A sharp threshold for rainbow connection in small-world networks

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Abstract. An edge-colored graph $G$ is rainbow connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection of a connected graph $G$, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow connected.

We prove that $p = \sqrt{\ln n / n}$ is a sharp threshold function for the property $rc(S(n, p, H)) \leq 2$ in the small-world networks. As by-products, our extension of the concept of independence in graph theory and generalized small-world network models are of independent interest.

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1. INTRODUCTION

We utilize the terminology and notation of [19] in this letter. An interesting connectivity concept of a graph was recently introduced in [3] and has attracted attention of some researchers. An edge-colored graph $G$ is referred to as rainbow connected if any two vertices are connected by a path whose edges have distinct colors. A rainbow connected graph must be connected, and conversely, any connected graph has a trivial edge coloring that makes it rainbow connected. The rainbow connection of a connected graph $G$, denoted $rc(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow connected.

An easy observation is that if $G$ has $n$ vertices then $rc(G) \leq n - 1$, since one may color the edges of a given spanning tree of $G$ with distinct colors, and color the remaining edges with one of the already used colors. It is also known that $rc(G) = 1$ if and only if $G$ is a complete graph, and that $rc(G) = n - 1$ if and only if $G$ is a tree. Note that $rc(G) \geq diam(G)$, where $diam(G)$ denotes the diameter of $G$. The behavior of $rc(G)$ with respect to the minimum degree $\delta(G)$ has been dealt with in the work [2, 13], of which a primary result is $rc(G) \leq 20n/\delta(G)$. Related concepts such as rainbow path [6], rainbow tree [5] and rainbow $k$-connectivity [4] have also been investigated recently.

A natural and intriguing direction to explore is the random graph scenarios [12, 17]. Let $G(n, p)$ be the classical random graph with $n$ vertices and edge probability $p$. For

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a graph property $\mathcal{A}$, we say that $G(n, p)$ satisfies $\mathcal{A}$ almost surely if the probability that $G(n, p)$ satisfies $\mathcal{A}$ tends to 1 as $n$ tends to infinity. A function $f(n)$ is called a sharp threshold function for the property $\mathcal{A}$ if there are two positive constants $C$ and $c$ such that $G(n, p)$ satisfies $\mathcal{A}$ almost surely for $p \geq Cf(n)$ and $G(n, p)$ almost surely does not satisfy $\mathcal{A}$ for $p \leq cf(n)$. A remarkable feature of random graphs is that all monotone graph properties have sharp thresholds (see e.g. [1, 10, 11]).

The parameter $rc(G)$ is monotone non-increasing in the sense that if we add an edge to $G$ we cannot increase its rainbow connection. The authors of [2] show that $p = \sqrt{\ln n / n}$ is a sharp threshold function for the property $rc(G(n, p)) \leq 2$. In this note, we propose a generalized small-world network model and explore the threshold of rainbow connection of it. The small-world network is a model with two important characteristics: the clustering effect and the small-world phenomenon, which was originally introduced by Watts and Strogatz [18] as a model of real world complex networks. It has since been the subject of considerable research interest within the physics community, see e.g. [7, 14–16] and references therein.

The rest of the note is organized as follows. In Section 2, we present some necessary notions including the generalized small-world model and state our sharp threshold result. The proofs are given in Section 3.

2. NOTIONS AND MAIN RESULT

Watts-Strogatz (WS) rewiring model [18] and its variant Newman-Watts (NW) model [16] are classical small-world network models. The NW model can be regarded as the union of an Erdős-Rényi random graph $G(n, p)$ and a $2k$-regular lattice. It is known that the NW model and the WS model are, essentially, the same. A natural extension would be to use a general sparse graph to replace the low-dimensional regular lattices.

Let $S(n, p, H)$ be a small-world network that is the union of a random graph $G(n, p)$ and a graph $H$ on $n$ vertices. Note that $S(n, p, H)$ is not necessarily connected when $H$ is not connected. When $H$ is a regular lattice, we then obtain the NW model.

Next, we need to extend the classical notion of independence in graph theory to distant $l$-independence. A subset $X$ of vertices in a graph $G$ is called distant $l$-independent for some $l \in \mathbb{N}$, if the distance between any two vertices in $V$ is larger than $l$. Thus, a distant $1$-independent set is independent in the classical sense. Recall that there is another generalization of independence, called $k$-independence [8, 9], which requires the induced subgraph has maximum degree less than $k$. The relative strength relationship of these three concepts can be described as follows:

\[ k\text{-independence} < \text{independence} < \text{distant } l\text{-independence}. \]

Now we are on the stage to state our main result.
Theorem 1. Let $H$ be a graph on $n$ vertices, which contains a distant 2-independent set of order $\Theta(n^\varepsilon)$ for some $\varepsilon > 0$. For the small-world network $S(n, p, H)$, $p = \sqrt{\ln n}/n$ is a sharp threshold function for the property $rc(S(n, p, H)) \leq 2$.

Clearly, a $2k$-regular lattice with $k \ll n^\alpha$ for some $\alpha \in (0, 1)$ serves as an eligible graph $H$ in Theorem 1. Therefore, the above result holds for both WS and NW models.

3. Proof of Theorem 1

In this section, we will provide a proof of Theorem 1 as per the reasoning in [2]. As mentioned in Section 1, $rc(G) \geq 2$ for any non-complete graph $G$. The following lemma gives a sufficient condition for $rc(G) = 2$.

Lemma 1. ([2]) If $G$ is a non-complete graph on $n$ vertices and any two vertices of $G$ have at least $2\ln n$ common neighbors, then $rc(G) = 2$.

Proof of Theorem 1. For the first part of the theorem, we need to prove that for a sufficiently large constant $C$, the small-world network $S(n, p, H)$ with $p = C \sqrt{\ln n}/n$ almost surely has $rc(G) = 2$. Recall that $rc(G)$ is monotone non-increasing, we need only to prove this for the random graph $G(n, p)$. By Lemma 1, it suffices to show that almost surely any two vertices of $G(n, p)$ have at least $2\ln n$ common neighbors.

Fix a pair of vertices $x, y$, and the probability that $z$ is a common neighbor of them is $C^2 \ln n/n$. Let random variable $X$ represents the number of common neighbors of $x$ and $y$. Accordingly, we get $EX = (n-2)(C^2 \ln n/n)$. By using the Chernoff bound (e.g. [12] pp.26), for large enough $C$, we have

$$P(X < 2\ln n) \leq e^{-\frac{C^2 \ln n}{4}}.$$  

Since there are $\binom{n}{2}$ pairs of vertices in $G(n, p)$, the union bound readily yields the result.

For the other direction, it suffices to show that for a sufficiently small constant $c$, the small-world network $S(n, p, H)$ with $p = c \sqrt{\ln n}/n$ almost surely has $diam(S(n, p, H)) \geq 3$. By the assumption in Theorem 1, fix a distant 2-independent set $X$ of order $\Theta(n^\varepsilon)$ for some $\varepsilon < 1/4$ in $H$, and let $Y$ be the remaining $n - \Theta(n^\varepsilon)$ vertices. Let $A$ be the event that $X$ induces an independent set in the small-world network $S(n, p, H)$. Let $B$ be the event that there exists a pair of vertices $x, y \in X$ with no common neighbor in $Y$. Consequently, it suffices to prove that (i) $P(A) \to 1$; and (ii) $P(B) \to 1$, as $n \to \infty$.

For (i): For $c$ sufficiently small we obtain

$$P(A) = (1 - p)^{\Theta(n^\varepsilon)} = (1 - c \sqrt{\ln n}/n)^{\Theta(n^\varepsilon)} \approx e^{\Theta\left(c^2 \ln n\right)} \to 1.$$
For (ii): For a pair \( x, y \in X \), the probability that \( x, y \) have a common neighbor in \( Y \) is shown to be given by

\[
1 - \left( 1 - \frac{c^2 \ln n}{n} \right)^{n - \Theta(n^\epsilon)} \sim (1 - n^{-c^2}).
\]

Since the vertex set \( X \) can be divided into \( \Theta(n^\epsilon)/2 = \Theta(n^\epsilon) \) pairs, the probability that all \( \Theta(n^\epsilon) \) pairs have a common neighbor is

\[
1 - P(B) = \left( 1 - \left( 1 - \frac{c^2 \ln n}{n} \right)^{n - \Theta(n^\epsilon)} \right)^{\Theta(n^\epsilon)} \sim \left( 1 - n^{-c^2} \right)^\Theta(n^\epsilon) \sim e^{-\Theta(n^\epsilon)/nc^2}. \tag{3.1}
\]

For sufficiently small \( c \), the right hand side of (3.1) tends to zero, which thus completes the proof. \( \square \)

REFERENCES


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