Miskolc Mathematical Notes
Vol. 13 (2012), No 2, pp. 493-497

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2012.347

A sharp threshold for rainbow connection in small-world networks

Y. Shang

Miskolc Mathematical Notes
Vol. 13 (2012), No. 2, pp. 493–497

HU e-ISSN 1787-2413

A SHARP THRESHOLD FOR RAINBOW CONNECTION IN SMALL-WORLD NETWORKS

Y. SHANG

Received 30 March, 2011

Abstract. An edge-colored graph G is rainbow connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection of a connected graph G, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected. We prove that $p = \sqrt{\ln n/n}$ is a sharp threshold function for the property $rc(S(n,p,H)) \leq 2$ in the small-world networks. As by-products, our extension of the concept of independence in graph theory and generalized small-world network models are of independent interest.

2000 *Mathematics Subject Classification:* 05C82; 05C15; 05C40

Keywords: rainbow connection, edge coloring, small world, networks

1. INTRODUCTION

We utilize the terminology and notation of [19] in this letter. An interesting connectivity concept of a graph was recently introduced in [3] and has attracted attention of some researchers. An edge-colored graph G is referred to as rainbow connected if any two vertices are connected by a path whose edges have distinct colors. A rainbow connected graph must be connected, and conversely, any connected graph has a trivial edge coloring that makes it rainbow connected. The rainbow connection of a connected graph G, denoted $rc(G)$, is the smallest number of colors that are needed in order to make G rainbow connected.

An easy observation is that if G has n vertices then $rc(G) \leq n-1$, since one may color the edges of a given spanning tree of G with distinct colors, and color the remaining edges with one of the already used colors. It is also known that $rc(G) = 1$ if and only if G is a complete graph, and that $rc(G) = n-1$ if and only if G is a tree. Note that $rc(G) \geq diam(G)$, where $diam(G)$ denotes the diameter of G. The behavior of $rc(G)$ with respect to the minimum degree $\delta(G)$ has been dealt with in the work [2, 13], of which a primary result is $rc(G) \leq 20n/\delta(G)$. Related concepts such as rainbow path [6], rainbow tree [5] and rainbow k-connectivity [4] have also been investigated recently.

A natural and intriguing direction to explore is the random graph scenarios [12,17]. Let $G(n,p)$ be the classical random graph with n vertices and edge probability p. For a graph property $\mathcal{A}$, we say that $G(n,p)$ satisfies $\mathcal{A}$ almost surely if the probability that $G(n,p)$ satisfies $\mathcal{A}$ tends to 1 as n tends to infinity. A function $f(n)$ is called a sharp threshold function for the property $\mathcal{A}$ if there are two positive constants C and c such that $G(n,p)$ satisfies $\mathcal{A}$ almost surely for $p \geq Cf(n)$ and $G(n,p)$ almost surely does not satisfy $\mathcal{A}$ for $p \leq cf(n)$. A remarkable feature of random graphs is that all monotone graph properties have sharp thresholds (see e.g. [1, 10, 11]).

The parameter $rc(G)$ is monotone non-increasing in the sense that if we add an edge to G we cannot increase its rainbow connection. The authors of [2] show that $p = \sqrt{\ln n/n}$ is a sharp threshold function for the property $rc(G(n,p)) \leq 2$. In this note, we propose a generalized small-world network model and explore the threshold of rainbow connection of it. The small-world network is a model with two important characteristics: the clustering effect and the small-world phenomenon, which was originally introduced by Watts and Strogatz [18] as a model of real world complex networks. It has since been the subject of considerable research interest within the physics community, see e.g. [7, 14–16] and references therein.

The rest of the note is organized as follows. In Section 2, we present some necessary notions including the generalized small-world model and state our sharp threshold result. The proofs are given in Section 3.

2. Notions and main result

Watts-Strogatz (WS) rewiring model [18] and its variant Newman-Watts (NW) model [16] are classical small-world network models. The NW model can be regarded as the union of an Erdős-Rényi random graph $G(n,p)$ and a $2k$-regular lattice. It is known that the NW model and the WS model are, essentially, the same. A natural extension would be to use a general sparse graph to replace the low-dimensional regular lattices.

Let $S(n,p,H)$ be a small-world network that is the union of a random graph $G(n,p)$ and a graph H on n vertices. Note that $S(n,p,H)$ is not necessarily connected when H is not connected. When H is a regular lattice, we then obtain the NW model.

Next, we need to extend the classical notion of independence in graph theory to distant l-independence. A subset X of vertices in a graph G is called distant l-independent for some $l \in \mathbb{N}$, if the distance between any two vertices in V is larger than l. Thus, a distant 1-independent set is independent in the classical sense. Recall that there is another generalization of independence, called k-independence [8, 9], which requires the induced subgraph has maximum degree less than k. The relative strength relationship of these three concepts can be described as follows:

$$k\text{-independence} \; < \; \text{independence} \; < \; \text{distant } l\text{-independence}.$$

Now we are on the stage to state our main result.

Theorem 1. *Let H be a graph on n vertices, which contains a distant 2-independent set of order $\Theta(n^{\varepsilon})$ for some $\varepsilon > 0$. For the small-world network $S(n, p, H)$, $p = \sqrt{\ln n/n}$ is a sharp threshold function for the property $rc(S(n, p, H)) \leq 2$.*

Clearly, a $2k$-regular lattice with $k \ll n^{\alpha}$ for some $\alpha \in (0, 1)$ serves as an eligible graph H in Theorem 1. Therefore, the above result holds for both WS and NW models.

3. Proof of Theorem 1

In this section, we will provide a proof of Theorem 1 as per the reasoning in [2]. As mentioned in Section 1, $rc(G) \geq 2$ for any non-complete graph G. The following lemma gives a sufficient condition for $rc(G) = 2$.

Lemma 1. *([2]) If G is a non-complete graph on n vertices and any two vertices of G have at least $2\ln n$ common neighbors, then $rc(G) = 2$.*

Proof of Theorem 1. For the first part of the theorem, we need to prove that for a sufficiently large constant C, the small-world network $S(n, p, H)$ with $p = C\sqrt{\ln n/n}$ almost surely has $rc(G) = 2$. Recall that $rc(G)$ is monotone non-increasing, we need only to prove this for the random graph $G(n, p)$. By Lemma 1, it suffices to show that almost surely any two vertices of $G(n, p)$ have at least $2\ln n$ common neighbors.

Fix a pair of vertices x, y, and the probability that z is a common neighbor of them is $C^2 \ln n/n$. Let random variable X represents the number of common neighbors of x and y. Accordingly, we get $EX = (n - 2)(C^2 \ln n/n)$. By using the Chernoff bound (e.g. [12] pp.26), for large enough C, we have

$$P(X < 2\ln n) \leq P\left(X < EX - \frac{C^2 \ln n}{4}\right) \leq e^{-\frac{C^2 \ln n}{32}} = o(n^{-2}).$$

Since there are $\binom{n}{2}$ pairs of vertices in $G(n, p)$, the union bound readily yields the result.

For the other direction, it suffices to show that for a sufficiently small constant c, the small-world network $S(n, p, H)$ with $p = c\sqrt{\ln n/n}$ almost surely has $diam(S(n, p, H)) \geq 3$. By the assumption in Theorem 1, fix a distant 2-independent set X of order $\Theta(n^{\varepsilon})$ for some $\varepsilon < 1/4$ in H, and let Y be the remaining $n - \Theta(n^{\varepsilon})$ vertices. Let $\mathcal{A}$ be the event that X induces an independent set in the small-world network $S(n, p, H)$. Let $\mathcal{B}$ be the event that there exists a pair of vertices $x, y \in X$ with no common neighbor in Y. Consequently, it suffices to prove that (i) $P(\mathcal{A}) \to 1$; and (ii) $P(\mathcal{B}) \to 1$, as $n \to \infty$.

For (i): For c sufficiently small we obtain

$$P(\mathcal{A}) = (1 - p)^{\binom{\Theta(n^{\varepsilon})}{2}} = (1 - c\sqrt{\ln n/n})^{\binom{\Theta(n^{\varepsilon})}{2}}$$
$$\sim e^{-\frac{c\sqrt{\ln n}}{\Theta\left(n^{\frac{1}{2}-2\varepsilon}\right)}} \to 1,$$

as $n \to \infty$, since $0 < \varepsilon < 1/4$.

For (ii): For a pair $x, y \in X$, the probability that x, y have a common neighbor in Y is shown to be given by

$$1 - \left(1 - \frac{c^2 \ln n}{n}\right)^{n - \Theta(n^\varepsilon)} \sim (1 - n^{-c^2}).$$

Since the vertex set X can be divided into $\Theta(n^\varepsilon)/2 = \Theta(n^\varepsilon)$ pairs, the probability that all $\Theta(n^\varepsilon)$ pairs have a common neighbor is

$$1 - P(\mathscr{B}) = \left(1 - \left(1 - \frac{c^2 \ln n}{n}\right)^{n - \Theta(n^\varepsilon)}\right)^{\Theta(n^\varepsilon)} \sim \left(1 - n^{-c^2}\right)^{\Theta(n^\varepsilon)} \sim e^{-\frac{\Theta(n^\varepsilon)}{n^{c^2}}}. \quad (3.1)$$

For sufficiently small c, the right hand side of (3.1) tends to zero, which thus completes the proof. □

REFERENCES

[1] B. Bollobás and A. Thomason, "Threshold functions," *Combinatorica*, vol. 7, pp. 35–38, 1987.

[2] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, and R. Yuster, "On rainbow connection," *Electron. J. Comb.*, vol. 15, no. 1, pp. 13, Research Paper R57, 2008.

[3] G. Chartrand, G. L. Johns, K. A. McKeon, and P. Zhang, "Rainbow connection in graphs," *Math. Bohem.*, vol. 133, no. 1, pp. 85–98, 2008.

[4] G. Chartrand, G. L. Johns, K. A. McKeon, and P. Zhang, "The rainbow connectivity of a graph," *Networks*, vol. 54, no. 2, pp. 75–81, 2009.

[5] G. Chartrand, F. Okamoto, and P. Zhang, "Rainbow trees in graphs and generalized connectivity," *Networks*, vol. 55, no. 4, pp. 360–367, 2010.

[6] D. j. Dellamonica, C. Magnant, and D. M. Martin, "Rainbow paths," *Discrete Math.*, vol. 310, no. 4, pp. 774–781, 2010.

[7] R. Durrett, *Random graph dynamics*, ser. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge: Cambridge University Press, 2007, vol. 20.

[8] J. F. Fink and M. S. Jacobson, "n-domination in graphs," in *Graph theory with applications to algorithms and computer science*, ser. Proc. 5th Int. Conf., Kalamazoo/Mich., 1984. New York: John Wiley & Sons, 1985, pp. 283–300.

[9] J. F. Fink and M. S. Jacobson, "On n-domination, n-dependence and forbidden subgraphs," in *Graph theory with applications to algorithms and computer science*, ser. Proc. 5th Int. Conf., Kalamazoo/Mich. 1984. New York: John Wiley & Sons, 1985, pp. 301–311.

[10] E. Friedgut, "Hunting for sharp thresholds," *Random Struct. Algorithms*, vol. 26, no. 1-2, pp. 37–51, 2005.

[11] E. Friedgut and G. Kalai, "Every monotone graph property has a sharp threshold," *Proc. Am. Math. Soc.*, vol. 124, no. 10, pp. 2993–3002, 1996.

[12] S. Janson, T. Łuczak, and A. Ruciński, *Random graphs*, ser. Wiley-Interscience Series in Discrete Mathematics and Optimization. New York: Wiley, 2000.

[13] M. Krivelevich and R. Yuster, "The rainbow connection of a graph is (at most) reciprocal to its minimum degree," *J. Graph Theory*, vol. 63, no. 3, pp. 185–191, 2010.

[14] M. E. Newman, "Models of the small world," *J. Stat. Phys.*, vol. 101, no. 3-4, pp. 819–841, 2000.

[15] M. E. J. Newman, C. Moore, and D. J. Watts, "Mean-field solution of the small-world network model," *Phys. Rev. Lett.*, vol. 84, pp. 3201–3204, 2000.

[16] M. E. J. Newman and D. J. Watts, "Renormalization group analysis of the small-world network model," *Phys. Lett., A*, vol. 263, no. 4-6, pp. 341–346, 1999.

[17] Y. Shang, “Sharp concentration of the rainbow connection of random graphs,” *Notes Number Theory Discrete Math.*, vol. 16, no. 4, pp. 25–28, 2010.

[18] D. J. Watts and S. H. Strogatz, “Collective dynamics of ”small-world” networks,” *Nature*, vol. 393, pp. 440–442, 1998.

[19] D. B. West, *Introduction to graph theory*. Prentice Hall, 2000.

f

Author’s address

Y. Shang
University of Texas at San Antonio, Institute for Cyber Security, San Antonio, TX 78249, USA
E-mail address: shylmath@hotmail.com