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ABOUT ONE CHARACTERISTIC INITIAL VALUE PROBLEM WITH PREHISTORY AND ITS INVESTIGATION

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Abstract. We have built a constructive method for investigation approximate solution of nonlinear characteristic initial value problem with prehistory and suggested the one algorithm acceleration convergence of consequent approximation, established sufficient condition of subsistence the unique regular solution of the investigated problem and its equality.

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1. INTRODUCTION

In examining the process of different nature (gas sorption, drying by the air flow, pipes heating by a stream of hot water, etc.) there are cases, when in relevant mathematical models (problems) not all output data (boundary or initial condition) are known, thus they need to be found through the solution some auxiliary problems, which are mathematical models of processes that proceed to research.

These problems should be named as problems with prehistory. This work is dedicated to investigation of the such problem.

2. PROBLEM SETTINGS AND REASONING

Let in phases space xOt there is specified domain $D = D_1 \cup D_2$, where

$$D_1 = \{(x,t)\,|x \in (x_0, x_1], t \in (g(x), t_1]\},$$
$$D_2 = \{(x,t)\,|x \in (x_0, x_1], t \in (t_1, t_2]\},$$

and $t = g(x) \Longleftrightarrow x = \kappa(t)$ — ”free” curve and $g'(x) > 0$, $t_i = g(x_i)$, $i = 0, 1$, $0 \le x_0 < x_1$, $0 \le t_0 < t_1 < t_2$.

Problem settings [1]: in the function space $C^*(\overline{D}_2) = C^{(1.1)}(D_2) \cap C(\overline{D}_2)$ to find the solution of non-linear differential equation,

$$D^{(1.1)}U(x,t) = f\left(x, t, U(x,t), D^{(0.1)}U(x,t)\right) := f\,[U(x,t)], \tag{2.1}$$

which satisfies the conditions

$$U(x_0,t)=\varphi(t),\ t\in[t_1,t_2],\qquad\qquad (2.2)$$
$$U(x,t_1)=V(x,t_1),\ x\in[x_0,x_1],$$

where $V(x,t)\in C^*(\overline{D}_1)$ is the solution the Darboux problem [8]

$$D^{(1.1)}V(x,t)+a_1(x,t)D^{(1.0)}V(x,t)+a_2(x,t)D^{(0.1)}V(x,t)$$
$$=\omega(x,t,V(x,t)):=\omega[V(x,t)],\qquad\qquad (2.3)$$
$$V(x_0,t)=\mu(t),\ t\in(g(x),t_1],$$
$$V(x,g(x))=\psi(x),\ x\in(x_0,x_1],\qquad\qquad (2.4)$$

and the appropriate condition of consistency is true

$$\varphi(t_1)=\mu(t_1),\ \psi(x_0)=\mu(t_0).\qquad\qquad (2.5)$$

Under the following conditions, we consider, that $f[U(x,t)]\in C(\overline{B}_2)$, $f:\overline{B}_2\longrightarrow\mathbb{R}$, $\overline{B}_2\subset\mathbb{R}^4$, $\Pi p_{xOt}\overline{B}_2=\overline{D}_2$, $\omega[V(x,t)]\in C(\overline{B}_1)$, $\omega:\overline{B}_1\longrightarrow\mathbb{R}$, $\overline{B}_1\subset\mathbb{R}^3$, $\Pi p_{xOt}\overline{B}_1=$ $=\overline{D}_1$, $a_1(x,t)\in C^{(1.0)}(D_1)$, $a_2(x,t)\in C^{(0.1)}(D_1)$, and the given functions are $\varphi(t)\in C^1[t_1,t_2]$, $\mu(t)\in C^1[t_0,t_1]$, $\psi(x)\in C^1[x_0,x_1]$.

It's easy to make sure in the equity of the following.

Lemma 1. *Let $f[U(x,t)]\in C(\overline{B}_2)$, $\omega[V(x,t)]\in C(\overline{B}_1)$, $a_1(x,t)\in C^{(1.0)}(D_1)$, $a_2(x,t)\in C^{(0.1)}(D_1)$ and the condition has been fulfilled.*

$$D^{(1.0)}a_1(x,t)=D^{(0.1)}a_2(x,t),\ (x,t)\in D_1.\qquad\qquad (2.6)$$

Thus, boundary value problem (2.1)–(2.5) can be represented in equivalent integral form [7]

$$U_s(x,t)=\Phi_s(x,t)+\varepsilon_s T_{1,s}f_1[U_1(\xi,\eta)]\qquad\qquad (2.7)$$
$$+T_sf_s[U_s(\xi,\eta)],(x,t)\in\overline{D}_s,\ s=1,2,$$

where

$$\varepsilon_s=\begin{cases}0, & for\ s=1,\\ 1, & for\ s=2,\end{cases}$$

and

$$\Phi_1(x,t):=\mu(t)exp\left(\int_x^{x_0}a_2(\xi,t)d\xi\right)+[\psi(x)$$
$$-\mu(g(x))exp\left(\int_x^{x_0}a_2(\xi,g(x))d\xi\right)\Bigg]\,exp\left(\int_t^{g(x)}a_1(x,\eta)d\eta\right),$$

$$\Phi_2(x,t) := \varphi(t) + \mu(t_1)\left[exp\left(\int_x^{x_0} a_2(\xi,t_1)d\xi\right) - 1\right] + [\psi(x)$$

$$-\mu(g(x))exp\left(\int_x^{x_0} a_2(\xi,g(x))d\xi\right)\right] exp\left(\int_{t_1}^{g(x)} a_1(x,\eta)d\eta\right),$$

$$T_1 f_1\left[U_1(\xi,\eta)\right] := \int_{x_0}^{x}\int_{g(x)}^{t} K(x,t;\xi,\eta) f_1\left[U_1(\xi,\eta)\right] d\eta d\xi$$

$$U_1(t,x) := V(t,x), K(x,t;\xi,\eta) := exp\left(\int_t^{\eta} a_1(\xi,\tau)d\tau + \int_x^{\xi} a_2(\zeta,t)d\zeta\right),$$

$$f_1\left[U_1(x,t)\right] := \omega\left[U_1(x,t)\right] + \left[D^{(0.1)}a_2(x,t) + a_1(x,t)a_2(x,t)\right]U_1(x,t),$$

$$U_2(x,t) := U(x,t), f_2\left[U_2(x,t)\right] := f\left[U(x,t)\right],$$

$$T_2 f_2\left[U_2(\xi,\eta)\right] := \int_{t_1}^{t}\int_{x_0}^{x} f_2\left[U_2(\xi,\eta)\right] d\xi d\eta,$$

$$T_{1,2} f_1\left[U_1(\xi,\eta)\right] := \int_{x_0}^{x}\int_{g(x)}^{t_1} K(x,t_1;\xi,\eta) f_1\left[U_1(\xi,\eta)\right] d\eta d\xi$$

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3. Construction investigation method and approximate solution for system (2.7)

Let‘s set the sufficient condition of existence and one way solution for integral equation systems (2.7) and make up the method of their approximate solution.

For this effect let‘s introduce the space of function $C_1(\overline{B}_s)$.

Definition 1. Supposing that, functions $f_s\left[U_s(x,t)\right] \in C_1(\overline{B}_s)$, if they satisfy following condition [3, 5]:

(1) $f_s\left[U_s(x,t)\right] \in C(\overline{B}_s)$, $s = 1,2$;

(2) in the space of function $C(\overline{B}_{1,s})$, $\overline{B}_{1,s} \subset \mathbb{R}^{2+2s}$, $\Pi p_{xOt}\overline{B}_{1,s} = \overline{D}_s, s = 1,2$, there exist functions

$$H_1(x,t,V_1(x,t);Z_1(x,t)) := H_1[V_1(x,t);Z_1(x,t)],$$

$$H_2(x,t,V_2(x,t),D^{(0.1)}V_2(x,t);Z_2(x,t),D^{(0.1)}Z_2(x,t)) := H_2[V_2(x,t);Z_2(x,t)],$$

such as

(a) $H_s[U_s(x,t);U_s(x,t)] \equiv f_s\left[U_s(x,t)\right]$, $(x,t) \in \overline{D}_s$, $s = 1,2$,

(b) for any continuously differential pair of functions $V_s(x,t)$, $Z_s(x,t) \in \overline{B}_{1,s}$, which satisfy conditions

$$V_s(x,t) \geq Z_s(x,t), D^{(0.1)}V_2(x,t) \geq D^{(0.1)}Z_2(x,t),$$

$(x,t) \in \overline{D}_s$, in $\overline{B}_{1,s}$ the inequality

$$H_s[V_s(x,t);Z_s(x,t)] \geq H_s[Z_s(x,t);V_s(x,t)], (x,t) \in \overline{D}_s, \tag{3.1}$$

is proven;

(3) the functions $H_s[V_s(x,t);Z_s(x,t)]$ in $\overline{B}_{1,s}$ satisfy the Lipschitz condition, i.e. for any functions $V_{s,\kappa}(x,t), Z_{s,\kappa}(x,t) \in \overline{B}_{1,s}, \kappa = 1,2$, from $C^{(0.1)}(\overline{D}_s)$ space the conditions

$$\begin{aligned}
|H_1[V_{1,1}(x,t);V_{1,2}(x,t)] & - H_1[Z_{1,1}(x,t);Z_{1,2}(x,t)]| \\
& \leq L_1 \left(|W_{1,1}(x,t)| + |W_{1,2}(x,t)|\right), \\
|H_2[V_{2,1}(x,t);V_{2,2}(x,t)] & - H_2[Z_{2,1}(x,t);Z_{2,2}(x,t)]| \\
& \leq L_2 \left(|W_{2,1}(x,t)| + |W_{2,2}(x,t)| + \left|D^{(0.1)}W_{2,1}(x,t)\right| \right. \\
& \qquad \left. + \left|D^{(0.1)}W_{2,2}(x,t)\right|\right),
\end{aligned}$$

are preserved, where L_s is the Lipschitz constant and

$$W_{s,\kappa}(x,t) = V_{s,\kappa}(x,t) - Z_{s,\kappa}(x,t), \qquad \kappa, s = 1,2.$$

Note, that if functions $f_s[U_s(x,t)] \in C(\overline{B}_s)$ and have in the domain $\overline{B}_s$ first order partial derivatives, then they always belong to space $C_1(\overline{B}_s)$. The opposite statement is not true.

Let functions $Z_{1,p}(x,t), V_{1,p}(x,t) \in C(\overline{D}_1), Z_{2,p}(x,t), V_{2,p}(x,t) \in C^*(\overline{D}_2)$, belong to the domain $\overline{B}_{1,s}, s = 1,2, p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

We will introduce the following:

$$\begin{aligned}
f_s^p(x,t) &:= H_s\left[Z_{s,p}(x,t);V_{s,p}(x,t)\right], \\
f_{s,p}(x,t) &:= H_s\left[V_{s,p}(x,t);Z_{s,p}(x,t)\right], \\
\alpha_{s,p}(x,t) &:= Z_{s,p}(x,t) - \Phi_{s,p}(x,t) - \varepsilon_s T_{1,s} f_1^p(\xi,\eta) - T_s f_s^p(\xi,\eta), \\
\beta_{s,p}(x,t) &:= V_{s,p}(x,t) - \Phi_{s,p}(x,t) - \varepsilon_s T_{1,s} f_{1,p}(\xi,\eta) - T_s f_{s,p}(\xi,\eta), \\
W_{s,p}(x,t) &= Z_{s,p}(x,t) - V_{s,p}(x,t), (x,t) \in \overline{D}_s, s = 1,2, p \in \mathbb{N}_0.
\end{aligned} \tag{3.2}$$

Let's build the sequences of functions $\{Z_{s,p}(x,t)\}, \{V_{s,p}(x,t)\}$ according to [4, 6]

$$\begin{aligned}
Z_{s,p+1}(x,t) &= \Phi_s(x,t) + \varepsilon_s T_{1,s} f_1^p(\xi,\eta) + T_s f_s^p(\xi,\eta), \\
V_{s,p+1}(x,t) &= \Phi_s(x,t) + \varepsilon_s T_{1,s} f_{1,p}(\xi,\eta) + T_s f_{s,p}(\xi,\eta), \\
& (x,t) \in \overline{D}_s, s = 1,2, p \in \mathbb{N}_0,
\end{aligned} \tag{3.3}$$

where zero approximation $Z_{s,0}(x,t), V_{s,0}(x,t) \in \overline{B}_{1,s}$ could be considered as arbitrary functions from $C(\overline{D}_1), C^*(\overline{D}_2)$, which satisfies conditions (2.4), (2.5), (2.2) and the inequalities

$$\begin{gathered} W_{s,0}(x,t) \geq 0,\ D^{0.1}W_{2,0}(x,t) \geq 0,\ \alpha_{s,0}(x,t) \geq 0,\ \beta_{s,0}(x,t) \leq 0, \\ D^{(0.1)}\alpha_{2,0}(x,t) \geq 0,\ D^{(0.1)}\beta_{2,0}(x,t) \leq 0, \\ (x,t) \in \overline{D}_s, s = 1,2. \end{gathered} \tag{3.4}$$

Definition 2. The functions $Z_{1,0}(x,t)$, $V_{1,0}(x,t) \in C(\overline{D}_1)$, $Z_{2,0}(x,t)$, $V_{2,0}(x,t) \in C^*(\overline{D}_2)$, of the domain $\overline{B}_{1,s}$, $s = 1,2$, and satisfy the conditions (2.4), (2.5), (2.2) and inequalities (3.4), could be called the comparison function of the problem (2.1)–(2.5) [7].

From (3.2), (3.3) the validity of the following equations follows:

$$\begin{aligned} & Z_{s,p}(x,t) - Z_{s,p+1}(x,t) = \alpha_{s,p}(x,t), \\ & V_{s,p}(x,t) - V_{s,p+1}(x,t) = \beta_{s,p}(x,t), \\ & W_{s,p+1}(x,t) = \varepsilon_s T_{1,s}\left(f_1^p(\xi,\eta) - f_{1,p}(\xi,\eta)\right) + T_s\left(f_s^p(\xi,\eta) - f_{s,p}(\xi,\eta)\right), \\ & \alpha_{s,p+1}(x,t) = \varepsilon_s T_{1,s}\left(f_1^p(\xi,\eta) - f_1^{p+1}(\xi,\eta)\right) + T_s\left(f_s^p(\xi,\eta) - f_s^{p+1}(\xi,\eta)\right), \\ & \beta_{s,p+1}(x,t) = \varepsilon_s T_{1,s}\left(f_{1,p}(\xi,\eta) - f_{1,p+1}(\xi,\eta)\right) + T_s\left(f_{s,p}(\xi,\eta) - f_{s,p+1}(\xi,\eta)\right), \\ & \qquad (x,t) \in \overline{D}_s, s = 1,2, p \in \mathbb{N}_0 \end{aligned} \tag{3.5}$$

From the last equations considering the conditions (3.1), (3.4), by the method of mathematical induction, we have confirmed that following inequalities are true

$$\begin{aligned} & \alpha_{s,p}(x,t) \geq 0,\ \beta_{s,p}(x,t) \leq 0, \\ & D^{(0,1)}\alpha_{2,p}(x,t) \geq 0,\ D^{(0,1)}\beta_{2,p}(x,t) \leq 0, \\ & D^{(0,\kappa-1)}V_{s,p}(x,t) \leq D^{(0,\kappa-1)}V_{s,p+1}(x,t) \leq D^{(0,\kappa-1)}Z_{s,p+1}(x,t) \\ & \qquad \leq D^{(0,\kappa-1)}Z_{s,p}(x,t), \end{aligned} \tag{3.6}$$

for all $p \in \mathbb{N}_0$ and $(x,t) \in \overline{D}_s$, $s = 1,2$ ($\kappa = 1$ in $s = 1$ and $\kappa = 1,2$ in $s = 2$)

The following lemma takes place

Theorem 1. *Let the functions $f_s[U_s(x,t)] \in C_1(\overline{B}_s), a_1(x,t) \in C^{(1.0)}(D_1), a_2(x,t) \in C^{(0.1)}(D_1)$ and the condition (2.6) holds, and for the domain $\overline{B}_{1,s}, s = 1,2$ there exist comparison functions $Z_{s,0}(x,t)$ and $V_{s,0}(x,t)$ of the problem (2.1)–(2.5).*

Then for any $p \in \mathbb{N}$ and $s = 1,2$ functions $Z_{s,p}(x,t)$ and $V_{s,p}(x,t)$, which had been built according to (3.3), the inequalities (3.6) are true in the domain $\overline{D}_s$.

Let show that the built this way sequence of functions $\{Z_{s,p}(x,t)\}$ and $\{V_{s,p}(x,t)\}$ converge uniformly at $(x,t) \in \overline{D}_s$ to the unique regular solution of problem (2.1)–(2.5).

Because of inequalities (3.6) at $(x,t) \in \overline{D}_s$ in order to

$$\lim_{p\longrightarrow\infty} Z_{s,p}(x,t) = \lim_{p\longrightarrow\infty} V_{s,p}(x,t)$$

it is enough to show, that $\lim\limits_{p\longrightarrow\infty} W_{s,p}(x,t) = 0$.

Let

$$\sup_{\overline{D}_1\times\overline{D}_1} K(x,t;\xi,\eta) \leq K, \max_{s}\left[\sup_{\overline{D}_s}|W_{s,0}(x,t)|, \sup_{\overline{D}_2}\left|D^{(0.1)}W_{2,0}(x,t)\right|\right] \leq d,$$
$$\sup_{\overline{D}}[1, x-x_0+t-t_0] = l, \max(2KL_1, 4L_2) = L.$$

Indeed: from (3.5), by the method of mathematical induction, we insure that the estimates are true

$$\max_{S}\left\{\sup_{\overline{D}_s} W_{s,p}(x,t), \sup_{\overline{D}_2}|W_{2,p_t}(x,t)|\right\} \leq \frac{[Ll(t-t_0+x-x_0)]^p}{p!}d,\ p\in\mathbb{N}. \tag{3.7}$$

From the estimates (3.7) follows, that

$$\lim_{p\longrightarrow\infty} W_{s,p}(x,t) = 0,\ \lim_{p\longrightarrow\infty} D^{(0.1)}W_{2,p}(x,t) = 0,$$

that

$$U_s(x,t) := \lim_{p\longrightarrow\infty} Z_{s,p}(x,t) = \lim_{p\longrightarrow\infty} V_{s,p}(x,t), (x,t)\in\overline{D}_s, s=1,2,$$
$$\gamma(x,t) := \lim_{p\longrightarrow\infty} D^{(0.1)}Z_{2,p}(x,t) = \lim_{p\longrightarrow\infty} D^{(0.1)}V_{2,p}(x,t), (x,t)\in\overline{D}_2.$$

Is not difficult to make sure that $\gamma(x,t) = D^{(0.1)}U_2(x,t)$, for it in (3.3) it would be enough to go the limit for $p \longrightarrow \infty$ and differentiate the obtained result in t under $s=2$. After that the limit functions $U_s(x,t)$ are the solutions of the system integral equations (2.7) and under $(x,t)\in\overline{D}_s, s=1,2$, and so the boundary-value problem (2.1)–(2.5).

Theorem 2. *Let the condition of Theorem 1 to be true. Then, the sequences of functions $\{Z_{s,p}(x,t)\}$ and $\{V_{s,p}(x,t)\}$, have been built according to (3.3), (3.4):*

(1) *uniformly convergence to the unique solution of the corresponding integral equation in system (2.7) for $(x,t)\in\overline{D}_s$, $s=1,2$;*
(2) *the true estimates (3.7);*
(3) *in $\overline{B}_{s,1}$ the inequalities hold*

$$\begin{aligned} D^{(0.\kappa-1)}V_{s,p}(x,t) \leq D^{(0.\kappa-1)}V_{s,p+1}(x,t) &\leq D^{(0.\kappa-1)}U_s(x,t) \\ &\leq D^{(0.\kappa-1)}Z_{s,p+1}(x,t) \leq D^{(0.\kappa-1)}Z_{s,p}(x,t), \\ &(x,t)\in\overline{D_s}, s=1,2, \end{aligned} \tag{3.8}$$

$$\kappa = \begin{cases} 1, s = 1, \\ 1, 2, s = 2, \end{cases} \tag{3.9}$$

for all $p \in \mathbb{N}_0$, where $U_s(x,t)$ – is the unique solution of the corresponding integral equation in (2.7) for $(x,t) \in \overline{D_s}, s = 1, 2$.

The uniqueness of the integral equations system solution (2.7) for $(x,t) \in \overline{D_s}$, $s = 1, 2$, and appropriateness of inequalities (3.8) have been proved from the opposite.

Corollary 1. *Let the conditions of Theorem 1 to be hold. Then in the domain $\overline{D}$ there exist a unique regular solution of the boundary-value problem (2.1)–(2.5) and for it the inequalities (3.8) are true.*

Corollary 2. *Let the conditions of Theorem 1 to be hold, $\mu(t) = \psi(x) = 0$, $t \in [g(x), t_1]$, $x \in [x_0, x_1]$, $\varphi(t) = 0$, $t \in [t_1, t_2]$, in particular $f_s[U_s(x,t)] = H_s[U_s(x,t); 0]$, $(x,t) \in \overline{D_s}, s = 1, 2$.*

Then, if $f_s[0] \leq (\geq) 0$ in the domain $\overline{B}_s, s = 1, 2$, the solution of the boundary-value problem (2.1)–(2.5) for $(x,t) \in \overline{D}_s$, $s = 1, 2$ satisfies the inequalities [7]

$$U_s(x,t) \leq (\geq) 0, (x,t) \in \overline{D_s}, s = 1, 2.$$

4. Acceleration convergence of the method (3.3), (3.4)

Let‘s show one approach of acceleration convergence iteration method (3.3), (3.4). With this aim let‘s build sequence of functions $Z_{s,p}(x,t)$, $V_{s,p}(x,t)$ according to [2, 7]

$$\begin{aligned} &Z_{s,p+1}(x,t) = \Phi_s(x,t) + \varepsilon_s T_{1,s} \overline{f}_1^p(\xi,\eta) + T_s \overline{f}_s^p(\xi,\eta), \\ &V_{s,p+1}(x,t) = \Phi_s(x,t) + \varepsilon_s T_{1,s} \overline{f}_{1,p}(\xi,\eta) + T_s \overline{f}_{s,p}(\xi,\eta), (x,t) \in \overline{D}_s, s = 1, 2, p \in \mathbb{N}_0, \end{aligned} \tag{4.1}$$

where

$$\begin{gathered} \overline{f}_s^p(x,t) := H_s\left[\overline{Z}_{s,p}(x,t); \overline{V}_{s,p}(x,t)\right], \\ \overline{f}_{s,p}(x,t) := H_s\left[\overline{V}_{s,p}(x,t); \overline{Z}_{s,p}(x,t)\right], s = 1, 2, (x,t) \in \overline{D}_s. \\ D^{(0.\kappa-1)}\overline{Z}_{s,p}(x,t) := D^{(0.\kappa-1)} Z_{s,p}(x,t) - d_{s,p}^{(\kappa-1)}(x,t) D^{(0.\kappa-1)} W_{s,p}(x,t), \\ D^{(0.\kappa-1)}\overline{V}_{s,p}(x,t) := D^{(0.\kappa-1)} V_{s,p}(x,t) + q_{s,p}^{(\kappa-1)}(x,t) D^{(0.\kappa-1)} W_{s,p}(x,t), \\ s = 1, 2, (x,t) \in \overline{D}_s, \\ \kappa = \begin{cases} 1, s = 1, \\ 1, 2, s = 2, \end{cases} \end{gathered}$$

and consistent functions $d_{s,p}^{(\kappa-1)}(x,t)$, $q_{s,p}^{(\kappa-1)}(x,t)$, $(x,t)\in\overline{D}_s$, satisfy conditions

$$0\le d_{s,p}^{(\kappa-1)}(x,t)\le 0,5;\ 0\le q_{s,p}^{(\kappa-1)}(x,t)\le 0,5, \tag{4.2}$$

$(x,t)\in\overline{D}_s,\ s=1,2, p\in\mathbb{N}_0$.

As the zero approximation $Z_{s,0}(x,t)$, $V_{s,0}(x,t)$ we take comparison function of the problem (2.1)–(2.5).

We have noted, that if $D^{(0.\kappa-1)}Z_{s,p}(x,t)$ and $D^{(0.\kappa-1)}V_{s,p}(x,t)$ belong to the domain $\overline{B}_{1,s}$, then

$$D^{(0.\kappa-1)}\overline{Z}_{s,p}(x,t), D^{(0.\kappa-1)}\overline{V}_{s,p}(x,t)\in\overline{B}_{1,s}.$$

If at every iteration step (4.1), (3.4), (4.2) functions $d_{s,p}^{(\kappa-1)}(x,t)$, $q_{s,p}^{(\kappa-1)}(x,t)$ satisfy the conditions

$$\begin{gathered}D^{(0.\kappa-1)}\left[Z_{s,p}(x,t)-Z_{s,p+1}(x,t)\right]-d_{s,p}^{(\kappa-1)}(x,t)D^{(0.\kappa-1)}W_{s,p}(x,t)\ge 0,\\ D^{(0.\kappa-1)}\left[V_{s,p}(x,t)-V_{s,p+1}(x,t)\right]+q_{s,p}^{(\kappa-1)}(x,t)D^{(0.\kappa-1)}W_{s,p}(x,t)\le 0,\\ (x,t)\in\overline{D}_s, s=1,2,\end{gathered}\tag{4.3}$$

then it‘s easy to see, that functions $Z_{s,p}(x,t)$ and $V_{s,p}(x,t)$ built according to (4.1), (3.4), (4.2), (4.4), satisfy the inequalities (3.6)

Lemma 2. *Let the conditions of Theorem 1 to be hold. The set of functions $d_{s,p}^{(\kappa-1)}(x,t)$ and $q_{s,p}^{(\kappa-1)}(x,t)$, $(x,t)\in\overline{D}_s, s=1,2, p\in\mathbb{N}_0$, which belong to space $C(\overline{D}_s)$ and satisfy conditions (4.2), (4.4), is non-empty.*

Proof. Indeed, let us put on each step of iteration in (4.1), (3.4)

$$d_{s,p}^{(\kappa-1)}(x,t)=\begin{cases}D^{(0.\kappa-1)}\alpha_{s,p}(x,t)\left[\rho_{s,p}^{(\kappa-1)}(x,t)\right]^{-1}, & \text{for } D^{(0.\kappa-1)}W_{s,p}(x,t)\ne 0,\\ 0, & \text{for } D^{(0.\kappa-1)}W_{s,p}(x,t)=0,\end{cases}$$
$$q_{s,p}^{(\kappa-1)}(x,t)=\begin{cases}-D^{(0.\kappa-1)}\beta_{s,p}(x,t)\left[\rho_{s,p}^{(\kappa-1)}(x,t)\right]^{-1}, & \text{for } D^{(0.\kappa-1)}W_{s,p}(x,t)\ne 0,\\ 0, & \text{for } D^{(0.\kappa-1)}W_{s,p}(x,t)=0,\end{cases}\tag{4.4}$$

where $\rho_{s,p}^{(\kappa-1)}(x,t):=D^{(0.\kappa-1)}\left[\alpha_{s,p}(x,t)-\beta_{s,p}(x,t)+W_{s,p}(x,t)\right]$.

It is obvious, that chosen functions $d_{s,p}^{(\kappa-1)}(x,t)$ and $q_{s,p}^{(\kappa-1)}(x,t)$ according to (4.4) satisfy the conditions (4.2) and under (3.1), (4.2) we have received

$$\begin{aligned}&D^{(0.\kappa-1)}\left[Z_{s,p}(x,t)-Z_{s,p-1}(x,t)\right]-\frac{D^{(0.\kappa-1)}\alpha_{s,p}(x,t)}{\rho_{s,p}^{(\kappa-1)}(x,t)}D^{(0.\kappa-1)}W_{s,p}(x,t)\\&\quad=D^{(0.\kappa-1)}\alpha_{s,p}(x,t)+D^{(0.\kappa-1)}\left[\varepsilon_sT_{1,s}\left(f_1^p(\xi,\eta)-\overline{f}_1^p(\xi,\eta)\right)\right.\end{aligned}$$

$$+T_s\left(f_s^p(\xi,\eta)-\overline{f}_s^p(\xi,\eta)\right)\Big]-\frac{D^{(0.\kappa-1)}\alpha_{s,p}(x,t)}{\rho_{s,p}^{(\kappa-1)}(x,t)}D^{(0.\kappa-1)}W_{s,p}(x,t)$$
$$\geq D^{(0.\kappa-1)}\alpha_{s,p}(x,t)\left[1-\frac{D^{(0.\kappa-1)}W_{s,p}(x,t)}{\rho_{s,p}^{(\kappa-1)}(x,t)}\right]\geq 0,\ (x,t)\in\overline{D}_s.$$

All other inequalities (4.4) are proofed analogically for $(x,t)\in\overline{D}_s, s=1,2,$ and Lemma 2 could have been proven. □

Let $Z_{s,p}(x,t)$ and $V_{s,p}(x,t)$ are constructed by some method of the problem comparison function (2.1)–(2.5). We could denote the following approximations to the solution of the system (2.7), constructed according to the method (4.1)–(4.4) via $Z^*_{s,p+1}(x,t)$ and $V^*_{s,p+1}(x,t)$, then from (3.3), (4.1) we have obtained

$$Z_{s,p+1}(x,t)-Z^*_{s,p+1}(x,t)=\varepsilon_s T_{1,s}\left[f_1^p(\xi,\eta)-\overline{f}_1^p(\xi,\eta)\right]$$
$$+T_s\left[f_s^p(\xi,\eta)-\overline{f}_s^p(\xi,\eta)\right]\geq 0,$$

similarly, that $V_{s,p+1}(x,t)\leq V^*_{s,p+1}(x,t)$, for $(x,t)\in\overline{D}_s$ the following inequalities take place

$$V_{s,p+1}(x,t)\leq V^*_{s,p+1}(x,t)\leq Z^*_{s,p+1}(x,t)\leq Z_{s,p+1}(x,t),$$

thus the convergence of the method (4.1)–(4.4) is not slower than the convergence of the iterative process (3.3), (3.4).

Theorem 3. *Let the conditions of Theorem 1 to be true.*

Then the sequences of function $\{Z_{s,p}(x,t)\}$ and $\{V_{s,p}(x,t)\}$, built according to (3.4), (4.1)–(4.4); have the following characteristics

(1) *are uniform convergent to the unique regular solution of the problem (2.1)–(2.5) for $(x,t)\in\overline{D}$;*
(2) *the inequalities (3.6) are true in the domain $\overline{B}_{s,1}$;*
(3) *the convergence of the method (3.4), (4.1)–(4.4) is not slower than the convergence of the iterative process (3.3), (3.4).*

Note, that depending on the choice of functions $d_{s,p}^{(\kappa-1)}(x,t)$ and $q_{s,p}^{(\kappa-1)}(x,t)$, which satisfy the condition (4.2), (4.4), we could obtain different approximate research methods and approximate solution for the differential equations theory problems.

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