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CONTROLLABILITY AND HYERS-ULAM STABILITY RESULTS OF INITIAL VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS VIA GENERALIZED PROPORTIONAL-CAPUTO FRACTIONAL DERIVATIVE

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Abstract. This paper concerns the investigation of controllability and Hyers-Ulam stability of initial value problems for fractional differential equations via generalized proportional-Caputo fractional derivatives. The main results are obtained by means of the Krasnoselskii's fixed point theorem.

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1. INTRODUCTION

Fractional differential equations have been studied extensively in the literature because of their applications in various fields of engineering and science; see the books [16, 18, 22]. In the few past years, the variety of definitions of fractional operators has become visible to those interested in fractional calculus.Here, we focus on the most ranking kinds including Liouville, Caputo, Hadamard, Atangana-Baleanu, Caputo-Fabrizio derivatives and etc. For instance, see the books [15, 23] and the papers [3, 8, 19–21] and the references quoted therein.

Recently, Khalil et al. [14] introduce a new definition of fractional derivative, called the conformable fractional derivative, with an obstacle that it does not tend to the original function as the order α tends to zero. The new definition has attracted good efforts of many researchers to establish some useful results; see, for example, [5–7, 17, 26].

In control theory, a proportional derivative controller for controller output u at time t with two tuning parameters has the algorithm

$$u(t) = \kappa_P \mathcal{E}(t) + \kappa_d \frac{d}{dt}\mathcal{E}(t),$$

where κ_P and κ_d are the proportional control parameter and the derivative control parameter, respectively. The function $\mathcal{E}$ is the error between the state variable and the process variable. This control law enables Dawei et al. [11] to present the control of complex networks models.

Inspired by the above concept of the proportional derivative controller, Anderson et al. [2] were able to define the proportional (conformable) derivative of order α by

$$ {}_0^P D_t^\alpha g(t) = k_1(\alpha,t)g(t) + k_0(\alpha,t)g'(t), $$

where g is differentiable function and $k_0, k_1 : [0,1]\times\mathbb{R} \to [0,\infty)$ are continuous functions of the variable t and the parameter $\alpha \in [0,1]$ which satisfy the following conditions for all $t \in \mathbb{R}$:

$$ \lim_{\alpha\to 0^+} k_0(\alpha,t) = 0, \quad \lim_{\alpha\to 1^-} k_0(\alpha,t) = 1, \quad k_0(\alpha,t) \neq 0, \ \alpha \in (0,1], \tag{1.1} $$

$$ \lim_{\alpha\to 0^+} k_1(\alpha,t) = 1, \quad \lim_{\alpha\to 1^-} k_1(\alpha,t) = 0, \quad k_1(\alpha,t) \neq 0, \ \alpha \in [0,1). \tag{1.2} $$

This newly defined local derivative tends to the original function as the order α tends to zero and hence improved the conformable derivatives. In [13], Jarad et al. discussed a special case of the proportional derivatives when $k_1(\alpha,t) = 1-\alpha$ and $k_0(\alpha,t) = \alpha$.

In [4], Baleanu et al. proposed a new more general proportional fractional derivative as a linear combination of a Riemann-Liouville integral and a Caputo derivative, also they obtained some amazing results relevant to the newly hybrid fractional operator such as the Laplace transform and its inversion. Further, they solved some differential equations involved that new hybrid derivative and got the solution in terms of a new bivariate Mittag-Leffler function.

Inspired by the new results in [4], we investigate initial value problems for fractional differential equations via generalized proportional-Caputo fractional derivatives. Precisely, we consider the following IVP:

$$ \begin{cases} {}_0^{PC} D_t^\alpha x(t) = f(t,x(t)) + Bu(t), \quad t \in J = [0,b], \ b < \infty, \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.3} $$

where ${}_0^{PC} D_t^\alpha$ denotes the proportional-Caputo fractional derivative of order $\alpha \in (0,1]$, the function $f : J\times\mathbb{R} \to \mathbb{R}$ is continuous, the control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space, and B is a bounded linear operator from U to $\mathbb{R}$.

Controllability is one of the fundamental notions of modern control theory, which enables one to steer the control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls where initial and final state may vary over the entire space. The problem of controllability of nonlinear systems represented by fractional differential equations has been extensively studied by several authors; see, for example, [1, 9, 10, 25] and the references therein.

To our knowledge there are no similar contributions to the controllability and Hyers-Ulam stability of fractional differential equations via generalized proportional-Caputo fractional derivatives.

2. Preliminaries

In this section we collect some definitions, properties and propositions of the new generalized proportional-Caputo hybrid fractional derivative.

Definition 1 ([4])**.** The proportional-Caputo hybrid fractional derivative of order $\alpha \in (0,1)$ of a differentiable function $g(t)$ is given by

$$ {}_{0}^{PC}D_t^{\alpha} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \left(k_1(\alpha,\tau)g(t) + k_0(\alpha,\tau)g'(t)\right)(t-\tau)^{-\alpha}\, d\tau, \tag{2.1} $$

where the function space domain is given by requiring that g is differentiable and both g and g' are locally L^1 functions on the positive reals.

Definition 2 ([4])**.** The inverse operator of the proportional-Caputo hybrid fractional derivative of order $\alpha \in (0,1)$ is given by

$$ {}_{0}^{PC}I_t^{\alpha} g(t) = \int_0^t \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) \frac{{}_{0}^{RL}D_u^{1-\alpha} g(u)}{k_0(\alpha,u)}\, du, \tag{2.2} $$

where ${}_{0}^{RL}D_u^{1-\alpha}$ denotes the Riemann-Liouville fractional derivative of order $1-\alpha$ and is given by

$$ {}_{0}^{RL}D_u^{1-\alpha} g(u) = \frac{1}{\Gamma(\alpha)} \frac{d}{du} \int_0^u (u-s)^{\alpha-1} g(s)\, ds. \tag{2.3} $$

For more details, we refer the reader to the book of Kilbas et al. [15].

Proposition 1 ([4])**.** *The following inversion relations*

$$ {}_{0}^{PC}D_t^{\alpha}\, {}_{0}^{PC}I_t^{\alpha} g(t) = g(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \lim_{t\to 0} {}_{0}^{RL}I_t^{\alpha} g(t), \tag{2.4} $$

$$ {}_{0}^{PC}I_t^{\alpha}\, {}_{0}^{PC}D_t^{\alpha} g(t) = g(t) - \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) g(0) \tag{2.5} $$

are satisfied.

Proposition 2 ([4])**.** *The proportional-Caputo hybrid fractional derivative operator* ${}_{0}^{PC}D_t^{\alpha}$ *is non-local and singular.*

Remark 1 ([4]). In the limiting cases $\alpha \to 0$ and $\alpha \to 1$, we recover the following special cases:

$$ \lim_{\alpha\to 0} {}_{0}^{PC}D_t^{\alpha} g(t) = \int_0^t g(\tau)\, d\tau, $$
$$ \lim_{\alpha\to 1} {}_{0}^{PC}D_t^{\alpha} g(t) = g(t). $$

Theorem 1 (Krasnoselskii's fixed point theorem, [24]). *Let Ω be a closed convex and non-empty subset of a Banach space $\mathbb{X}$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$, be two operators such that*

(i) $\mathcal{P}_1 x + \mathcal{P}_2 y \in \Omega$, *for all* $x, y \in \Omega$,
(ii) $\mathcal{P}_1$ *is compact and continuous,*
(iii) $\mathcal{P}_2$ *is a contraction mapping.*

Then there exists $z \in \Omega$ such that $z = \mathcal{P}_1 z + \mathcal{P}_2 z$.

3. Controllability Results

In this section, we employ the generalized proportional Caputo fractional derivative operator to discuss the controllability of the IVP (1.3).

Let $C(J,\mathbb{R})$ be the Banach space of all real-valued continuous functions from J into $\mathbb{R}$ equipped by the norm $\|x\| = \sup_{t\in[0,T]} |x(t)|$.

Firstly, we consider the following auxiliary lemma.

Lemma 1. *Let $0 < \alpha \le 1$ and $h \in C(J,\mathbb{R})$. Then the solution of the following linear fractional differential equation*

$$\begin{cases} {}_0^{PC}D_t^{\alpha} x(t) = h(t), \quad t \in J, \\ x(0) = x_0, \end{cases} \tag{3.1}$$

is equivalent to the Volterra integral equation

$$\begin{aligned} x(t) &= \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) x_0 \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} h(\tau)\, d\tau\, du. \end{aligned} \tag{3.2}$$

Proof. Applying the operator ${}_0^{PC}I_t^{\alpha}(\cdot)$ on both sides of (3.1), we get

$${}_0^{PC}I_t^{\alpha}\, {}_0^{PC}D_t^{\alpha} x(t) = {}_0^{PC}I_t^{\alpha} h(t).$$

Using (2.2) and (2.3) together with Proposition 1, we get

$$x(t) - \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) x(0) = \int_0^t \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) \frac{{}_0^{RL}D_u^{1-\alpha} h(u)}{k_0(\alpha,u)}\, du.$$

In view of the following elementary fact:

$${}_0^{RL}D_u^{1-\alpha} h(u) = {}_0^{RL}I_u^{-(1-\alpha)} h(u) = {}_0^{RL}I_u^{\alpha-1} h(u) = \frac{1}{\Gamma(\alpha-1)} \int_0^u (u-\tau)^{\alpha-2} h(\tau)\, d\tau,$$

one can easily obtain the desired integral equation (3.3). The converse follows by direct computation. This completes the proof. □

By virtue of Lemma 1, the solution of the IVP (1.3) is given by

$$x(t) = \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)x_0 + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}f(\tau,x(\tau))\,d\tau\,du + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}Bu_x(\tau)\,d\tau\,du. \quad (3.3)$$

The following definition is helpful in the discussion of the controllability of the IVP (1.3).

Definition 3. The IVP (1.3) is said to be controllable on the interval J if, for every $x_0, x_1 \in \mathbb{R}$, there exists a control $u \in L^2(J,U)$ such that a solution x of equation (1.3) satisfies $x(b) = x_1$.

The following assumptions will be imposed.

($\mathbf{A1}$) The function $f : J\times\mathbb{R}\to\mathbb{R}$ is continuous.

($\mathbf{A2}$) There exists a constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \le L|x-y|, \quad \text{for all } t\in J,\ x,y\in\mathbb{R}.$$

($\mathbf{A3}$) The linear operator $W : L^2(J,U)\to\mathbb{R}$, defined by

$$Wu = \frac{1}{\Gamma(\alpha-1)}\int_0^b\int_0^u \exp\left(-\int_u^b \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}Bu(\tau)\,d\tau\,du$$

has an induced inverse operator W^{-1} which takes values in $L^2(J,U)/\ker W$, where the kernel space of W is defined by $\ker W = \{x\in L^2(J,U) : Wx = 0\}$ and there exist constants $M_1, M_2 > 0$ such that $\|B\| \le M_1$ and $\|W^{-1}\| \le M_2$.

Now we formulate the first main theorem of the paper.

Theorem 2. *If the assumptions* ($\mathbf{A1}$) – ($\mathbf{A3}$) *are satisfied. Then the IVP (1.3) is controllable on J, provided that*

$$\frac{M_1 M_2 b^{2\alpha} L}{M_k^2\Gamma^2(\alpha+1)} < 1, \quad (3.4)$$

where $\inf_{t\in J}|k_0(\alpha,t)| = M_k \neq 0$.

Proof. Set $\sup_{t\in J}|f(t,0)| = M_f < \infty$.

We consider the set $\mathcal{B}_r = \{x\in C(J,\mathbb{R}) : \|x\| \le r\}$ with $r \ge \dfrac{\Lambda_1}{1-\Lambda_2}$.

For the purpose of expediency, we define the two constants $\Lambda_1 > 0$ and $0 < \Lambda_2 < 1$ as

$$\Lambda_1 = |x_0| + \frac{M_f b^\alpha}{M_k\Gamma(\alpha+1)} + \frac{M_1M_2b^\alpha}{M_k\Gamma(\alpha+1)}\left[|x_1| + |x_0| + \frac{M_f b^\alpha}{M_k\Gamma(\alpha+1)}\right],$$

$$\Lambda_2 = \frac{Lb^\alpha}{M_k\Gamma(\alpha+1)}\Big(1+\frac{M_1M_2b^\alpha}{M_k\Gamma(\alpha+1)}\Big).$$

Define the control $u_x(t)$ by

$$u_x(t) = W^{-1}\Big[x_1 - \exp\Big(-\int_0^b \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\Big)x_0 + \frac{1}{\Gamma(\alpha-1)}\int_0^b\int_0^u \exp\Big(-\int_u^b \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\Big)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}f(\tau,x(\tau))\,d\tau\,du\Big](t),\ t\in J.$$

Later, we shall use the following estimates:

$$\begin{aligned}
\|u_x\| &= \sup_{t\in J}|u_x(t)| \\
&\le M_2\sup_{t\in J}\Big\{|x_1|+|x_0|+\frac{1}{\Gamma(\alpha-1)}\int_0^b\int_0^u\frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|}|(f(\tau,x(\tau))-f(\tau,0))+f(\tau,0)|\,d\tau\,du\Big\} \\
&\le M_2\sup_{t\in J}\Big\{|x_1|+|x_0|+\frac{1}{\Gamma(\alpha-1)}\int_0^b\int_0^u\frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|}\big(|f(\tau,x(\tau))-f(\tau,0)|+|f(\tau,0)|\big)\,d\tau\,du\Big\} \\
&\le M_2\sup_{t\in J}\Big\{|x_1|+|x_0|+\frac{1}{\Gamma(\alpha-1)}\int_0^b\int_0^u\frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|}\big(L|x(\tau)|+|f(\tau,0)|\big)\,d\tau\,du\Big\} \\
&\le M_2\Big[|x_1|+|x_0|+\frac{b^\alpha}{M_k\Gamma(\alpha+1)}\big(L\|x\|+M_f\big)\Big],
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
\|u_x-u_y\| &= \sup_{t\in J}|u_x(t)-u_y(t)| \\
&\le M_2\sup_{t\in J}\left\{\frac{1}{\Gamma(\alpha-1)}\int_0^b\int_0^u\frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|}|f(\tau,x(\tau))-f(\tau,y(\tau))|\,d\tau\,du\right\} \\
&\le \frac{M_2L}{M_k\Gamma(\alpha-1)}\sup_{t\in J}\left\{\int_0^b\int_0^u(u-\tau)^{\alpha-2}|x(\tau)-y(\tau)|\,d\tau\,du\right\} \\
&\le \frac{M_2Lb^\alpha}{M_k\Gamma(\alpha+1)}\|x-y\|.
\end{aligned} \tag{3.6}$$

Using the control $u_x(t)$, we define the operators $\mathcal{P}_1,\mathcal{P}_2$ on $\mathcal{B}_r$ as:

$$(\mathcal{P}_1x)(t) = \exp\Big(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\Big)x_0 + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\Big(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\Big)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}f(\tau,x(\tau))\,d\tau\,du,$$

$$(\mathcal{P}_2x)(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\Big(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\Big)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}Bu_x(\tau)\,d\tau\,du.$$

Clearly, one can notice that $(\mathcal{P}_1 x + \mathcal{P}_2 x)(b) = x_1$. This means that u_x steers the IVP (1.3) from x_0 to x_1 in finite time b, which implies that the IVP (1.3) is controllable on J.

The proof is divided into three main steps.

Step 1. $\mathcal{P}_1 x + \mathcal{P}_2 y \in \mathcal{B}_r, \quad \forall x, y \in \mathcal{B}_r$.
For each $t \in J$ and $x, y \in \mathcal{B}_r$, using (3.5), one has

$$
\begin{aligned}
\|\mathcal{P}_1 x + \mathcal{P}_2 y\| &= \sup_{t\in J} |(\mathcal{P}_1 x)(t) + (\mathcal{P}_2 y)(t)| \\
&\le \sup_{t\in J} \Bigg\{ \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) |x_0| \\
&\quad + \frac{1}{\Gamma(\alpha-1)} \int_0^b \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) \\
&\quad \times \frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|} |(f(\tau,x(\tau)) - f(\tau,0)) + f(\tau,0)| \, d\tau \, du \\
&\quad + \frac{1}{\Gamma(\alpha-1)} \int_0^b \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) \frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|} |Bu_y(\tau)| \, d\tau \, du \Bigg\} \\
&\le |x_0| + \frac{1}{M_k\Gamma(\alpha-1)} \int_0^b \int_0^u (u-\tau)^{\alpha-2} \left(L\|x\| + M_f\right) d\tau \, du \\
&\quad + \frac{1}{M_k\Gamma(\alpha-1)} \int_0^b \int_0^u (u-\tau)^{\alpha-2} \|B\| \|u_y\| \, d\tau \, du \\
&\le |x_0| + \frac{b^\alpha}{M_k\Gamma(\alpha+1)} \left(Lr + M_f\right) \\
&\quad + \frac{M_1 M_2 b^\alpha}{M_k\Gamma(\alpha+1)} \left[|x_1| + |x_0| + \frac{b^\alpha}{M_k\Gamma(\alpha+1)} \left(Lr + M_f\right)\right] \\
&\le |x_0| + \frac{M_f b^\alpha}{M_k\Gamma(\alpha+1)} + \frac{M_1 M_2 b^\alpha}{M_k\Gamma(\alpha+1)} \left[|x_1| + |x_0| + \frac{M_f b^\alpha}{M_k\Gamma(\alpha+1)}\right] \\
&\quad + \frac{L b^\alpha}{M_k\Gamma(\alpha+1)} \left(1 + \frac{M_1 M_2 b^\alpha}{M_k\Gamma(\alpha+1)}\right) r \\
&= \Lambda_1 + \Lambda_2 r \le r.
\end{aligned}
$$

Thus, we conclude that $\mathcal{P}_1 x + \mathcal{P}_2 y \in \mathcal{B}_r$.

Step 2. $\mathcal{P}_1$ is compact and continuous. Firstly, we show that $\mathcal{P}_1$ is continuous. Let $\{x_n\}$ be a sequence such that $x_n \to x$ as $n \to \infty$ in $\mathcal{B}_r$. Thus, for each $t \in J$, we have

$$
\begin{aligned}
\|\mathcal{P}_1 x_n - \mathcal{P}_1 x\| &= \sup_{t\in J} |(\mathcal{P}_1 x_n)(t) - (\mathcal{P}_1 x)(t)| \\
&\le \frac{1}{M_k\Gamma(\alpha-1)} \int_0^t \int_0^u (u-\tau)^{\alpha-2} \|(f(\cdot,x_n(\cdot)) - f(\cdot,x(\cdot))\| \, d\tau \, du.
\end{aligned}
$$

Therefore, the continuity of f implies that $\mathcal{P}_1$ is continuous.

Next, we show that $\mathcal{P}_1$ is uniformly bounded on $\mathcal{B}_r$. For each $t \in J$ and $x \in \mathcal{B}_r$, one has

$$\begin{aligned}\|\mathcal{P}_1 x\| &= \sup_{t\in J} |(\mathcal{P}_1 x)(t)| \\ &\leq \sup_{t\in J}\Bigg\{ \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)|x_0| + \frac{1}{\Gamma(\alpha-1)}\int_0^b\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right) \\ &\quad \times \frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|}|(f(\tau,x(\tau))-f(\tau,0))+f(\tau,0)|\,d\tau\,du\Bigg\} \\ &\leq |x_0| + \frac{b^\alpha}{M_k\Gamma(\alpha+1)}\big(Lr+M_f\big),\end{aligned}$$

which implies that $\mathcal{P}_1$ is uniformly bounded on $\mathcal{B}_r$.

It remains to show that $\mathcal{P}_1$ is equicontinuous. For each $t_1,t_2 \in J,\ t_1 < t_2$ and $x \in \mathcal{B}_r$, one obtain that:

$$\begin{aligned}&\|(\mathcal{P}_1 x)(t_2)-(\mathcal{P}_1 x)(t_1)\| \\ &\leq \left\|\exp\left(-\int_0^{t_2}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)x_0 - \exp\left(-\int_0^{t_1}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)x_0\right\| \\ &\quad + \frac{1}{\Gamma(\alpha-1)}\Bigg\|\int_0^{t_2}\int_0^u\left[\exp\left(-\int_0^{t_2}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)-\exp\left(-\int_0^{t_1}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\right] \\ &\quad \times \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}(f(\tau,x(\tau))\,d\tau\,du\Bigg\| \\ &\quad + \frac{1}{\Gamma(\alpha-1)}\left\|\int_{t_1}^{t_2}\int_0^u \exp\left(-\int_0^{t_1}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}(f(\tau,x(\tau))\,d\tau\,du\right\| \\ &= \left\|\frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)}x_0\exp\left(-\int_0^{\xi}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)(t_2-t_1)\right\| \\ &\quad + \frac{1}{\Gamma(\alpha-1)}\Bigg\|\int_0^{t_2}\int_0^u\frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)}\exp\left(-\int_0^{\xi}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right) \\ &\quad (t_2-t_1)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}(f(\tau,x(\tau))\,d\tau\,du\Bigg\| \\ &\quad + \frac{1}{\Gamma(\alpha-1)}\left\|\int_{t_1}^{t_2}\int_0^u \exp\left(-\int_0^{t_1}\frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}(f(\tau,x(\tau))\,d\tau\,du\right\| \\ &\leq \left|\frac{k_1(\alpha,\xi)}{k_0(\alpha,\xi)}x_0\right|(t_2-t_1)+\frac{\bar f}{\Gamma(\alpha+1)}\left|\frac{k_1(\alpha,\xi)}{k_0^2(\alpha,\xi)}\right|t_2(t_2-t_1)+\frac{\bar f}{\Gamma(\alpha+1)}\left|\frac{1}{k_0(\alpha,\xi)}\right|(t_2^\alpha-t_1^\alpha),\end{aligned}$$

where $\bar{f} = \sup_{t\in J\times\mathcal{B}_r} |f(t,x(t))|$ and $\xi \in (t_1,t_2)$. As $t_1 \to t_2$, the right hand side of the above inequality tends to zero independently of $x \in \mathcal{B}_r$. As a consequence of the Arzelà-Ascoli theorem, we deduce that $\mathcal{P}_1$ is compact on $\mathcal{B}_r$.

Step 3. $\mathcal{P}_2$ is a contraction on $\mathcal{B}_r$.
For each $t \in J$ and $x,y \in \mathcal{B}_r$, using (3.6), one has

$$\begin{aligned}
\|\mathcal{P}_2x-\mathcal{P}_2y\| &= \sup_{t\in J}|(\mathcal{P}_2x)(t)-(\mathcal{P}_2y)(t)|\\
&= \sup_{t\in J}\left\{\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}B(u_x(\tau)-u_y(\tau))\,d\tau\,du\right\}\\
&\le \frac{M_1b^{\alpha}}{M_k\Gamma(\alpha+1)}\|u_x-u_y\|\\
&\le \frac{M_1M_2b^{2\alpha}L}{M_k^2\Gamma^2(\alpha+1)}\|x-y\|.
\end{aligned}$$

In view of the condition (3.4), we conclude that $\mathcal{P}_2$ is a contraction mapping.

Therefore, all the assumptions of Krasnoselskii's fixed point theorem (Theorem 1) are satisfied. Hence, the IVP (1.3) is controllable on J. This completes the proof. □

4. Hyers-Ulam Stability

Here, we elucidate Hyers-Ulam stability of the IVP (1.3). We begin with the following essential definition.

Definition 4 ([12])**.** The integral equation (3.3) is said to be Hyers-Ulam stable, if there exists a constant $\mu > 0$ satisfying: for every $\varepsilon > 0$, if

$$\begin{aligned}
\Bigg| x(t) &- \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)x_0\\
&-\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}f(\tau,x(\tau))\,d\tau\,du\\
&-\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}Bu_x(\tau)\,d\tau\,du\Bigg| \le \varepsilon,
\end{aligned}$$

there exists a continuous function $x^*(t)$ satisfying

$$\begin{aligned}
x^*(t) =& \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)x_0\\
&+\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}f(\tau,x^*(\tau))d\tau du\\
&+\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}Bu_{x^*}(\tau)d\tau du, \qquad (4.1)
\end{aligned}$$

such that

$$|x(t)-x^{*}(t)|\leq \mu\,\varepsilon, \quad \forall\, t\in J.$$

Theorem 3. *Assume that the assumptions* $(\mathbf{A1})-(\mathbf{A3})$ *are satisfied. Then the IVP (1.3) is Hyers-Ulam stable.*

Proof. With the help of Theorem 2, let $x(t)$ be unique solution of (3.3) and $x^{*}(t)$ be any other solution satisfying (4.1). Then, by a similar way in the proof of Theorem 2 and by virtue of (3.6), one has

$$\begin{aligned}
|x(t)-x^{*}(t)| &\leq \frac{1}{M_k\Gamma(\alpha-1)}\int_0^t\int_0^u (u-\tau)^{\alpha-2}\,|f(\tau,x(\tau))-f(\tau,x^{*}(\tau))|\;d\tau\,du\\
&\quad+\frac{M_1}{M_k\Gamma(\alpha-1)}\int_0^t\int_0^u (u-\tau)^{\alpha-2}\,|u_x(\tau)-u_{x^{*}}(\tau)|\;d\tau\,du\\
&\leq \frac{L}{M_k\Gamma(\alpha-1)}\int_0^t\int_0^u (u-\tau)^{\alpha-2}\,|x(\tau)-x^{*}(\tau)|\;d\tau\,du\\
&\quad+\frac{M_1}{M_k\Gamma(\alpha-1)}\int_0^t\int_0^u (u-\tau)^{\alpha-2}\,|u_x(\tau)-u_{x^{*}}(\tau)|\;d\tau\,du\\
&\leq \left(\frac{b^{\alpha}L}{M_k\Gamma(\alpha+1)}+\frac{M_1M_2b^{2\alpha}L}{M_k^2\Gamma^2(\alpha+1)}\right)\|x-x^{*}\|\\
&=\mu\,\|x-x^{*}\|,
\end{aligned}$$

where

$$\mu:=\frac{b^{\alpha}L}{M_k\Gamma(\alpha+1)}+\frac{M_1M_2b^{2\alpha}L}{M_k^2\Gamma^2(\alpha+1)}.$$

Therefore, the integral equation (3.3) is Hyers-Ulam stable. Consequently, the IVP (1.3) is Hyers-Ulam stable. The proof is finished. □

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