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ON SUMS WITH GENERALIZED HARMONIC, HYPERHARMONIC AND SPECIAL NUMBERS

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Abstract. In this paper, we establish interesting sums including generalized harmonic numbers and special numbers by using generating functions of these numbers and some combinatorial identities.

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1. INTRODUCTION

Harmonic numbers are important in various branches of combinatorics and number theory. The harmonic numbers are defined by

$$H_0 = 0, \ \ H_n = \sum_{i=1}^{n} \frac{1}{i}, \text{ for } n = 1, 2, \cdots .$$

The first few harmonic numbers are $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \cdots$. These numbers have been generalized by some authors.

In [7], for every ordered pair $(\alpha, n) \in \mathbb{R}^+ \times \mathbb{N}$, the generalized harmonic numbers $H_n(\alpha)$ are defined by

$$H_0(\alpha) = 0, \ \ H_n(\alpha) = \sum_{i=1}^{n} \frac{1}{i\alpha^i}, \text{ for } n = 1, 2, \cdots .$$

For $\alpha = 1$, the usual harmonic numbers are $H_n(1) = H_n$ and the generating function of the generalized harmonic numbers is

$$\sum_{n=1}^{\infty} H_n(\alpha) x^n = -\frac{\ln\left(1 - \frac{x}{\alpha}\right)}{1 - x}.$$

In [10], for the generalized harmonic numbers $H_n(\alpha)$, the authors defined the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$ as follows:

Definition 1. For $r < 0$ or $n \leq 0$, $H_n^r(\alpha) = 0$ and for $n \geq 1$, the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$ are defined by

$$H_n^r(\alpha) = \sum_{i=1}^{n} H_i^{r-1}(\alpha), \quad r \geq 1,$$

where $H_n^0(\alpha) = \frac{1}{n\alpha^n}$.

For $\alpha = 1$, $H_n^r(1) = H_n^r$ are the hyperharmonic numbers of order r. The generating function of the generalized hyperharmonic numbers of order r is

$$\sum_{n=1}^{\infty} H_n^r(\alpha) x^n = -\frac{\ln\left(1 - \frac{x}{\alpha}\right)}{(1-x)^r}. \tag{1.1}$$

In [8, 13], the generalized harmonic numbers $H(n,r)$ of rank r are defined as for $n \geq 1$, $r \geq 0$,

$$H(n,r) = \sum_{1 \leq n_0 + n_1 + \cdots + n_r \leq n} \frac{1}{n_0 n_1 \cdots n_r}$$

or, equivalently, as

$$H(n,r) = \frac{(-1)^{r+1}}{n!} \left(\frac{d^n}{dx^n} \frac{[\ln(1-x)]^{r+1}}{1-x} \right) \Bigg|_{x=0}.$$

It is clear that $H(n,0) = H_n$.

Some special numbers in areas of number theory that we will use in the future are given as follows:

The Cauchy numbers of order r, showed by C_n^r, are defined by the generating functions to be

$$\left(\frac{x}{\ln(1+x)} \right)^r = \sum_{n=0}^{\infty} C_n^r \frac{x^n}{n!}. \tag{1.2}$$

The Daehee numbers of order r, showed by D_n^r, are defined by the generating functions to be

$$\left(\frac{\ln(1+x)}{x} \right)^r = \sum_{n=0}^{\infty} D_n^r \frac{x^n}{n!}. \tag{1.3}$$

For $r = 1$, $D_n^1 = D_n$ are called Daehee numbers. It is clear that

$$D_0 = 1, D_1 = -\frac{1}{2}, \cdots, D_n = (-1)^n \frac{n!}{n+1}. \tag{1.4}$$

The derangement numbers, denoted by d_n, are defined by the generating functions to be

$$\frac{e^{-x}}{1-x} = \sum_{n=0}^{\infty} d_n \frac{x^n}{n!}.$$

It is well known that Stirling numbers play an important role in combinatorial analysis. The Stirling numbers of the first kind $s(n,i)$ are given by

$$x^{\underline{n}} = \sum_{i=0}^{n} s(n,i)x^i,$$

where for $n \geq 0$, $s(n,0) = \delta_{n0}$, δ_{ni} is the Kronecker delta [1, 14]. $x^{\underline{n}}$ stands for the falling factorial defined by $x^{\underline{n}} = x(x-1)\ldots(x-n+1)$.

The Stirling numbers of the first kind $s(n,k)$ satisfy the recurrence relation

$$s(n+1,k) = s(n,k-1) - ns(n,k),$$

and the generating function of these numbers is given by

$$\frac{\ln(1+x)^k}{k!} = \sum_{n=0}^{\infty} s(n,k)\frac{x^n}{n!}.$$

The signed Stirling numbers of the first kind $|s(n,k)|$ are defined such that the number of permutations of n elements which contain exactly k permutation cycles is the nonnegative number

$$|s(n,k)| = (-1)^{n-k} s(n,k).$$

This means that $s(n,k) = 0$ for $k > n$ and $s(n,n) = 1$ and the generating function of these numbers [2,9] is given by

$$\frac{(-\ln(1-x))^k}{k!} = \sum_{n=0}^{\infty} |s(n,k)|\frac{x^n}{n!}. \tag{1.5}$$

In [2], the authors obtained many relations between the Stirling numbers of first kind and the generalized harmonic numbers $H(n,r)$ of rank r. For example, for any positive integers n,r,

$$\sum_{i=1}^{n} \frac{(-1)^{i+1}}{i!} H(n+1,i) = H_n, \qquad \sum_{i=r}^{n-1} \frac{r!}{i!} s(i,r) H_{n-i}^{i+1} = H(n,r).$$

In [12], the authors gave some identities including the hyperharmonic, the Daehee and the derangement numbers and derived some nonlinear differential equations from the generating function of the hyperharmonic numbers. For example, for any positive integers n,r,

$$D_n = n! \sum_{i=0}^{n} \binom{r}{n-i} (-1)^i H_{i+1}^r, \qquad \sum_{i=0}^{n} \frac{(-1)^{n-i}}{(n-i)!} H_i^r = \sum_{i=0}^{n} \frac{d_{n-i}}{(n-i)!} H_i^{r-1}.$$

Harmonic numbers and generalized harmonic numbers have been studied since the distant past and are involved in a wide range of diverse fields such as analysis, computer science and various branches of number theory. Recently, there are a lot of works about infinite series which are associated with the Riemann Zeta function, the Digamma (and Polygamma) functions, the harmonic (and the generalized harmonic) numbers and also properties, generating functions including higher-order harmonic numbers by using umbral type method [3,4,11].

In [5], the authors derived several generating functions involving harmonic numbers by making use of an interesting approach based on the umbral calculus.

In [18], the authors obtained closed-form expressions for three families of combinatorial series associated with the Digamma (or Psi) function $\psi(z)$ and the generalized harmonic number $H_n(z)=\sum_{j=1}^{n} 1/(z+j)$. They also considered several illustrative examples and applications of their main results.

A. Sofo and H. M. Srivastava extended some results of Euler related sums [15,16]. In [16], they investigated products of the shifted harmonic numbers and the reciprocal binomial coefficients and in [15], integral and closed-form representations of sums with products of harmonic numbers and binomial coefficients were developed in terms of Polygamma functions.

In [6], the authors gave the properties of a class of generalized harmonic numbers $H(n,r)$. By means of the method of coefficients, the authors established some identities involving $H(n,r)$. They obtained the asymptotic expansion of certain sums involving these numbers and inverse of binomial coefficients by Laplace's method.

In [17], the author obtained a set of identities for finite sums of products of harmonic numbers higher-order and reciprocal binomial coefficients.

In this paper, we establish interesting sums including the generalized harmonic numbers and special numbers by using generating functions of these numbers and some combinatorial identities. For example, for any positive integers n,r and m,

$$n!H_n^r(\alpha)=(-1)^{n-1}\sum_{i=0}^{n-1}\binom{n}{i}\frac{(n-i)\,(-r)^{\underline{n-i-1}}}{\alpha^{i+1}}D_i,$$

$$H(n,r)=\sum_{i=0}^{n}\sum_{j=0}^{i}\binom{m-1}{n-i}(-1)^{n-j-r}\frac{H_j^m(\alpha)\,s(i-j,r)r!}{\alpha^{i-j}\,(i-j)!}.$$

2. Some identities involving generalized hyperharmonic numbers and Daehee numbers

In this section, using the generating functions of the generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$, the Daehee numbers D_n, the derangement numbers d_n and the Cauchy numbers C_n, we will give some interesting sums including these numbers.

Theorem 1. *For any positive integers n,r, we have*

$$D_n=\alpha^{n+1}n!\sum_{i=0}^{n}\binom{r}{n-i}(-1)^i H_{i+1}^r(\alpha), \tag{2.1}$$

$$H_n(\alpha)=\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\frac{D_i}{\alpha^{i+1}}. \tag{2.2}$$

Proof. From (1.3), we have

$$\sum_{n=0}^{\infty}\frac{D_n}{\alpha^{n+1}}\frac{x^n}{n!}=\frac{\ln\left(1+\frac{x}{\alpha}\right)}{x}=\left(-\frac{\ln\left(1+\frac{x}{\alpha}\right)}{(1+x)^r}\right)\left(-\frac{1}{x}\right)(1+x)^r.$$

By (1.1) and Binomial theorem, we write

$$\sum_{n=0}^{\infty}\frac{D_n}{\alpha^{n+1}}\frac{x^n}{n!}=\left(-\frac{1}{x}\right)\sum_{i=1}^{\infty}(-1)^i H_i^r(\alpha)x^i\sum_{j=0}^{\infty}\binom{r}{j}x^j.$$

and by product of generating functions, we get

$$\sum_{n=0}^{\infty}\frac{D_n}{\alpha^{n+1}}\frac{x^n}{n!}=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{r}{n-i}(-1)^i H_{i+1}^r(\alpha)x^n.$$

Comparing the coefficients on both sides, we get the desired result.

Now, we will give second equality. Then from (1.4), we have

$$\begin{aligned}\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\frac{D_i}{\alpha^{i+1}}&=\sum_{i=0}^{n-1}\frac{(-1)^i}{i!}\frac{(-1)^i i!}{(i+1)\alpha^{i+1}}\\&=\sum_{i=0}^{n-1}\frac{1}{(i+1)\alpha^{i+1}}=H_n(\alpha),\end{aligned}$$

as claimed result. □

Corollary 1. *For any positive integers n,r, we have*

$$H_n(\alpha)=\sum_{i=0}^{n-1}\sum_{j=0}^{i}\binom{r}{i-j}(-1)^{i+j}H_{j+1}^r(\alpha).$$

Proof. Combining (2.1) and (2.2), the result is clearly given. □

Theorem 2. *For any positive integers n,r, we have*

$$n!H_n^r(\alpha)=(-1)^{n-1}\sum_{i=0}^{n-1}\binom{n}{i}\frac{(n-i)(-r)^{\underline{n-i-1}}}{\alpha^{i+1}}D_i,$$

$$\sum_{i=0}^{n}\frac{(-1)^{n-i}}{(n-i)!}H_i^r(\alpha)=\sum_{i=0}^{n}\frac{d_{n-i}}{(n-i)!}H_i^{r-1}(\alpha).$$

Proof. By (1.1), we write

$$\sum_{n=1}^{\infty}H_n^r(\alpha)x^n=-\frac{\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^r}=\frac{1}{\alpha}\frac{\ln\left(1-\frac{x}{\alpha}\right)}{-\frac{x}{\alpha}}\frac{x}{(1-x)^r},$$

and from (1.3) and Binomial theorem,

$$\sum_{n=1}^{\infty}H_n^r(\alpha)x^n=\frac{1}{\alpha}\sum_{i=0}^{\infty}\frac{(-1)^i D_i}{\alpha^i}\frac{x^i}{i!}\sum_{n=0}^{\infty}(-1)^n(-r)^{\underline{n}}\frac{x^{n+1}}{n!}$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i D_i}{\alpha^{i+1}} \frac{x^i}{i!} \sum_{n=1}^{\infty} (-1)^{n-1} (-r)^{\underline{n-1}} \frac{x^n}{(n-1)!}$$
$$= \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \frac{(-1)^i D_i}{\alpha^{i+1} i!} \frac{(-1)^{n-i-1} (n-i) (-r)^{\underline{n-i-1}}}{(n-i)!} x^n$$
$$= \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \binom{n}{i} \frac{(-1)^{n-1} (n-i) (-r)^{\underline{n-i-1}}}{\alpha^{i+1} n!} D_i x^n.$$

Thus, we have the desired result.

We will give the other sum. By multiplying the generating function of generalized hyperharmonic numbers of order r, $H_n^r(\alpha)$ by e^{-x}, we get

$$-\frac{\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^r} e^{-x} = \sum_{n=0}^{\infty} H_n^r(\alpha) x^n \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} x^i$$
$$= \sum_{n=0}^{\infty} \sum_{i=0}^{n} H_i^r(\alpha) \frac{(-1)^{n-i}}{(n-i)!} x^n \tag{2.3}$$

and

$$-\frac{\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^r} e^{-x} = \frac{-\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^{r-1}} \frac{e^{-x}}{1-x}$$
$$= \sum_{n=0}^{\infty} H_n^{r-1}(\alpha) x^n \sum_{i=0}^{\infty} \frac{d_i}{i!} x^i$$
$$= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{d_{n-i}}{(n-i)!} H_i^{r-1}(\alpha) x^n. \tag{2.4}$$

From (2.3) and (2.4), we have the desired identity. Hence we have the proof. □

Theorem 3. *For any positive integers n, r, k, we have*

$$D_n^r = n! \alpha^{n+1} \sum_{i=0}^{n} \sum_{j=0}^{i} \frac{(-1)^j k^{\underline{n-i}}}{\alpha^{i-j} (i-j)! (n-i)!} D_{i-j}^{r-1} H_{j+1}^k(\alpha),$$
$$n! H_n^r(\alpha) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} \binom{n}{i} \binom{n-i}{j} \frac{(-1)^{n-1} (-r)^{\underline{n-i-j-1}} (n-i-j)}{\alpha^{i+j+1}} D_i^r C_j^{r-1}.$$

Proof. From (1.3), we write

$$\sum_{n=0}^{\infty} \frac{D_n^r}{\alpha^n} \frac{x^n}{n!} = \left(\frac{\ln\left(1+\frac{x}{\alpha}\right)}{\frac{x}{\alpha}}\right)^r = \alpha \frac{-\ln\left(1+\frac{x}{\alpha}\right)}{(-x)(1+x)^k} \left(\frac{\ln\left(1+\frac{x}{\alpha}\right)}{\frac{x}{\alpha}}\right)^{r-1} (1+x)^k.$$

By (1.1), (1.3) and Binomial theorem, using product of generating functions, we have

$$\sum_{n=0}^{\infty} \frac{D_n^r}{\alpha^n} \frac{x^n}{n!} = \alpha \sum_{n=0}^{\infty} (-1)^n H_{n+1}^k(\alpha) x^n \sum_{j=0}^{\infty} \frac{D_j^{r-1}}{\alpha^j} \frac{x^j}{j!} \sum_{i=0}^{\infty} k^{\underline{i}} \frac{x^i}{i!}$$

$$= \alpha \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j H_{j+1}^k(\alpha) \frac{D_{n-j}^{r-1}}{(n-j)!\alpha^{n-j}} x^n \sum_{i=0}^{\infty} k^{\underline{i}} \frac{x^i}{i!}$$
$$= \alpha \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{i} \frac{(-1)^j k^{\underline{n-i}}}{\alpha^{i-j}(n-i)!(i-j)!} H_{j+1}^k(\alpha) D_{i-j}^{r-1} x^n.$$

Comparing the coefficients on both sides, we get the desired result. Similarly, from (1.2) and (1.3), the proof of the other result is clearly obtained. □

3. Some identities involving generalized harmonic numbers of rank r and Stirling numbers of the first kind

In this section, inspiring from the definition of $H(n,r)$ of rank r in their works [8, 13], $H(n,r,\alpha)$ are defined as for $n \geq 1$, $r \geq 0$,

$$H(n,r,\alpha) = \sum_{1 \leq n_0+n_1+\cdots+n_r \leq n} \frac{1}{n_0 n_1 \cdots n_r \alpha^{n_0+n_1+\cdots+n_r}} \tag{3.1}$$

or, equivalently, as

$$H(n,r,\alpha) = \frac{(-1)^{r+1}}{n!} \left(\frac{d^n}{dx^n} \frac{\left[\ln\left(1-\frac{x}{\alpha}\right)\right]^{r+1}}{1-x} \right) \Bigg|_{x=0}.$$

For $\alpha = 1$, $H(n,r,1) = H(n,r)$.

TABLE 1. $H(n,r,2)$

n \ r	0	1	2	3	4
1	$\frac{1}{2}$	0			
2	$\frac{5}{8}$	$\frac{1}{4}$	0		
3	$\frac{2}{3}$	$\frac{3}{8}$	$\frac{1}{8}$	0	
4	$\frac{131}{192}$	$\frac{83}{192}$	$\frac{7}{32}$	$\frac{1}{16}$	0
5	$\frac{661}{960}$	$\frac{11}{24}$	$\frac{35}{128}$	$\frac{1}{8}$	$\frac{1}{32}$

The generating function of the generalized harmonic numbers of rank r, $H(n,r,\alpha)$ is given by

$$\sum_{n=0}^{\infty} H(n,r,\alpha)x^n = \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{1-x}. \tag{3.2}$$

Now, we will give some sums with the help of the generating functions.

Theorem 4. *For any positive integers n, r, we have*

$$\begin{aligned} H(n,r,\alpha) &= (-1)^{n-r}\sum_{i=0}^{n} \frac{(-1)^i s(n-i,r)r!}{\alpha^{n-i}(n-i)!} H_i(\alpha) \\ &= (-1)^{r+1}\sum_{i=0}^{n} \frac{(-1)^i s(i,r+1)(r+1)!}{\alpha^i i!}. \end{aligned}$$

Proof. By (1.5) and (3.2), we consider

$$\begin{aligned} \sum_{n=0}^{\infty} H(n,r,\alpha)x^n &= \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{1-x} \\ &= \frac{-\ln\left(1-\frac{x}{\alpha}\right)}{1-x}\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^r \\ &= \sum_{n=0}^{\infty} H_n(\alpha)x^n \sum_{i=0}^{\infty} \frac{(-1)^{i-r} s(i,r)r!}{\alpha^i i!} x^i \\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{n} \frac{(-1)^{n-i-r} s(n-i,r)r!}{\alpha^{n-i}(n-i)!} H_i(\alpha)x^n. \end{aligned}$$

Comparing the coefficients on both sides, we arrive at the desired result. Similarly, using $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$, the other result is obtained. □

Theorem 5. *For any positive integers n, r, m, we have*

$$H(n,r,\alpha) = \sum_{i=0}^{n}\sum_{j=0}^{i} \binom{m-1}{n-i}(-1)^{n-j-r} \frac{H_j^m(\alpha)s(i-j,r)r!}{\alpha^{i-j}(i-j)!},$$

$$D_n^{r+1} = n!\alpha^{n+1}\sum_{i=0}^{n}\sum_{j=0}^{i} \binom{k}{n-i} \frac{(-1)^j D_{i-j}^r H_{j+1}^k(\alpha)}{\alpha^{i-j}(i-j)!}.$$

Proof. By (1.1), (1.5), (3.2) and Binomial theorem, we write

$$\begin{aligned} \sum_{n=0}^{\infty} H(n,r,\alpha)x^n &= \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{1-x} \\ &= \frac{-\ln\left(1-\frac{x}{\alpha}\right)}{(1-x)^m}\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^r (1-x)^{m-1} \end{aligned}$$

$$= \sum_{n=0}^{\infty} H_n^m(\alpha) x^n \sum_{j=0}^{\infty} \frac{(-1)^{j-r} s(j,r) r!}{\alpha^j j!} x^j \sum_{i=0}^{\infty} \binom{m-1}{i} (-1)^i x^i,$$

and using product of generating functions, equals

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(-1)^{n-j-r} s(n-j,r) r!}{\alpha^{n-j}(n-j)!} H_j^m(\alpha) x^n \sum_{i=0}^{\infty} \binom{m-1}{i} (-1)^i x^i$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{m-1}{n-i} (-1)^{n-j-r} \frac{H_j^m(\alpha) s(i-j,r) r!}{\alpha^{i-j}(i-j)!} x^n.$$

Thus, comparing the coefficients on both sides, the desired result is given.

Similarly, considering

$$\left(\frac{\ln\left(1+\frac{x}{\alpha}\right)}{\frac{x}{\alpha}} \right)^{r+1} = \alpha \frac{-\ln\left(1+\frac{x}{\alpha}\right)}{-x(1+x)^k} \left(\frac{\ln\left(1+\frac{x}{\alpha}\right)}{\frac{x}{\alpha}} \right)^{r} (1+x)^k$$

and the generating function of Daehee numbers, the proof of the other equality is complete. □

Theorem 6. *For any positive integers n, r, we have*

$$(-1)^{n+1-r} s(n+1,r) r! = \alpha^{n+1}(n+1)! \left(H(n+1,r-1,\alpha) - H(n,r-1,\alpha) \right).$$

Proof. By (1.5) and (3.2), we have

$$\sum_{n=r}^{\infty} \frac{(-1)^{n-r} s(n,r) r!}{\alpha^n n!} x^{n-1} = \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^r}{x} = \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^r}{1-x} \left(\frac{1}{x} - 1 \right)$$

$$= \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^r}{x(1-x)} - \frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^r}{1-x}$$

$$= \sum_{n=0}^{\infty} H(n,r-1,\alpha) x^{n-1} - \sum_{n=0}^{\infty} H(n,r-1,\alpha) x^n,$$

and from $H(0,r,\alpha) = 0$, equals

$$\sum_{n=1}^{\infty} H(n,r-1,\alpha) x^{n-1} - \sum_{n=0}^{\infty} H(n,r-1,\alpha) x^n$$

$$= \sum_{n=0}^{\infty} H(n+1,r-1,\alpha) x^n - \sum_{n=0}^{\infty} H(n,r-1,\alpha) x^n$$

$$= \sum_{n=0}^{\infty} \left(H(n+1,r-1,\alpha) - H(n,r-1,\alpha) \right) x^n. \tag{3.3}$$

With the help of $s(k,r) = 0$ for $0 \le k < r$, from here,

$$\sum_{n=r}^{\infty} \frac{(-1)^{n-r} s(n,r) r!}{\alpha^n n!} x^{n-1} = \sum_{n=r-1}^{\infty} \frac{(-1)^{n+1-r} s(n+1,r) r!}{\alpha^{n+1}(n+1)!} x^n$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^{n+1-r}s(n+1,r)r!}{\alpha^{n+1}(n+1)!}x^{n}. \tag{3.4}$$

From (3.3) and (3.4), we have the proof. □

Theorem 7. *For any positive integers n,r, we have*

$$\sum_{i=0}^{n}\binom{n-i+r}{r}\frac{(-1)^{i-r-1}s(i,r+1)(r+1)!}{\alpha^{i}i!}=\sum_{i=0}^{n}\binom{n-i+r-1}{r-1}H(i,r,\alpha),$$

$$\sum_{i=0}^{n}\binom{n}{i}\frac{(-1)^{i-r}s(i,r)r!}{\alpha^{i}}d_{n-i}=n!\sum_{i=0}^{n}\frac{(-1)^{n-i}}{(n-i)!}H(i,r-1,\alpha).$$

Proof. Observed that

$$\left(\frac{-\ln\left(1-\frac{x}{\alpha}\right)}{1-x}\right)^{r+1}=\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}\frac{1}{(1-x)^{r+1}}.$$

From $\sum_{n=0}^{\infty}\binom{n}{r}x^{n}=\frac{x^{r}}{(1-x)^{r+1}}$ and the generating functions of Stirling numbers of the first kind, we have

$$\left(\frac{-\ln\left(1-\frac{x}{\alpha}\right)}{1-x}\right)^{r+1}=\sum_{n=0}^{\infty}\frac{(-1)^{n-r-1}s(n,r+1)(r+1)!}{\alpha^{n}n!}x^{n}\sum_{i=0}^{\infty}\binom{i+r}{r}x^{i}$$

$$=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n-i+r}{r}\frac{(-1)^{i-r-1}s(i,r+1)(r+1)!}{\alpha^{i}i!}x^{n}, \tag{3.5}$$

and

$$\left(\frac{-\ln\left(1-\frac{x}{\alpha}\right)}{1-x}\right)^{r+1}=\frac{\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}{1-x}\frac{1}{(1-x)^{r}}$$

$$=\sum_{n=0}^{\infty}H(n,r,\alpha)x^{n}\sum_{i=0}^{\infty}\binom{i+r-1}{r-1}x^{i}$$

$$=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n-i+r-1}{r-1}H(i,r,\alpha)x^{n}. \tag{3.6}$$

By (3.5) and (3.6), we have the result. The proof of the other result is similar to the proof. Thus, the proof is complete. □

Theorem 8. *For any positive integers n,r, we have*

$$\sum_{i=0}^{n}H(i,r,\alpha)H(n-i,r,\alpha)$$

$$=\sum_{i=0}^{n}\left\{(1+i)H(i+1,r,\alpha)\frac{(-1)^{n-i-r-1}s(n-i,r+1)(r+1)!}{\alpha^{n-i}(n-i)!}\right.$$

$$-\frac{r+1}{\alpha}\sum_{j=0}^{i}\frac{(-1)^{j-r}s(j,r)r!}{\alpha^{n-i+j}j!}H(i-j,r,\alpha)\Bigg\}.$$

Proof. Define a function as

$$A(x) := \sum_{n=0}^{\infty} H(n,r,\alpha)x^n = (-1)^{r+1}\left(\frac{1-x}{\left(\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}}\right)^{-1}.$$

Derivating of $A(x)$ respect to x, we write

$$A'(x) = (-1)^r (A(x))^2\left(\frac{-1}{\left(\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}} + \frac{(-1)^{r+1}(r+1)(A(x))^{-1}}{(\alpha-x)\ln\left(1-\frac{x}{\alpha}\right)}\right)$$
$$= \frac{(-1)^{r+1}(A(x))^2}{\left(\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}} - \frac{(r+1)A(x)}{(\alpha-x)\ln\left(1-\frac{x}{\alpha}\right)},$$

and from here,

$$(A(x))^2 = \left(A'(x) + \frac{(r+1)A(x)}{(\alpha-x)\ln\left(1-\frac{x}{\alpha}\right)}\right)\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1}$$
$$= A'(x)\left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r+1} - \left(-\ln\left(1-\frac{x}{\alpha}\right)\right)^{r}\frac{(r+1)A(x)}{(\alpha-x)}$$
$$= \sum_{n=0}^{\infty}(n+1)H(n+1,r,\alpha)x^n\sum_{i=0}^{\infty}\frac{(-1)^{i-r-1}s(i,r+1)(r+1)!}{\alpha^i i!}x^i$$
$$-\frac{r+1}{\alpha}\sum_{n=0}^{\infty}\frac{(-1)^{n-r}s(n,r)r!}{\alpha^n n!}x^n\sum_{j=0}^{\infty}H(j,r,\alpha)x^j\sum_{i=0}^{\infty}\left(\frac{1}{\alpha}\right)^i x^i.$$

Using product of generating functions, we get

$$(A(x))^2 = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{(i+1)H(i+1,r,\alpha)(-1)^{n-i-r-1}s(n-i,r+1)(r+1)!}{\alpha^{n-i}(n-i)!}x^n$$
$$-\frac{r+1}{\alpha}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{(-1)^{j-r}s(j,r)r!H(i-j,r,\alpha)}{\alpha^{n-i+j}j!}x^n. \tag{3.7}$$

After that, we also have

$$(A(x))^2 = \sum_{n=0}^{\infty}\sum_{i=0}^{n}H(i,r,\alpha)H(n-i,r,\alpha)x^n. \tag{3.8}$$

By combining (3.7) and (3.8), the proof is over. □

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