\int_\Omega\frac{\zeta}{m(x)}\left(|z(x,\rho,t)|^{m(x)}-|z(x,\rho,0)|^{m(x)}\right)dxd\rho\\&\qquad=\int_0^t\int_\Omega\frac{\zeta}{m(x)\tau}\left(|z(x,0,s)|^{m(x)}-|z(x,1,s)|^{m(x)}\right)dxds.\end{aligned}\tag{3.7}$$

Combining (3.6) and (3.7), we have

$$\begin{aligned}&\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\Delta u\|^2+\int_0^t\|\nabla u_t\|^2ds+\int_0^1\int_\Omega\frac{\zeta}{m(x)}|z(x,\rho,t)|^{m(x)}dxd\rho\\&+\mu_1\int_0^t\int_\Omega|u_t(s)|^{m(x)}dxds+\mu_2\int_0^t\int_\Omega|z(x,1,s)|^{m(x)-2}z(x,1,s)u_t(s)dxds\\&+\int_0^t\int_\Omega\frac{\zeta}{m(x)\tau}\left(|z(x,1,s)|^{m(x)}-|u_t(s)|^{m(x)}\right)dxds\\&=\frac{1}{2}\int_\Omega u_1^2dx+\frac{1}{2}\int_\Omega|\Delta u_0|^2dx+\int_0^1\int_\Omega\frac{\zeta}{m(x)}|f_0(x,-\tau\rho)|^{m(x)}dxd\rho\\&+b\int_0^t\int_\Omega|v|^{p(x)-2}vu_t(s)dxds\end{aligned}\tag{3.8}$$

Utilizing Young's inequality and (1.2), we get

$$\begin{aligned}&-\mu_2\int_\Omega|z(x,1,s)|^{m(x)-2}z(x,1,s)u_t(s)dxds\\&\qquad\le\frac{|\mu_2|}{m^-}\int_\Omega|u_t(s)|^{m(x)}dx+\frac{(m^+-1)|\mu_2|}{m^+}\int_\Omega|z(x,1,s)|^{m(x)}dx.\end{aligned}\tag{3.9}$$

Applying Young's inequality and Sobolev embedding $H_0^2(\Omega)\hookrightarrow L^{\frac{2n}{n-4}}(\Omega)$, we obtain

$$\begin{aligned}\left|\int_\Omega|v|^{p(x)-2}vu_t(s)dx\right|&\le\frac{\varepsilon}{4}\int_\Omega|u_t(s)|^2dx+\frac{1}{\varepsilon}\int_\Omega|v|^{2(p(x)-1)}dx\\&\le\frac{\varepsilon}{4}\int_\Omega|u_t(s)|^2dx+\frac{c_e}{\varepsilon}\left\{\|\Delta v\|^{2(p^--1)}+\|\Delta v\|^{2(p^+-1)}\right\},\end{aligned}\tag{3.10}$$

here, c_e is the embedding constant. We should inserting (3.9) and (3.10) into (3.8), then, we have

$$\begin{aligned}
&\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\Delta u\|^2+\int_0^t\|\nabla u_t\|^2 ds+\int_0^1\int_\Omega\frac{\zeta}{m(x)}|z(x,\rho,t)|^{m(x)}dxd\rho\\
&+\left(\mu_1-\frac{|\mu_2|}{m^-}-\frac{\zeta}{m^-\tau}\right)\int_0^t\int_\Omega|u_t(s)|^{m(x)}dxds\\
&+\left(\frac{\zeta}{m^+\tau}-\frac{(m^+-1)|\mu_2|}{m^+}\right)\int_0^t\int_\Omega|z(x,1,s)|^{m(x)}dxds\\
\leq&\frac{1}{2}\int_\Omega u_1^2dx+\frac{1}{2}\int_\Omega|\Delta u_0|^2dx+\int_0^1\int_\Omega\frac{\zeta}{m(x)}|f_0(x,-\tau\rho)|^{m(x)}dxd\rho\\
&+\frac{\varepsilon cT}{4}\sup_{(0,T)}\int_\Omega|u_t|^2dx+\frac{c_ec}{\varepsilon}\left\{\int_0^T\|\Delta v\|^{2(p^--1)}ds+\int_0^T\|\Delta v\|^{2(p^+-1)}ds\right\}.
\end{aligned}$$

By (3.2), we have

$$\begin{aligned}
&\frac{1}{2}\sup_{(0,T)}\|u_t\|^2+\frac{1}{2}\sup_{(0,T)}\|\Delta u\|^2+\frac{\zeta}{m^+}\|z(x,\rho,t)\|^{m(\cdot)}_{L^{m(\cdot)}(\Omega\times(0,1))}\\
&\qquad\leq\frac{1}{2}\int_\Omega u_1^2dx+\frac{1}{2}\int_\Omega|\Delta u_0|^2dx+\frac{\zeta}{m^-}\int_0^1\int_\Omega|f_0(x,-\tau\rho)|^{m(x)}dxd\rho\\
&\qquad\qquad+\frac{\varepsilon cT}{4}\sup_{(0,T)}\|u_t\|^2+\frac{c_ecT}{\varepsilon}\left\{\|v\|_X^{2(p^--1)}+\|v\|_X^{2(p^+-1)}\right\}.
\end{aligned}$$

By taking ε such that $\varepsilon cT=1$, we get

$$\begin{aligned}
\|u\|_X^2\leq&\frac{c^*}{2}\int_\Omega u_1^2dx+\frac{c^*}{2}\int_\Omega|\Delta u_0|^2dx+\frac{c^*\zeta}{m^-}\int_0^1\int_\Omega|f_0(x,-\tau\rho)|^{m(x)}dxd\rho\\
&+c_*T\left\{\|v\|_X^{2(p^--1)}+\|v\|_X^{2(p^+-1)}\right\},
\end{aligned}$$

where $\frac{1}{c^*}=\min\left\{\frac{1}{4},\frac{\zeta}{m^+}\right\}$ and $c_*=\frac{c^*c_ec}{\varepsilon}$. Here, we choose $M>0$ large enough, such that $\|v\|_X\leq M$, then

$$c^*\int_\Omega u_1^2dx+c^*\int_\Omega|\Delta u_0|^2dx+\frac{2c^*\zeta}{m^-}\int_0^1\int_\Omega|f_0(x,-\tau\rho)|^{m(x)}dxd\rho\leq M^2$$

and T sufficiently small such that

$$T\leq\frac{1}{2c_*\left(M^{2(p^--2)}+M^{2(p^+-2)}\right)}.$$

As a result, we have

$$\|u\|_X^2\leq M^2.$$

Thus we have $K\colon Z \to Z$, where

$$Z = \{u \in X \text{ such that } \|u\|_X \leq M\}.$$

Next, we show that K is a contraction mapping. For this purpose, we let $K(v^1) = u^1$ and $K(v^2) = u^2$ and set $u = u^1 - u^2$ and $w = w^1 - w^2$ then u and w satisfy

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta u_t + \mu_1 \left|u_t^1(x,t)\right|^{m(x)-2} u_t^1(x,t) & \\ -\mu_1 \left|u_t^2(x,t)\right|^{m(x)-2} u_t^2(x,t) & \\ +\mu_2 \left|z^1(x,1,t)\right|^{m(x)-2} z^1(x,1,t) & \text{in } \Omega \times (0,T), \\ -\mu_2 \left|z^2(x,1,t)\right|^{m(x)-2} z^2(x,1,t) & \\ = b\left|v^1\right|^{p(x)-2} v^1 - b\left|v^2\right|^{p(x)-2} v^2 & \\ u(x,t) = \frac{\partial u(x,t)}{\partial v} = 0 & \text{on } \partial\Omega \times (0,T), \\ z(x,0,t) = u_t(x,t) & \text{in } \Omega \times (0,T), \\ z(x,\rho,0) = 0 & \text{in } \Omega \times (0,1), \\ u(x,0) = 0,\ u_t(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{3.11}$$

We multiply equation (3.11) by u_t and integrate over $\Omega \times (0,t)$, we get

$$\begin{aligned} &\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \int_0^t \|\nabla u_t\|^2 ds \\ &\qquad + \mu_1 \int_0^t \int_\Omega \left(\left|u_t^1(s)\right|^{m(x)-2} u_t^1(s) - \left|u_t^2(s)\right|^{m(x)-2} u_t^2(s) \right) u_t(s)\, dxds \\ &+ \mu_2 \int_0^t \int_\Omega \left(\left|z^1(x,1,s)\right|^{m(x)-2} z^1(x,1,s) - \left|z^2(x,1,s)\right|^{m(x)-2} z^2(x,1,s) \right) u_t(s)\, dxds \\ &\qquad\qquad = \int_0^t \int_\Omega (h(v_1) - h(v_2))\, u_t(s)\, dxds, \end{aligned}$$

where $h(v) = b|v|^{p(x)-2} v$. Since the function $u \to |u|^{m(x)-2} u$ is increasing, we conclude that

$$\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \int_0^t \|\nabla u_t\|^2 ds \leq \int_0^t \int_\Omega (h(v_1) - h(v_2))\, u_t(s)\, dxds. \tag{3.12}$$

Thanks to (3.4), Young's inequality and Sobolev embedding, we obtain

$$\begin{aligned} &\int_\Omega |h(v_1) - h(v_2)|\, |u_t(s)|\, dx = \int_\Omega \left|h'(\rho)\right| \|v\|\, |u_t(s)|\, dx \\ &\leq \frac{\delta_0}{2} \int_\Omega |u_t(s)|^2 dx + \frac{1}{2\delta_0} \int_\Omega \left|h'(\rho)\right|^2 |v|^2 dx \leq \frac{\delta_0}{2} \int_\Omega |u_t(s)|^2 dx \\ &+ \frac{b^2 (p^+ - 1)^2}{2\delta_0} \left[\left(\int_\Omega |\rho|^{\frac{n(p^- - 2)}{2}} dx \right)^{\frac{4}{n}} + \left(\int_\Omega |\rho|^{\frac{n(p^+ - 2)}{2}} dx \right)^{\frac{4}{n}} \right] \left(\int_\Omega |v|^{\frac{2n}{n-4}} dx \right)^{\frac{n-4}{n}} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\delta_0}{2}\|u_t(s)\|^2 + \frac{b^2(p^+-1)^2 c_e}{2\delta_0}\left[\|\Delta\rho\|^{2(p^--2)} + \|\Delta\rho\|^{2(p^+-2)}\right]\|\Delta v\|^2 \\
&\leq \frac{\delta_0}{2}\|u_t(s)\|^2 + \frac{b^2(p^+-1)^2 c_e}{\delta_0}\left(M^{2(p^--2)} + M^{2(p^+-2)}\right)\|\Delta v\|^2,
\end{aligned} \tag{3.13}$$

where $v = v_1 - v_2$ and $\rho = \vartheta v_1 + (1-\vartheta)v_2$, $0 \leq \vartheta \leq 1$.

By inserting (3.13) into (3.12) and choosing δ_0 small enough, we obtain

$$\|u\|_X^2 \leq d\,\|v\|_X^2, \tag{3.14}$$

where $d = \frac{4b^2(p^+-1)^2 c_e T}{\delta_0}\left(M^{2(p^--2)} + M^{2(p^+-2)}\right)$.

Now, we choose T small enough so that $0 < d < 1$. Thus, (3.14) indicates that K is a contraction. The Banach fixed theorem shows that the existence of a unique $u \in Z$ satisfying $K(u) = u$. Obviously, it is a solution of (3.3).

Uniqueness: Assume that (3.3) have two solutions (u^1, z^1), (u^2, z^2). We define $\widetilde{u} = u^1 - u^2$ and $\widetilde{z} = z^1 - z^2$, then $(\widetilde{u}, \widetilde{z})$ satisfy

$$\begin{cases}
\widetilde{u}_{tt} + \Delta^2\widetilde{u} - \Delta\widetilde{u}_t + \mu_1\left|u_t^1(t)\right|^{m(x)-2}u_t^1(t) & \\
-\mu_1\left|u_t^2(t)\right|^{m(x)-2}u_t^2(x,t) & \\
+\mu_2\left|z^1(x,1,t)\right|^{m(x)-2}z^1(x,1,t) & \text{in } \Omega\times(0,T), \\
-\mu_2\left|z^2(x,1,t)\right|^{m(x)-2}z^2(x,1,t) & \\
= bu^1\left|u^1\right|^{p(x)-2} - bu^2\left|u^2\right|^{p(x)-2} & \\
\tau\widetilde{z}_t(x,\rho,t) + \widetilde{z}_\rho(x,\rho,t) \ = 0 & \text{in } \Omega\times(0,1)\times(0,T), \\
\widetilde{u}(x,t) = \frac{\partial\widetilde{u}(x,t)}{\partial\nu} = 0 & \text{in } \partial\Omega\times(0,T), \\
\widetilde{z}(x,0,t) = \widetilde{u}_t(x,t) & \text{in } \Omega\times(0,T), \\
\widetilde{z}(x,\rho,0) = 0 & \text{in } \Omega\times(0,1), \\
\widetilde{u}(x,0) = 0\ \widetilde{u}_t(x,0) = 0 & \text{in } \Omega.
\end{cases} \tag{3.15}$$

We multiply the first equation in (3.15) by $\widetilde{u}_t$ and integrate over Ω, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega\left|\widetilde{u}_t\right|^2dx + \int_\Omega\left|\Delta\widetilde{u}\right|^2dx\right\} + \int_\Omega\left|\nabla\widetilde{u}_t\right|^2dx \\
&+\mu_1\int_\Omega\left(\left|u_t^1(t)\right|^{m(x)-2}u_t^1(t) - \left|u_t^2(t)\right|^{m(x)-2}u_t^2(x,t)\right)\widetilde{u}_t(t)\,dx \\
&+\mu_2\int_\Omega\left(\left|z^1(x,1,t)\right|^{m(x)-2}z^1(x,1,t) - \left|z^2(x,1,t)\right|^{m(x)-2}z^2(x,1,t)\right)\widetilde{u}_t(t)\,dx \\
&= b\int_\Omega\left(u^1\left|u^1\right|^{p(x)-2} - bu^2\left|u^2\right|^{p(x)-2}\right)\widetilde{u}_t(t)\,dx.
\end{aligned} \tag{3.16}$$

Multiplying the second equation in (3.15) by $\widetilde{z}$ and integrating over $\Omega\times(0,1)$, we have

$$\frac{\tau}{2}\frac{d}{dt}\int_0^1\int_\Omega\left|\widetilde{z}(x,\rho,t)\right|^2dxd\rho+\frac{1}{2}\left(\left\|\widetilde{z}(x,1,t)\right\|^2-\left\|\widetilde{u}_t(t)\right\|^2\right)=0. \tag{3.17}$$

Combining (3.16) and (3.17), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega\left|\widetilde{u}_t(t)\right|^2dx+\int_\Omega\left|\Delta\widetilde{u}(t)\right|^2dx+\tau\left\|\widetilde{z}(x,\rho,t)\right\|^2_{L^2(\Omega\times(0,1))}\right\}+\int_\Omega\left|\nabla\widetilde{u}_t\right|^2dx\\&\quad+\frac{1}{2}\left\|\widetilde{z}(x,1,t)\right\|^2+\mu_1\int_\Omega\left(\left|u_t^1(t)\right|^{m(x)-2}u_t^1(t)-\left|u_t^2(t)\right|^{m(x)-2}u_t^2(t)\right)\widetilde{u}_t(t)\,dx\\&\quad+\mu_2\int_\Omega\left(\left|z^1(x,1,t)\right|^{m(x)-2}z^1(x,1,t)-\left|z^2(x,1,t)\right|^{m(x)-2}z^2(x,1,t)\right)\widetilde{u}_t(t)\,dx\\&\qquad=b\int_\Omega\left(u^1\left|u^1\right|^{p(x)-2}-bu^2\left|u^2\right|^{p(x)-2}\right)\widetilde{u}_t(t)\,dx+\frac{1}{2}\left\|\widetilde{u}_t(t)\right\|^2.\end{aligned} \tag{3.18}$$

Since the function $y\to|y|^{m(\cdot)-2}y$ is increasing, we get

$$\int_\Omega\left(\left|u_t^1(t)\right|^{m(x)-2}u_t^1(t)-\left|u_t^2(t)\right|^{m(x)-2}u_t^2(t)\right)\widetilde{u}_t(t)\,dx\geq0, \tag{3.19}$$

$$\int_\Omega\left(\left|z^1(x,1,t)\right|^{m(x)-2}z^1(x,1,t)-\left|z^2(x,1,t)\right|^{m(x)-2}z^2(x,1,t)\right)\widetilde{u}_t(t)\,dx\geq0. \tag{3.20}$$

By using (3.18), (3.19) and (3.20), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega\left\|\widetilde{u}_t(t)\right\|^2+\left\|\Delta\widetilde{u}(t)\right\|^2+\tau\left\|\widetilde{z}(x,\rho,t)\right\|^2_{L^2(\Omega\times(0,1))}\right\}+\frac{1}{2}\left\|\widetilde{z}(x,1,t)\right\|^2\\\leq c\left(\left\|\widetilde{u}_t(t)\right\|^2+\left\|\Delta\widetilde{u}(t)\right\|^2\right)\end{aligned}$$

which implies that $\widetilde{u}=0$, $\widetilde{z}=0$. □

4. Blow up

In this part, for the case $b>0$, we establish the blow-up result for our main problem (1.1). Now we introduce, as in the work of [15], the new function

$$z(x,\rho,t)=u_t(x,t-\tau\rho),\qquad x\in\Omega,\qquad\rho\in(0,1),\qquad t>0,$$

which implies that

$$\tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0,\qquad x\in\Omega,\qquad\rho\in(0,1),\qquad t>0.$$

Consequently, problem (1.1) is equivalent to:

$$\begin{cases} u_{tt}+\Delta^2 u-\Delta u_t+\mu_1 u_t(x,t)\,|u_t(x,t)|^{m(x)-2} & \\ +\mu_2 z(x,1,t)\,|z(x,1,t)|^{m(x)-2} & \text{in } \Omega\times(0,\infty), \\ =bu\,|u|^{p(x)-2} & \\ \tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0 & \text{in } \Omega\times(0,1)\times(0,\infty), \\ z(x,\rho,0)=f_0(x,-\rho\tau) & \text{in } \Omega\times(0,1), \\ u(x,t)=\frac{\partial u(x,t)}{\partial v}=0 & \text{on } \partial\Omega\times[0,\infty), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x) & \text{in } \Omega. \end{cases} \tag{4.1}$$

We define the energy functional of (4.1) as

$$E(t)=\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\Delta u\|^2+\int_0^1\int_\Omega \frac{\xi(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho-b\int_\Omega \frac{|u|^{p(x)}}{p(x)}dx,$$

for $t\geq 0$, where ξ is a continuous function satisfies

$$\tau|\mu_2|(m(x)-1)<\xi(x)<\tau(\mu_1 m(x)-|\mu_2|),\qquad x\in\overline{\Omega}. \tag{4.2}$$

The following lemma gives that, under the condition $\mu_1>|\mu_2|$, $E(t)$ is nonincreasing.

Lemma 8. *Let (u,z) be a solution of (4.1). Then there exists some $C_0>0$ such that*

$$E'(t)\leq -C_0\int_\Omega\left(|u_t|^{m(x)}+|z(x,1,t)|^{m(x)}\right)dx\leq 0.$$

Proof. We multiply the first equation in (4.1) by u_t, integrate over Ω, then multiplying the second equation of (4.1) by $\frac{1}{\tau}\xi(x)\,|z|^{m(x)-2}z$ and integrate over $\Omega\times(0,1)$, summing up, we get

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\Delta u\|^2+\int_0^1\int_\Omega \frac{\xi(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho-b\int_\Omega\frac{|u|^{p(x)}}{p(x)}dx\right] \\ &=-\|\nabla u_t\|^2-\mu_1\int_\Omega|u_t|^{m(x)}dx-\frac{1}{\tau}\int_\Omega\int_0^1\xi(x)\,|z(x,\rho,t)|^{m(x)-2}zz_\rho(x,\rho,t)\,d\rho dx \\ &\qquad -\mu_2\int_\Omega u_t z(x,1,t)\,|z(x,1,t)|^{m(x)-2}dx. \end{aligned} \tag{4.3}$$

Next, we estimate the last two terms of the right-hand side of (4.3) as following,

$$\begin{aligned} &-\frac{1}{\tau}\int_\Omega\int_0^1\xi(x)\,|z(x,\rho,t)|^{m(x)-2}zz_\rho(x,\rho,t)\,d\rho dx \\ &\qquad=-\frac{1}{\tau}\int_\Omega\int_0^1\frac{\partial}{\partial\rho}\left(\frac{\xi(x)\,|z(x,\rho,t)|^{m(x)}}{m(x)}\right)d\rho dx \\ &\qquad=\frac{1}{\tau}\int_\Omega\frac{\xi(x)}{m(x)}\left(|z(x,0,t)|^{m(x)}-|z(x,1,t)|^{m(x)}\right)dx \end{aligned}$$

$$= \int_{\Omega} \frac{\xi(x)}{\tau m(x)} |u_t|^{m(x)} dx - \int_{\Omega} \frac{\xi(x)}{\tau m(x)} |z(x,1,t)|^{m(x)}.$$

Using the Young's inequality, $q = \frac{m(x)}{m(x)-1}$ and $q' = m(x)$ for the last term to obtain

$$|u_t| |z(x,1,t)|^{m(x)-1} \leq \frac{1}{m(x)} |u_t|^{m(x)} + \frac{m(x)-1}{m(x)} |z(x,1,t)|^{m(x)}.$$

Consequently, we deduce that

$$\begin{aligned} &-\mu_2 \int_{\Omega} u_t z |z(x,1,t)|^{m(x)-2} dx \\ &\qquad \leq |\mu_2| \left(\int_{\Omega} \frac{1}{m(x)} |u_t(t)|^{m(x)} dx + \int_{\Omega} \frac{m(x)-1}{m(x)} |z(x,1,t)|^{m(x)} dx \right). \end{aligned}$$

So

$$\begin{aligned} \frac{dE(t)}{dt} \leq &- \int_{\Omega} \left[\mu_1 - \left(\frac{\xi(x)}{\tau m(x)} + \frac{|\mu_2|}{m(x)} \right) \right] |u_t(t)|^{m(x)} dx \\ &- \int_{\Omega} \left(\frac{\xi(x)}{\tau m(x)} - \frac{|\mu_2|(m(x)-1)}{m(x)} \right) |z(x,1,t)|^{m(x)} dx. \end{aligned}$$

As a result, for all $x \in \overline{\Omega}$, the relation (4.2) satisfies,

$$f_1(x) = \mu_1 - \left(\frac{\xi(x)}{\tau m(x)} + \frac{|\mu_2|}{m(x)} \right) > 0,$$

$$f_2(x) = \frac{\xi(x)}{\tau m(x)} - \frac{|\mu_2|(m(x)-1)}{m(x)} > 0.$$

Since $m(x)$, and hence $\xi(x)$, is bounded, we infer that $f_1(x)$ and $f_2(x)$ are also bounded. So, if we define

$$C_0(x) = \min\{f_1(x),\ f_2(x)\} > 0 \text{ for any } x \in \overline{\Omega},$$

and take $C_0(x) = \inf_{\overline{\Omega}} C_0(x)$, so $C_0(x) \geq C_0 > 0$. Hence,

$$E'(t) \leq -C_0 \left[\int_{\Omega} |u_t(t)|^{m(x)} dx + \int_{\Omega} |z(x,1,t)|^{m(x)} dx \right] \leq 0.$$

□

To prove the blow-up result, we assume that $E(0) < 0$ in addition to (1.2). Set $H(t) = -E(t)$, hence $H'(t) = -E'(t) \geq 0$,

$$0 < H(0) \leq H(t) \leq b \int_{\Omega} \frac{|u|^{p(x)}}{p(x)} dx \leq \frac{b}{p^-} \rho(u),$$

where

$$\rho(u) = \rho_{p(\cdot)}(u) = \int_{\Omega} |u|^{p(x)} dx.$$

Lemma 9 ([6, Lemma 3.2]). *Suppose that condition (1.2) satisfies.Then, depending on* Ω *only, there exists a positive* $C>1$*, such that*

$$\rho^{s/p^-}(u) \le C\left(\|\Delta u\|^2 + \rho(u)\right).$$

Then, we have following inequalities:

$$\|u\|_{p^-}^s \le C\left(\|\Delta u\|^2 + \|u(t)\|_{p^-}^{p^-}\right),$$

$$\rho^{s/p^-}(u) \le C\left(|H(t)| + \|u_t\|^2 + \rho(u) + \int_0^1 \int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho\right),$$

$$\|u\|_{p^-}^s \le C\left(|H(t)| + \|u_t\|^2 + \|u\|_{p^-}^{p^-} + \int_0^1 \int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho\right), \quad (4.4)$$

for any $u \in H_0^1(\Omega)$ *and* $2 \le s \le p^-$. *Let* (u,z) *be a solution of (4.1), then*

$$\rho(u) \ge C\|u\|_{p^-}^{p^-}, \qquad \int_\Omega |u|^{m(x)} dx \le C\left(\rho^{m^-/p^-}(u) + \rho^{m^+/p^-}(u)\right). \quad (4.5)$$

The blow-up result is given by the following theorem:

Theorem 2. *Let conditions (1.2) and (2.1) be provided and assume that* $E(0)<0$. *Then, the solution (4.1) blows up in finite time* T^**, and*

$$T^* \le \frac{1-\alpha}{\Psi\alpha[L(0)]^{\alpha/(1-\alpha)}},$$

where $L(t)$ *and* α *are given in (4.6) and (4.7), respectively.*

Proof. Define

$$L(t) = H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t dx + \frac{\varepsilon}{2}\|\nabla u\|^2, \quad (4.6)$$

where ε small to be chosen later and

$$0 \le \alpha \le \min\left\{\frac{p^- - 2}{2p^-}, \frac{p^- - m^-}{p^-(m^+ - 1)}, \frac{p^- - m^+}{p^-(m^+ - 1)}\right\}. \quad (4.7)$$

Differentiation $L(t)$ with respect to t, and using the first equation in (4.1), we obtain

$$\begin{aligned} L'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_\Omega \left[u_t^2 - |\Delta u|^2\right] dx \\ + \varepsilon b \int_\Omega |u|^{p(x)} dx - \varepsilon\mu_1 \int_\Omega u u_t(x,t)|u_t(x,t)|^{m(x)-2} dx \\ - \varepsilon\mu_2 \int_\Omega u z(x,1,t)|z(x,1,t)|^{m(x)-2} dx. \end{aligned}$$

By using the definition of the $H(t)$ and for $0<a<1$, such that

$$\begin{aligned}
L'(t) \geq\ & C_0(1-\alpha)H^{-\alpha}(t)\left[\int_\Omega |u_t|^{m(x)}dx+\int_\Omega |z(x,1,t)|^{m(x)}dx\right]\\
&+\varepsilon\left((1-a)p^-H(t)+\frac{(1-a)p^-}{2}\|u_t\|^2+\frac{(1-a)p^-}{2}\|\Delta u\|^2\right)\\
&+\varepsilon(1-a)p^-\int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho\\
&+\varepsilon\int_\Omega\left[u_t^2-|\Delta u|^2\right]dx+\varepsilon ab\int_\Omega |u|^{p(x)}dx\\
&-\varepsilon\mu_1\int_\Omega uu_t(x,t)|u_t(x,t)|^{m(x)-2}dx-\varepsilon\mu_2\int_\Omega uz(x,1,t)|z(x,1,t)|^{m(x)-2}dx.
\end{aligned}$$

Hence

$$\begin{aligned}
L'(t) \geq\ & C_0(1-\alpha)H^{-\alpha}(t)\left[\int_\Omega |u_t|^{m(x)}dx+\int_\Omega |z(x,1,t)|^{m(x)}dx\right]\\
&+\varepsilon(1-a)p^-H(t)+\varepsilon\frac{(1-a)p^-+2}{2}\|u_t\|^2+\varepsilon\frac{(1-a)p^--2}{2}\|\Delta u\|^2\\
&+\varepsilon(1-a)p^-\int_0^1\int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)}dxd\rho+\varepsilon ab\rho(u)\\
&-\varepsilon\mu_1\int_\Omega uu_t(x,t)|u_t(x,t)|^{m(x)-2}dx-\varepsilon\mu_2\int_\Omega uz(x,1,t)|z(x,1,t)|^{m(x)-2}dx.
\end{aligned}$$

Utilizing Young's inequality, we get

$$\int_\Omega |u_t|^{m(x)-1}|u|\,dx \leq \frac{1}{m^-}\int_\Omega \delta^{m(x)}|u|^{m(x)}dx+\frac{m^+-1}{m^+}\int_\Omega \delta^{-\frac{m(x)}{m(x)-1}}|u_t|^{m(x)}dx \tag{4.8}$$

and

$$\begin{aligned}
&\int_\Omega |z(x,1,t)|^{m(x)-1}|u|\,dx\\
&\qquad\leq \frac{1}{m^+}\int_\Omega \delta^{m(x)}|u|^{m(x)}dx+\frac{m^+-1}{m^+}\int_\Omega \delta^{-\frac{m(x)}{m(x)-1}}|z(x,1,t)|^{m(x)}dx.
\end{aligned} \tag{4.9}$$

As in [14], estimates (4.8) and (4.9) remain valid if δ is time-dependent. Let us choose δ so that

$$\delta^{-\frac{m(x)}{m(x)-1}}=kH^{-\alpha}(t),$$

where $k\geq 1$ is specified later, we obtain

$$\int_\Omega \delta^{-\frac{m(x)}{m(x)-1}}|u_t|^{m(x)}dx=kH^{-\alpha}(t)\int_\Omega |u_t|^{m(x)}dx, \tag{4.10}$$

$$\int_\Omega \delta^{-\frac{m(x)}{m(x)-1}}|z(x,1,t)|^{m(x)}dx=kH^{-\alpha}(t)|z(x,1,t)|^{m(x)}dx \tag{4.11}$$

and

$$\int_\Omega \delta^{m(x)} |u|^{m(x)} dx = \int_\Omega k^{1-m(x)} H^{\alpha(m(x)-1)}(t) |u|^{m(x)} dx$$
$$\leq \int_\Omega k^{1-m^-} H^{\alpha(m^+-1)}(t) \int_\Omega |u|^{m(x)} dx. \quad (4.12)$$

By using (4.5), we obtain

$$H^{\alpha(m^+-1)}(t) \int_\Omega |u|^{m(x)} dx$$
$$\leq C\left[(\rho(u))^{m^-/p^- + \alpha(m^+-1)} + (\rho(u))^{m^+/p^- + \alpha(m^+-1)}\right]. \quad (4.13)$$

From (4.7), we deduce that

$$s = m^- + \alpha p^- \left(m^+ - 1\right) \leq p^- \qquad \text{and} \qquad s = m^+ + \alpha p^- \left(m^+ - 1\right) \leq p^-.$$

Then, by using Lemma 9, satisfies

$$H^{\alpha(m^+-1)}(t) \int_\Omega |u|^{m(x)} dx \leq C\left(\|\Delta u\|^2 + \rho(u)\right). \quad (4.14)$$

Combining (4.8)-(4.14), we get

$$\begin{aligned} L'(t) \geq\ & (1-\alpha) H^{-\alpha}(t)\left[C_0 - \varepsilon\left(\frac{m^+-1}{m^+}\right)ck\right]\int_\Omega |u_t|^{m(x)} dx \\ & + (1-\alpha) H^{-\alpha}(t)\left[C_0 - \varepsilon\left(\frac{m^+-1}{m^+}\right)ck\right]\int_\Omega |z(x,1,t)|^{m(x)} dx \\ & + \varepsilon\left(\frac{(p^- - 2) - ap^-}{2} - \frac{C}{m^- k^{1-m^-}}\right)\|\Delta u\|^2 \\ & + \varepsilon(1-a) p^- H(t) + \varepsilon\frac{(1-a)p^- + 2}{2}\|u_t\|^2 + \varepsilon\left(ab - \frac{C}{m^- k^{1-m^-}}\right)\rho(u) \\ & + \varepsilon(1-a) p^- \int_0^1 \int_\Omega \frac{\xi(x)|z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho. \end{aligned} \quad (4.15)$$

Let us choose a small enough such that

$$\frac{(1-a)p^- + 2}{2} > 0$$

and k large enough so that

$$\frac{(p^- - 2) - ap^-}{2} - \frac{C}{m^- k^{1-m^-}} > 0 \qquad \text{and} \qquad ab - \frac{C}{m^- k^{1-m^-}} > 0.$$

Once k and a are fixed, picking ε small enough such that

$$C_0 - \varepsilon\left(\frac{m^+-1}{m^+}\right)ck > 0, \qquad C_0 - \varepsilon\left(\frac{m^+-1}{m^+}\right)ck > 0$$

and

$$L(0) = H^{1-\alpha}(0) + \varepsilon \int_\Omega u_0 u_1 dx + \frac{\varepsilon}{2} \|\nabla u_0\|^2 > 0.$$

Consequently, (4.15) yields

$$L'(t) \geq \varepsilon\eta \left[H(t) + \|u_t\|^2 + \|\Delta u\|^2 + \rho(u) + \int_0^1 \int_\Omega \frac{\xi(x) |z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho \right] \tag{4.16}$$

for a constant $\eta > 0$. Thus we get $L(t) \geq L(0) > 0$, $\forall t \geq 0$.

Now, for some constants σ, $\Gamma > 0$ we denote $L'(t) \geq \Gamma L^\sigma(t)$. On the other hand, applying Hölder inequality, we obtain

$$\left| \int_\Omega u u_t dx \right|^{1/(1-\alpha)} \leq C \|u\|_{p^-}^{1/(1-\alpha)} \|u_t\|_2^{1/(1-\alpha)},$$

and by using Young's inequality gives

$$\left| \int_\Omega u u_t dx \right|^{1/(1-\alpha)} \leq C \left[\|u\|_{p^-}^{mu/(1-\alpha)} + \|u_t\|_2^{\Theta/(1-\alpha)} \right],$$

where $1/\mu + 1/\Theta = 1$. From (4.7), the choice of $\Theta = 2(1-\alpha)$ will make $\mu/(1-\alpha) = 2/(1-2\alpha) \leq p^-$. Hence,

$$\left| \int_\Omega u u_t dx \right|^{1/(1-\alpha)} \leq C \left[\|u\|_{p^-}^s + \|u_t\|^2 \right],$$

where $s = \mu/(1-\alpha)$. From (4.4), we have

$$\left| \int_\Omega u u_t dx \right|^{1/(1-\alpha)} \leq C \left[|H(t)| + \|u_t\|^2 + \rho(u) + \int_0^1 \int_\Omega \frac{\xi(x) |z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho \right].$$

Hence, we get

$$\begin{aligned} L^{1/(1-\alpha)}(t) &= \left[H^{(1-\alpha)}(t) + \varepsilon \int_\Omega u u_t dx + \frac{\varepsilon}{2} \|\nabla u\|^2 \right]^{1/(1-\alpha)} \\ &\leq 2^{\alpha/(1-\alpha)} \left[H(t) + \left| \int_\Omega u u_t dx \right|^{1/(1-\alpha)} \right] \\ &\leq C \left[|H(t)| + \|u_t\|^2 + \|\Delta u\|^2 + \rho(u) + \int_0^1 \int_\Omega \frac{\xi(x) |z(x,\rho,t)|^{m(x)}}{m(x)} dx d\rho \right] \end{aligned}$$

So, for some $\Psi > 0$, from (4.16) we arrive

$$L'(t) \geq \Psi L^{1/(1-\alpha)}(t). \tag{4.17}$$

A simple integration of (4.17) over $(0,t)$ satisfies

$$L^{\alpha/(1-\alpha)}(t) \geq \frac{1}{L^{-\alpha/(1-\alpha)}(0) - \Psi\alpha t/(1-\alpha)},$$

which implies that the solution blows up in a finite time T^*, with

$$T^* \leq \frac{1-\alpha}{\Psi\alpha\left[L(0)\right]^{\alpha/(1-\alpha)}}.$$

As a result, the proof is completed. □

5. Conclusions

In recent years, there has been published much work concerning the wave equation with constant delay or time-varying delay. However, to the best of our knowledge, there was no blow-up result for the Petrovsky equation with delay term and variable exponents. Firstly, we have been obtained the local existence result by using the Faedo-Galerkin method. Later, we have been proved that blow-up of solutions for problem (1.1) under the sufficient conditions in a bounded domain.

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