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GROUND STATE SOLUTIONS FOR CHOQUARD EQUATION WITH LOGARITHMIC NONLINEARITY

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Abstract. In this paper, we study the following Choquard equation with logarithmic nonlinearity

$$\begin{cases} -\Delta u + V(x)u = k(I_\alpha * |u|^p)|u|^{p-2}u + H(x)u\log|u|, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}$$

where $\Omega\subset\mathbb{R}^3$ is a smooth bounded domain with boundary $\partial\Omega$, $k>0$ is real parameter, $\alpha\in(0,3), I_\alpha$ is the Riesz potential. Under some mild assumptions on V and H, we obtain a ground state solution by variational method and logarithmic inequality. In addition, we investigate the limit profiles of Choquard equations as $\alpha\to 0$ or $\alpha\to N$.

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Keywords: Choquard equation, logarithmic nonlinearity, variational method, ground state solution.

1. INTRODUCTION

In this paper, we consider the following Choquard equation

$$\begin{cases} -\Delta u + V(x)u = k(I_\alpha * |u|^p)|u|^{p-2}u + H(x)u\log|u|, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}\tag{1.1}$$

where $\Omega\subset\mathbb{R}^3$ is a smooth bounded domain with boundary $\partial\Omega$, $k>0$ is real parameter, $p\in(1+\alpha/3, 3+\alpha)$, $\alpha\in(0,3), I_\alpha$ is the Riesz potential, V,H are potential functions.

Choquard equation was firstly introduced by Pekar [16] in 1954 for describing the quantum mechanics of a polaron at rest, an electron trapped in its hole by Choquard [12]. Li and Tang [11] considered Choquard equation with the upper critical exponent, where the nonlinearity satisfies (f_0) $f\in C(\mathbb{R},\mathbb{R})$is odd, $(f_1)\lim_{t\to 0} f(t)/t = \lim_{t\to+\infty} f(t)/t^{2^*-1}=0$, where $2^*=2N/(N-2)$, (f_2) there exists $\mu>4$ such that

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$0 < \mu F(t) \leqslant f(t)t$ for all $t \neq 0$. When the nonlinear perturbation satisfies (f_0) $f(t) = o(t)$ as $t \to 0$, (f_1) $|f(t)| \leqslant a(|t| + |t|^{q-1})$ for some $a > 0$ and $q > 2$ with $1/q > 1/2 - 1/N$, (f_2) there exists $\mu > 4$ such that $0 < \mu F(t) \leqslant f(t)t$ for all $t \neq 0$, Choquard equations with lower critical exponent was studied in [22]. Multiple solutions to Kirchhoff type equations with Hardy-Littlewood-Sobolev critical nonlinearity was solved [5, 6]. Next, [10] proved the existence and concentrate behavior of ground state solutions for critical Choquard equations. Li, Gao and Liang [9] considered the existence and concentration of nontrivial nonnegative ground state solutions to Kirchhoff-type system with Hartree-type nonlinearity, where f satisfies (f_0) $f \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $C > 0$ such that $|f'(t)| \leqslant C(1 + |t|^{q-2})$ for all $t \in \mathbb{R}_+$, (f_1) $f(t)/t^{2p-1}$ is increasing on $(0, \infty)$, $\lim_{t\to\infty} f(t)/t^{2p-1} = \infty$ and $p \in (3 + \alpha)$. More results about Choquard equation, we can refer to [15, 20, 25].

Recently, logarithmic nonlinearity appears frequently in partial differential equations, which describe physical phenomena, for example continuum mechanics, phase transition phenomena, and population dynamics. On the other hand, for the parabolic equations, we can refer to [2, 3, 8, 14] and the references therein. For partial equations with logarithmic nonlinearity, we can refer to [1, 7, 13, 18, 21, 24] and the references therein. Specially, multiple solutions for the semilinear elliptic equations with logarithmic nonlinearity by Nehari manifold was studied in [18]. Liu and Xiao [13] analyzed ground state solutions for a fourth-order nonlinear elliptic problem with logarithmic nonlinearity, where $H = 1$. When $H \in C^1(\mathbb{R}^3), \min_{\mathbb{R}^3} H > 0$ and $\min_{\mathbb{R}^3}(V + H) > 0$, multiple solutions to Schrödinger equation with periodic potential and logarithmic nonlinearity was solved in [21]. Furthermore, the $\max_{\mathbb{R}^3} V \in (-1, \infty)$ and $H = 1$ with ground sate solutions was seen [1]. Le [24] studied a fractional p-Laplacian equation in the whole space with the sign-changing logarithmic nonlinearity by using Nehari manifold.

However, the existence of ground state solutions for Choquard equation with logarithmic nonlinearity has not been studied. In this paper, we assume that V and H satisfy:

(V): $V \in L^{3/2}(\Omega)$ and $|V^-|_{3/2} < S$, where $V^+ = \max\{V, 0\}, V^- = \min\{V, 0\}$, and $S = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \|u\|^2_{H_0^1} / |u|_6^2$;

(H): $H \in C(\overline{\Omega}), \mu := \inf_\Omega H > 0$, satisfying

$$\max_{\overline{\Omega}} H \leqslant 2\pi(1 - S^{-1}|V^-|_{3/2})/e^{-8|\Omega|^{1/2}+2}.$$

According to the condition (V), there exists $H \in C(\overline{\Omega})$ and $\min_\Omega(V + H) < 0$. So, our conditions are more complex than [1, 13, 21]. Notice that the logarithmic nonlinearity does not satisfy (f_0),(f_1) and (f_2), which means that the logarithmic nonlinear does not satisfy the subcritical or Ambrosetti-Rabinowitz condition. Therefore, the logarithmic nonlinearity can not be replaced by the general nonlinear term [9–11, 22]. Meanwhile, there is no logarithmic Sobolev inequality concerning to the logarithmic nonlinearity with H. In order to overcome the above difficulties, we introduce a new logarithmic inequality.

In this paper, it's worth noting that we obtain a new existence Theorem 3 on the ground state solution for the equation (1.1), which is quite different from these in the polynomial case. On the other hand, our method and conclusion are different from [18, 24]. In addition, we are interested in limit of behaviors of ground solution to (1.1) as either $\alpha \to 0$ or $\alpha \to N$.

Theorem 1. *Suppose that* (V) *and* (H) *hold. Then there exists a ground solution to* (1.1).

Remark 1. Notice that the logarithmic nonlinearity contains potential H. Therefore, the logarithmic Sobolev inequality cannot be applied directly. By condition (V), it is not difficult to find that our potential function is sign-change potential function. In addition, (1.1) contains Hatree nonlinearity. To get our conclusions, it is crucial to deal with the relationship between H, V and Hatree nonlinearity.

Theorem 2. *Let* $\{u_a\}$ *be a family of ground solution to* (1.1) *for* α *close to 0. Then there exists* u_0 *such that* $u_\alpha \to u_0$ *and* u_0 *is a nontrivial solution of the equation*

$$\begin{cases} -\Delta u + V(x)u = k|u|^{2p-2}u + H(x)u\log|u|, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Let $\{u_a\}$ *be a family of ground solution to* (1.1) *for* α *close to N. Then there exists* u_N *such that* $u_\alpha \to u_N$ *and* u_N *is a nontrivial solution of the equation*

$$\begin{cases} -\Delta u + V(x)u = k\left(\int_\Omega |u|^p\right)|u|^{p-2}u + H(x)u\log|u|, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

Remark 2. Seok [19] considered the limit profiles and uniqueness of ground states to the nonlinear Choquard equations, namely, $\alpha \to 0, \alpha \to N$. When logarithmic nonlinearity does not exist, as $\alpha \to 0$, it reduces to the Euler-Lagrange equation. On the other hand, inspired by [19], we are interested to consider the case of $\alpha \to N$. In addition, by studying the limit profiles, we can better understand the properties of logarithmic nonlinearity.

Theorem 3. *When* $k = 0$*, the equation* (1.1) *has a ground state solution.*

Remark 3. Similarly, we can also consider the following fractional Schrödinger equation

$$\begin{cases} (-\Delta)^s u + V(x)u = H(x)u\log|u|, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain with boundary $\partial\Omega$, $s \in (0,1)$. We assume that the potential functions H and V satisfy:

(V_1) $V \in L^{3/(2s)}(\Omega)$ and $|V^-|_{3/(2s)} < S$, where $V^+ = \max\{V,0\}, V^- = \min\{V,0\}$ and $S = \inf_{u \in H_0^s(\Omega)\setminus\{0\}} \|u\|^2_{H_0^s}/|u|^2_{6/(3-2s)}$.

(H_1) $H \in C(\overline{\Omega}), \mu := \inf_\Omega H > 0$, satisfying

$$\max_{\overline{\Omega}} H \leqslant \pi^s \left(1 - S^{-1}|V^-|_{3/2s}\right) / \left(\frac{\Gamma(\frac{3}{2s})}{(\frac{3}{2})} e^{\frac{2|\Omega|^{1/2}-2}{3}}\right)^{6/s}.$$

Assume that $(V_1), (H_1)$ hold, (1.4) has a ground state solution. A complete introduction to fractional Sobolev space and fractional logarithmic Sobolev inequality, we can refer to [4, 17, 23].

2. Preliminaries

In this paper, we make use of the following notation:

- $H_0^1(\Omega)$ is the usual Hilbert space with the norm

$$\|u\|_{H_0^1} = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{1/2}.$$

- $L^p(\Omega)(1 \leqslant p < \infty)$ is the Lebesgue space with the norm

$$|u|_p = \left(\int_\Omega |u|^p \mathrm{d}x\right)^{1/p}.$$

- $C, C_i, i = 1, 2, \dots,$ denote various positive constants.
- M denotes $\max_{\overline{\Omega}} H$.
- $|\Omega|$denotes the Lebesgue measure of Ω.
- From the Sobolev and Rellich embedding theorem, the embedding $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ is continuous for $s \in [2,6]$ and is compact for $s \in [2,6)$.

A weak solution to the equation (1.1) is a critical point of the following energy functional J defined on $H_0^1(\Omega)$ by

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega (|\nabla u|^2 + Vu^2)\mathrm{d}x - \frac{k}{2p}\int_\Omega (I_\alpha * |u|^p)|u|^p \mathrm{d}x \\ &\quad -\frac{1}{2}\int_\Omega Hu^2 \log|u|\mathrm{d}x + \frac{1}{4}\int_\Omega Hu^2\mathrm{d}x, \quad u \in H_0^1(\Omega). \end{aligned}$$

It is easy to proof that J is well defined on $H_0^1(\Omega)$ and $J \in C^1(H_0^1(\Omega), \mathbb{R})$. Furthermore,

$$\begin{aligned} \langle J'(u), v\rangle &= \int_\Omega (\nabla u \cdot \nabla v + Vuv)\mathrm{d}x - k\int_\Omega (I_\alpha * |u|^p)|u|^{p-2}uv\mathrm{d}x \\ &\quad - \int_\Omega Huv \log|u|\mathrm{d}x, \quad v \in H_0^1(\Omega). \end{aligned}$$

For each $u \in H_0^1(\Omega)$, by the Hölder inequality and the condition (V), we have

$$\begin{aligned}\int_\Omega |\nabla u|^2 \mathrm{d}x + \int_\Omega V u^2 \mathrm{d}x &\geqslant \int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega |V^-| u^2 \mathrm{d}x \\ &\geqslant \left(1 - S^{-1}|V^-|_{3/2}\right) \int_\Omega |\nabla u|^2 \mathrm{d}x := \delta \int_\Omega |\nabla u|^2 \mathrm{d}x,\end{aligned} \tag{2.1}$$

where $\delta = 1 - S^{-1}|V^-|_{3/2}$.

Proposition 1. *(Logarithmic Sobolev inequality [8, 14]) Let u be a function in $H^1(\mathbb{R}^3) \setminus \{0\}$ and $a > 0$ be a constant. Then*

$$2\int_{\mathbb{R}^3} u^2 \log \frac{|u|}{|u|_2} \mathrm{d}x + 3(1+\log a)|u|_2^2 \mathrm{d}x \leqslant \frac{a^2}{\pi} \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x.$$

For any $u \in H_0^1(\Omega) \setminus \{0\}$, we define $u(x) = 0$ for $x \in \mathbb{R}^3 \setminus \Omega$. According to the logarithmic Sobolev inequality, we have

$$2\int_\Omega u^2 \log \frac{|u|}{|u|_2} \mathrm{d}x + 3(1+\log a)|u|_2^2 \leqslant \frac{a^2}{\pi} \int_\Omega |\nabla u|^2 \mathrm{d}x. \tag{2.2}$$

Proposition 2. *(Hardy-Littlewood-Sobolev inequality [5, 6]) Let $s, t > 1$ and $\alpha \in (0,3)$ with $1/s + 1/r = 1 + \alpha/3$. Then there exists a sharp constant $C(\alpha, s, r) > 0$ such that for any $g \in L^s(\Omega)$ and $h \in L^r(\Omega)$*

$$\int_\Omega \int_\Omega \frac{g(x)h(y)}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y \leqslant C(\alpha, s, r)|g|_s |h|_r.$$

3. Proofs of main results

Lemma 1. *There exists a $C > 0$ such that,*

$$|t \log |t|| \leqslant C(1+t^2), \quad t \in \mathbb{R}. \tag{3.1}$$

Proof. Define

$$f(t) = \begin{cases} \frac{|t \log |t||}{1+t^2}, & t \neq 0. \\ 0, & t = 0. \end{cases}$$

Clearly,

$$\lim_{|t| \to 0} f(t) = 0, \quad \lim_{|t| \to \infty} f(t) = 0. \tag{3.2}$$

It follows from (3.2) that (3.1) holds. □

Lemma 2. *For any $u \in H_0^1(\Omega) \setminus \{0\}$, then we have*

$$\int_\Omega H u^2 \log \frac{|u|}{|u|_2} \mathrm{d}x \leqslant M \left(\frac{a^2}{2\pi} \int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{3(1+\log a)}{2} |u|_2^2 + 2|\Omega|^{1/2} |u|_2^2 \right).$$

Proof. For $\theta > 0$, we have

$$\log\theta < \theta. \tag{3.3}$$

For any $u \in H_0^1(\Omega)\setminus\{0\}$, then we have

$$\int_\Omega Hu^2\log\frac{|u|}{|u|_2}\mathrm{d}x = \int_{\Omega_1} Hu^2\log\frac{|u|}{|u|_2}\mathrm{d}x + \int_{\Omega\setminus\Omega_1} Hu^2\log\frac{|u|}{|u|_2}\mathrm{d}x, \tag{3.4}$$

where $\Omega_1 = \{x\in\Omega : |u(x)|/|u|_2 < 1\}$. According to the Hölder inequality and (3.3), we deduce

$$\int_{\Omega_1} Hu^2\log\frac{|u|}{|u|_2}\mathrm{d}x \leqslant M\int_{\Omega_1} u^2\log\frac{|u|_2}{|u|}\mathrm{d}x \leqslant M|\Omega|^{1/2}|u|_2^2. \tag{3.5}$$

Observing (2.2), (3.4) and (3.5) that

$$\begin{aligned}
\int_{\Omega\setminus\Omega_1} Hu^2\log\frac{|u|}{|u|_2}\mathrm{d}x &\leqslant M\left(\int_\Omega u^2\log\frac{|u|}{|u|_2}\mathrm{d}x - \int_{\Omega_1} u^2\mathrm{d}x\log\frac{|u|}{|u|_2}\mathrm{d}x\right) \\
&\leqslant M\left(\int_\Omega u^2\log\frac{|u|}{|u|_2}\mathrm{d}x + |\Omega|^{1/2}|u|_2^2\right) \\
&\leqslant M\left(\frac{a^2}{2\pi}\int_\Omega|\nabla u|^2\mathrm{d}x - \frac{3(1+\log a)}{2}|u|_2^2 + |\Omega|^{1/2}|u|_2^2\right).
\end{aligned} \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\int_\Omega Hu^2\log\frac{|u|}{|u|_2}\mathrm{d}x \leqslant M\left(\frac{a^2}{2\pi}\int_\Omega|\nabla u|^2\mathrm{d}x - \frac{3(1+\log a)}{2}|u|_2^2 + 2|\Omega|^{1/2}|u|_2^2\right).$$

□

Lemma 3. *The functional J has the mountain pass geometry:*

(i) *there exists $\rho > 0$ such that $\inf_{\|u\|=\rho} J(u) > 0$.*

(ii) *for any $u \in H_0^1(\Omega)\setminus\{0\}$, it holds $\lim_{t\to\infty} J(tu) = -\infty$.*

Proof. According to Hardy-Littlewood-Sobolev Inequality, we have

$$\int_\Omega (I_\alpha * |u|^p)|u|^p\mathrm{d}x \leqslant C_1|u|_{pr}^{2p}, \tag{3.7}$$

where $r = 6/(3+\alpha)$. For any $u\in H_0^1(\Omega)\setminus\{0\}$, by (2.1), (3.3), (3.7) and Lemma 2, we get

$$\begin{aligned}
J(u) &\geqslant \frac{1}{2}\int_\Omega \delta|\nabla u|^2\mathrm{d}x - \frac{1}{2}\int_\Omega Hu^2\log\frac{|u|}{|u|_2}\mathrm{d}x - \frac{1}{2}\int_\Omega Hu^2\log|u|_2\mathrm{d}x \\
&\quad - C_1k|u|_{pr}^{2p} \\
&\geqslant \frac{1}{4}\left(2\delta - \frac{Ma^2}{\pi}\right)\int_\Omega|\nabla u|^2\mathrm{d}x + \frac{1}{4}(3+3\log a - 4|\Omega|^{1/2})|u|_2^2 \\
&\quad - \frac{M}{2}(\int_\Omega u^2\mathrm{d}x)^{3/2} - C_2k\|u\|^{2p}.
\end{aligned} \tag{3.8}$$

According to the arbitrariness of a in logarithmic Sobolev inequality, we take $a = e^{\frac{4|\Omega|^{1/2}-3}{3}}$, we know

$$3+3\log a-4|\Omega|^{1/2}>0.$$

According to the condition (H), we have

$$2\delta-\frac{Ma^2}{\pi}>0.$$

When $\|u\|=\rho>0$ is small enough, according to (3.8) and the condition (H), we complete the proof of (i). For any $u\in H_0^1(\Omega)\setminus\{0\}$ and $t>0$, we have

$$\begin{aligned}
J(tu)&=\frac{1}{2}\int_\Omega(|\nabla tu|^2+V(tu)^2)\mathrm{d}x-\frac{k}{2p}\int_\Omega(I_\alpha*|tu|^p)|tu|^p\mathrm{d}x\\
&\quad-\frac{1}{2}\int_\Omega H(tu)^2\log|tu|\mathrm{d}x+\frac{1}{4}\int_\Omega H(tu)^2\mathrm{d}x\\
&=\frac{t^2}{2}\int_\Omega(|\nabla u|^2+Vu^2)\mathrm{d}x-\frac{t^{2p}k}{2p}\int_\Omega(I_\alpha*|u|^p)|u|^p\mathrm{d}x\\
&\quad-\frac{t^2\log t}{2}\int_\Omega Hu^2\mathrm{d}x-\frac{t^2}{2}\int_\Omega Hu^2\log|u|\mathrm{d}x+\frac{t^2}{4}\int_\Omega Hu^2\mathrm{d}x,
\end{aligned}$$

and the conclusion (ii) follows. □

Lemma 4. *Suppose that $\{u_n\}$ is a sequence in $H_0^1(\Omega)$ such that $u_n\rightharpoonup u$, then we have*

$$\lim_{n\to\infty}\int_\Omega(I_\alpha*|u_n|^p)|u_n|^p\mathrm{d}x=\int_\Omega(I_\alpha*|u|^p)|u|^p\mathrm{d}x.$$

For any $v\in H_0^1(\Omega)$, we have

$$\lim_{n\to\infty}\int_\Omega(I_\alpha*|u_n|^p)|u_n|^{p-1}u_nv\mathrm{d}x=\int_\Omega(I_\alpha*|u|^p)|u|^{p-1}uv\mathrm{d}x.$$

Proof. Firstly, by Fatou's lemma, then we have

$$\int_\Omega(I_\alpha*|u|^p)|u|^p\mathrm{d}x\leqslant\liminf_{n\to\infty}\int_\Omega(I_\alpha*|u_n|^p)|u_n|^p\mathrm{d}x. \tag{3.9}$$

From that $u_n\rightharpoonup u$ in $H_0^1(\Omega)$ and $6p/(3+\alpha)\in[2,6)$, we have that $u_n\to u$ in $L^{6p/(3+\alpha)}(\Omega)$. Therefore, $|u_n|^p\to|u|^p$ in $L^{6/(3+\alpha)}(\Omega)$. Combining with Proposition 2, we obtain

$$\int_\Omega[I_\alpha*(|u_n|^p-|u|^p|)]|u_n|^p\mathrm{d}x\leqslant C_{\alpha,p}||u_n|^p-|u|^p|_{6/(3+\alpha)}|u_n^p|_{6/(3+\alpha)}, \tag{3.10}$$

According to (3.10), as $n\to\infty$, we have

$$\int_\Omega[I_\alpha*(|u_n|^p-|u|^p|)]|u_n|^p\mathrm{d}x\to0. \tag{3.11}$$

Similarly, we get

$$\int_{\Omega}(I_\alpha * |u|^p)||u_n|^p - |u|^p|\mathrm{d}x = o(1). \tag{3.12}$$

According to (3.9), (3.11) and (3.12), we obtain

$$\lim_{n\to\infty}\int_{\Omega}[I_\alpha * (|u_n|^p)]|u_n|^p\mathrm{d}x = \int_{\Omega}[I_\alpha * (|u|^p)]|u|^p\mathrm{d}x.$$

It follows from Proposition 2 that $I_\alpha * (|u|^{p-2}uv) \in L^{6/(3+\alpha)}(\Omega)$.

As $n\to\infty$, we get

$$\begin{aligned}
&\left|\int_{\Omega}(I_\alpha * |u_n|^p)\ |u_n|^{p-1}u_n v\mathrm{d}x - \int_{\Omega}(I_\alpha * |u|^p)|u|^{p-1}uv\mathrm{d}x\right| \\
&\quad\leqslant \int_{\Omega}|(I_\alpha * |u_n|^p)[|u_n|^{p-1}u_n v - |u|^{p-1}uv]|\mathrm{d}x \\
&\qquad + \left|\int_{\Omega}[I_\alpha * |u_n|^p - I_\alpha * |u|^p]|u|^{p-1}uv\mathrm{d}x\right| \\
&\quad\leqslant |I_\alpha * |u_n|^p|_{6/(3-\alpha)}||u_n|^{p-1}u_n v - |u|^{p-1}uv|_{6/(3-\alpha)} \\
&\qquad + \left|\int_{\Omega}[I_\alpha * |u_n|^p - I_\alpha * |u|^p]|u|^{p-1}uv\mathrm{d}x\right| \to 0.
\end{aligned} \tag{3.13}$$

The proof is complete. □

Now, we prove that the functional J satisfies (PS) condition.

Lemma 5. *J satisfies (PS) condition.*

Proof. Assume that $\{u_n\}\subset H_0^1(\Omega)$ is a (PS) sequence certifies $|J(u_n)| < b$ for some positive constant b and $J'(u_n)\to 0$, as $n\to\infty$. We claim $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Indeed, it follows from that for n large enough that

$$\begin{aligned}
\frac{\mu}{4}\int_{\Omega}u_n^2\mathrm{d}x &\leqslant \frac{1}{4}\int_{\Omega}Hu_n^2\mathrm{d}x = J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle \\
&\quad - \left(\frac{1}{2} - \frac{1}{2p}\right)k\int_{\Omega}(I_\alpha * |u_n|^p)|u_n|^p\mathrm{d}x \leqslant b + o(1)\|u_n\|.
\end{aligned} \tag{3.14}$$

It follows from (3.3) and (3.14) that

$$\begin{aligned}
\int_{\Omega}H|u_n|^2\log|u_n|_2\mathrm{d}x &= \log|u_n|_2\int_{\Omega}H|u_n|^2\mathrm{d}x \leqslant M|u_n|_2\int_{\Omega}|u_n|^2\mathrm{d}x \\
&\leqslant \frac{8M}{\mu^{3/2}}(4b + o(1)\|u_n\|)^{3/2}.
\end{aligned} \tag{3.15}$$

setting $\gamma = \frac{1}{2} - \frac{1}{2p}$, according to (3.14), (3.15) and Lemma 2, we have

$$b + o(1)\|u_n\| \geqslant J(u_n) - \frac{1}{2p}\langle J'(u_n), u_n\rangle$$

$$
\begin{aligned}
&\geqslant \frac{\gamma}{2}\left(2\delta-\frac{Ma^2}{\pi}\right)\int_\Omega |\nabla u|^2\mathrm{d}x+\frac{\gamma}{2}(3+3\log a-4|\Omega|^{1/2})|u|_2^2\\
&\quad -\gamma\int_\Omega H|u_n|^2\log|u_n|_2\mathrm{d}x\\
&\geqslant \frac{\gamma}{2}\left(2\delta-\frac{Ma^2}{\pi}\right)\|u_n\|^2-\left|\frac{\gamma}{2\mu}(3+3\log a-4|\Omega|^{1/2})\right|(b+o(1)\|u_n\|)\\
&\quad -\frac{8M\gamma}{\mu^{3/2}}(4b+o(1)\|u_n\|)^{3/2},
\end{aligned}
$$

which concludes the claim. So, going if necessary to a subsequence (still denote $\{u_n\}$), we can assume that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$. It follows from (3.1) that

$$
\left|\int_\Omega (Hu_n\log|u_n|-Hu\log|u|)(u_n-u)\mathrm{d}x\right|\to 0. \tag{3.16}
$$

By Lemma 4, we have

$$
\left|\int_\Omega [(I_\alpha*|u_n|^p)|u_n|^{p-2}u_n-(I_\alpha*|u|^p)|u|^{p-2}u](u_n-u)\mathrm{d}x\right|\to 0. \tag{3.17}
$$

On the other hand, we have

$$
\begin{aligned}
\|u_n-u\|^2=&\langle J'(u_n)-J'(u),u_n-u\rangle+\int_\Omega (u_n\log|u_n|-u\log|u|)(u_n-u)\mathrm{d}x\\
&+\int_\Omega [(I_\alpha*|u_n|^p)|u_n|^{p-2}u_n-(I_\alpha*|u|^p)|u|^{p-2}u](u_n-u)\mathrm{d}x\\
&-\int_\Omega (Vu_n-Vu)(u_n-u)\mathrm{d}x.
\end{aligned}\tag{3.18}
$$

It is clear that

$$
\langle J'(u_n)-J'(u),u_n-u\rangle\to 0. \tag{3.19}
$$

and

$$
\int_\Omega (Vu_n-Vu)(u_n-u)\mathrm{d}x\to 0. \tag{3.20}
$$

Therefore, according to (3.16)-(3.20), we get $\|u_n-u\|\to 0$. The proof is complete. □

Proof of Theorem 1. According to Lemma 3 and Lemma 5, (1.1) exists a nontrivial solution. Set $K=\{u\in H_0^1(\Omega)\setminus\{0\}: J'(u)=0\}$, we know K is nonempty. So for any $u\in K$, then

$$
\begin{aligned}
J(u)-\frac{1}{2}\langle J'(u),u\rangle&=\left(\frac{1}{2}-\frac{1}{2p}\right)k\int_\Omega (I_\alpha*|u_n|^p)|u_n|^p\mathrm{d}x+\frac{1}{4}\int_\Omega Hu^2\mathrm{d}x\\
&\geqslant \frac{1}{4}\int_\Omega Hu^2\mathrm{d}x\geqslant 0.
\end{aligned}
$$

So we may define $c_\alpha = \inf\{J(u) : u \in K\}$. Let $\{u_n\} \subseteq K$ be such that $J(u_n) \to c_\alpha$, as $n \to \infty$. By Lemma 5, there exists a subsequence (still denote $\{u_n\}$) and $u_n \to u_0$ in $H_0^1(\Omega)$. By the continuation of J', we imply that $J(u_0) = c_\alpha$ and $J'(u_0) = 0$. We show that $u_0 \neq 0$. Since $u_n \in K$, we have

$$\begin{aligned}\|u_n\|^2 &= k\int_\Omega (I_\alpha * |u_n|^p)|u_n|^p \mathrm{d}x + \int_\Omega Hu_n^2 \log|u_n| \mathrm{d}x \\ &\leqslant C_2\|u_n\|^{2p} + C_4\|u_n\|^4.\end{aligned} \tag{3.21}$$

It follows from $u_n \in K$ and (3.21) that

$$1 \leqslant C_2\|u_n\|^{2p-2} + C_4\|u_n\|^2.$$

Therefore, we imply that $u_0 \neq 0$. It completes the proof. $\square$

Proof of Theorem 2. A weak solution to the equation (1.2) is a critical point of the following energy functional J_1 defined on $H_0^1(\Omega)$ by

$$J_1(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + Vu^2)\mathrm{d}x - \frac{k}{2p}\int_\Omega |u|^{2p}\mathrm{d}x - \frac{1}{2}\int_\Omega Hu^2\log|u| + \frac{1}{4}\int_\Omega Hu^2 \mathrm{d}x,$$

for all $u \in H_0^1(\Omega)$. It is easy to proof that J_1 is well defined on $H_0^1(\Omega)$ and $J \in C^1(H_0^1(\Omega), \mathbb{R})$. Furthermore,

$$\langle J_1'(u), v\rangle = \int_\Omega (\nabla u \cdot \nabla v + Vuv)\mathrm{d}x - k\int_\Omega |u|^{p-2}uv\mathrm{d}x - \int_\Omega Huv\log|u|\mathrm{d}x,$$

for all $v \in H_0^1(\Omega)$. $\square$

Lemma 6. *Fix $1 < p < \frac{2N}{N-2}$. Let $\{\alpha_j\} > 0$ be a sequence converging to 0 and let $\{u_j\} \subset H_0^1(\Omega)$ be a sequence converging weakly in $H_0^1(\Omega)$ to some $u_0 \in H_0^1(\Omega)$. Then, as $j \to \infty$, the following holds:*

$$\int_\Omega (I_{\alpha_j} * |u_j|^p)|u_j|^p \mathrm{d}x = \int_\Omega |u_0|^{2p}\mathrm{d}x.$$

For any $v \in H_0^1(\Omega)$, we have

$$\int_\Omega (I_{\alpha_j} * |u_j|^p)|u_j|^{p-1}u_j v\mathrm{d}x = \int_\Omega |u_0|^{2p-2}u_0 v\mathrm{d}x.$$

Proof. For the proof of the lemma, we refer to ([19], Proposition 2.7). $\square$

Lemma 7. *Let $\{u_a\}$ be a family of ground solution to (1.1) for α close to 0 and then there exists u_0 such that u_0 is a nontrivial solution of the equation (1.2).*

Proof. Let $\{u_a\}$ be a family of ground solutions to (1.1) for α close to 0. Similarly to proof of Theorem 1, define $c_{2p} = \inf\{J_1(u) : u \in K_1\}$, where $K_1 = \{u \in H_0^1(\Omega) \setminus \{0\} : J_1'(u) = 0\}$. Given $u \in H_0^1(\Omega) \setminus \{0\}$, as $\alpha \to 0$, we get

$$c_\alpha \leqslant \max_{t>0} J(tu) = \max_{t>0} \frac{t^2}{2}\int_\Omega (|\nabla u|^2 + Vu^2)\mathrm{d}x - \frac{t^{2p}k}{2p}\int_\Omega (I_\alpha * |u|^p)|u|^p \mathrm{d}x$$

$$
\begin{aligned}
&\quad -\frac{t^2\log t}{2}\int_\Omega Hu^2\mathrm{d}x-\frac{t^2}{2}\int_\Omega Hu^2\log|u|\mathrm{d}x+\frac{t^2}{4}\int_\Omega Hu^2\mathrm{d}x\\
&\to \max_{t>0}\frac{t^2}{2}\int_\Omega(|\nabla u|^2+Vu^2)\mathrm{d}x-\frac{t^{2p}k}{2p}\int_\Omega|u|^{2p}\mathrm{d}x\\
&\quad -\frac{t^2\log t}{2}\int_\Omega Hu^2\mathrm{d}x-\frac{t^2}{2}\int_\Omega Hu^2\log|u|\mathrm{d}x+\frac{t^2}{4}\int_\Omega Hu^2\mathrm{d}x\\
&=\max_{t>0}J_1(tu).
\end{aligned}
$$

Taking the infimum with respect to $u\in H_0^1(\Omega)\setminus\{0\}$, we deduce that

$$
\limsup_{\alpha\to 0}c_\alpha\leqslant c_{2p}. \tag{3.22}
$$

Since u_a is a ground solution to (1.1), by Lemma 5 and (3.22), we obtain that $\{u_a\}$ is bounded. So, there exists $\{u_j\}\subset H_0^1(\Omega)$ be a sequence converging weakly in $H_0^1(\Omega)$ to $u_0\in H_0^1(\Omega)$. We claim that u_0 is a weak solution of the equation (1.2). Indeed,

$$
\begin{aligned}
&\lim_{j\to\infty}\int_\Omega(\nabla u_j\cdot\nabla v+Vu_jv)\mathrm{d}x-k\int_\Omega(I_\alpha*|u_{\alpha_j}|^p)|u_j|^{p-2}uv\mathrm{d}x-\int_\Omega Hu_jv\log|u_j|\mathrm{d}x\\
&\quad=\int_\Omega(\nabla u_0\cdot\nabla v+Vu_0v)\mathrm{d}x-k\int_\Omega|u_0|^{p-2}u_0v\mathrm{d}x-\int_\Omega Hu_0v\log|u_0|\mathrm{d}x.
\end{aligned}\tag{3.23}
$$

Since u_j is a ground solution of equation (1.1), then we obtain

$$
\begin{aligned}
\langle J'(u_j),v\rangle&=\int_\Omega(\nabla u_j\cdot\nabla v+Vu_jv)\mathrm{d}x-k\int_\Omega(I_{\alpha_j}*|u_j|^p)|u_j|^{p-2}uv\mathrm{d}x\\
&\quad-\int_\Omega Hu_jv\log|u_j|\mathrm{d}x=0.
\end{aligned}\tag{3.24}
$$

It follows from (3.23) and (3.24) that

$$
\begin{aligned}
&\int_\Omega(\nabla u_0\cdot\nabla v+Vu_0v)\mathrm{d}x-k\int_\Omega|u_0|^{p-2}u_0v\mathrm{d}x-\int_\Omega Hu_0v\log|u_0|\mathrm{d}x\\
&\quad=0=\langle J_1'(u_0),v\rangle,
\end{aligned}\tag{3.25}
$$

which concludes the claim. According to (3.1), we have

$$
\lim_{j\to\infty}\int_\Omega Hu_j^2\log|u_j|\mathrm{d}x=\int_\Omega Hu_0^2\log|u_0|\mathrm{d}x. \tag{3.26}
$$

Since u_0 is a weak solution of the equation (1.2), by the (3.26) and Lemma 6, we imply

$$
\begin{aligned}
\lim_{j\to\infty}\|u_j\|&=\lim_{j\to\infty}\left(\int_\Omega(I_{\alpha_j}*|u_j|^p)|u_j|^p\mathrm{d}x+\int_\Omega Hu_j^2\log|u_j|\mathrm{d}x\right)\\
&=\int_\Omega|u_0|^{2p}\mathrm{d}x+\int_\Omega Hu_0^2\log|u_0|\mathrm{d}x=\|u_0\|.
\end{aligned}\tag{3.27}
$$

According to (3.27) and $u_j \rightharpoonup u_0$, we have $u_j \to u_0$. Since u_j is a ground solution of equation (1.1), we get

$$\|u_j\|^2 = k\int_\Omega (I_\alpha * |u_j|^p)|u_j|^p \mathrm{d}x + \int_\Omega H u_j^2 \log|u_j| \mathrm{d}x \leqslant C_2\|u_j\|^{2p} + C_4\|u_j\|^4, \quad (3.28)$$

Observe from (3.28) that

$$1 \leqslant C_2\|u_j\|^{2p-2} + C_4\|u_j\|^2. \quad (3.29)$$

According to (3.29) and $u_j \to u_0$, we imply that u_0 is a nontrivial solution of the equation (1.2). □

A weak solution to the equation (1.3) is a critical point of the following energy functional J_2 defined on $H_0^1(\Omega)$ by

$$J_2(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + Vu^2)\mathrm{d}x - \frac{k}{2p}\left(\int_\Omega |u|^p \mathrm{d}x\right)^2 - \frac{1}{2}\int_\Omega Hu^2\log|u|\mathrm{d}x + \frac{1}{4}\int_\Omega Hu^2\mathrm{d}x,$$

for all $u \in H_0^1(\Omega)$. It is easy to proof that J_2 is well defined on $H_0^1(\Omega)$ and $J_2 \in C^1(H_0^1(\Omega), \mathbb{R})$. Furthermore,

$$\langle J_2'(u), v\rangle = \int_\Omega (\nabla u\cdot\nabla v + Vuv)\mathrm{d}x - k\int_\Omega |u|^p \int_\Omega |u|^{p-2}uv\mathrm{d}x - \int_\Omega Huv\log|u|\mathrm{d}x,$$

for all $v \in H_0^1(\Omega)$.

Lemma 8. *Fix $1 < p < \frac{2N}{N-2}$. Let $\{\alpha_j\} > 0$ be a sequence converging to N and let $\{u_j\} \subset H_0^1(\Omega)$ be a sequence converging weakly in $H_0^1(\Omega)$ to some $u_N \in H_0^1(\Omega)$. Then, as $j \to \infty$, the following holds:*

$$\int_\Omega (I_{\alpha_j} * |u_j|^p)|u_j|^p \mathrm{d}x = \left(\int_\Omega |u_N|^p \mathrm{d}x\right)^2.$$

For any $v \in H_0^1(\Omega)$, we have

$$\int_\Omega (I_{\alpha_j} * |u_j|^p)|u_j|^{p-1}u_n v\mathrm{d}x = \int_\Omega |u_N|^p \int_\Omega |u_N|^{p-2}u_N v\mathrm{d}x.$$

Proof. For a proof of the lemma, we refer to ([19], Proposition 2.8). □

Lemma 9. *Let $\{u_a\}$ be a family of ground solution to (1.1) for α close to N and then there exists u_N such that u_N is a nontrivial solution of the equation (1.3).*

Proof. Let $\{u_a\}$ be a family of ground solution to (1.1) for α close to N. Similarly to proof of Theorem 1, define $c_p = \inf\{J_2(u) : u \in K_2\}$, where $K_2 = \{u \in H_0^1(\Omega)\setminus\{0\} : J_2'(u) = 0\}$. Given $u \in H_0^1(\Omega)\setminus\{0\}$, as $\alpha \to 0$, we have

$$c_\alpha \leqslant \max_{t>0} J(tu)$$

$$
\begin{aligned}
&= \max_{t>0} \frac{t^2}{2}\int_\Omega (|\nabla u|^2 + Vu^2)\mathrm{d}x - \frac{t^{2p}k}{2p}\int_\Omega (I_\alpha * |u|^p)|u|^p \mathrm{d}x \\
&\quad - \frac{t^2 \log t}{2}\int_\Omega Hu^2 \mathrm{d}x - \frac{t^2}{2}\int_\Omega Hu^2 \log|u| \mathrm{d}x + \frac{t^2}{4}\int_\Omega Hu^2 \mathrm{d}x \\
&\to \max_{t>0} \frac{t^2}{2}\int_\Omega (|\nabla u|^2 + Vu^2)\mathrm{d}x - \frac{t^{2p}k}{2p}(\int_\Omega |u|^p \mathrm{d}x)^2 \\
&\quad - \frac{t^2 \log t}{2}\int_\Omega Hu^2 \mathrm{d}x - \frac{t^2}{2}\int_\Omega Hu^2 \log|u| \mathrm{d}x + \frac{t^2}{4}\int_\Omega Hu^2 \mathrm{d}x \\
&= \max_{t>0} J_2(tu).
\end{aligned}
$$

Taking the infimum with respect to $u \in H_0^1(\Omega) \setminus \{0\}$, we deduce that

$$
\limsup_{\alpha\to 0} c_\alpha \leqslant c_p. \tag{3.30}
$$

According to Lemma 5 and (3.30), we obtain that $\{u_\alpha\}$ is bounded. So, there exists $\{u_j\} \subset H_0^1(\Omega)$ be a sequence converging weakly in $H_0^1(\Omega)$ to $u_N \in H_0^1(\Omega)$. We claim that u_N is a weak solution of the equation (1.1). Indeed,

$$
\begin{aligned}
&\lim_{j\to\infty} \int_\Omega (\nabla u_j \cdot \nabla v + V u_j v)\mathrm{d}x - k\int_\Omega (I_\alpha * |u_{\alpha_j}|^p)|u_j|^{p-2} uv \mathrm{d}x \\
&\quad - \int_\Omega H u_j v \log|u_j| \mathrm{d}x \\
&= \int_\Omega (\nabla u_N \cdot \nabla v + V u_N v)\mathrm{d}x - k\int_\Omega |u_N|^p \mathrm{d}x \int_\Omega |u_N|^{p-2} u_N v \mathrm{d}x \\
&\quad - \int_\Omega H u_N v \log|u_N| \mathrm{d}x.
\end{aligned} \tag{3.31}
$$

Since u_j is a ground solution of equation (1.1), we obtain

$$
\begin{aligned}
\langle J'(u_j), v\rangle &= \int_\Omega (\nabla u_j \cdot \nabla v + V u_j v)\mathrm{d}x - k\int_\Omega (I_{\alpha_j} * |u_j|^p)|u_j|^{p-2} uv \mathrm{d}x \\
&\quad - \int_\Omega H u_j v \log|u_j| \mathrm{d}x = 0.
\end{aligned} \tag{3.32}
$$

It follows from (3.31) and (3.32) that

$$
\begin{aligned}
&\int_\Omega (\nabla u_N \cdot \nabla v + V u_N v)\mathrm{d}x - k\int_\Omega |u_N|^p \mathrm{d}x \int_\Omega |u_N|^{p-2} u_N v \mathrm{d}x - \int_\Omega H u_N v \log|u_N| \mathrm{d}x \\
&= 0 = \langle J'(u_N), v\rangle,
\end{aligned}
$$

which concludes the claim. Since u_0 is a weak solution of the equation (1.3), according to (3.25) and Lemma 8, we have

$$
\lim_{j\to\infty} \|u_j\| = \lim_{j\to\infty} \left(\int_\Omega (I_{\alpha_j} * |u_j|^p)|u_j|^p \mathrm{d}x + \int_\Omega H u_j^2 \log|u_j| \mathrm{d}x \right)
$$

$$= \int_\Omega (|u_0|^p \mathrm{d}x)^2 + \int_\Omega H u_0^2 \log |u_0| \mathrm{d}x = \|u_0\|. \tag{3.33}$$

According to (3.33) and $u_j \rightharpoonup u_0$, we have $u_j \to u_0$. Since u_j is a ground solution of equation (1.1), we obtain

$$\begin{aligned} \|u_j\|^2 &= k\int_\Omega (I_\alpha * |u_j|^p)|u_j|^p \mathrm{d}x + \int_\Omega H u_j^2 \log |u_j| \mathrm{d}x \\ &\leqslant C_2\|u_j\|^{2p} + C_4\|u_j\|^4, \end{aligned} \tag{3.34}$$

By (3.34), we deduce

$$1 \leqslant C_2\|u_j\|^{2p-2} + C_4\|u_j\|^2. \tag{3.35}$$

According to (3.35) and $u_j \to u_0$, we imply that u_0 is a nontrivial solution of the equation (1.3). □

According to Lemma 7 and Lemma 9, we complete the proof of Theorem 2.

Proof of Theorem 3. When $k = 0$, define $K = \{u \in H_0^1(\Omega) \setminus \{0\} : J'(u) = 0\}$. It follows from Lemma 3 and Lemma 5 that K is nonempty. So for any $u \in K$, according to (2.1), the condition (H) and Lemma 2, we have

$$\begin{aligned} 0 &= \langle J'(u), u\rangle \\ &\geqslant \int_\Omega \delta|\nabla u|^2 \mathrm{d}x - \int_\Omega H u^2 \log \frac{|u|}{|u|_2} \mathrm{d}x - \int_\Omega H u^2 \log |u|_2 \mathrm{d}x + \int_\Omega H u^2 \mathrm{d}x - \int_\Omega H u^2 \mathrm{d}x \\ &\geqslant \frac{1}{4}\left(2\delta - \frac{Ma^2}{\pi}\right)\int_\Omega |\nabla u|^2 \mathrm{d}x + M\left(\frac{3(1+\log a)}{2} - 2|\Omega|^{1/2} - 1\right)|u|_2^2 \\ &\quad + (1 - \log |u|_2)\int_\Omega H u^2 \mathrm{d}x. \end{aligned} \tag{3.36}$$

Take $a = e^{\frac{4|\Omega|^{1/2}-1}{3}}$, we deduce

$$1 - \log |u|_2 \leqslant 0. \tag{3.37}$$

It follows from (3.37) that $e^1 \leqslant \int_\Omega u^2$. Next, we claim that J is bounded from below on K. For any $u \in K$, we obtain

$$J(u) - \frac{1}{2}\langle J'(u), u\rangle = \frac{1}{4}\int_\Omega H u^2 \mathrm{d}x \geqslant \frac{\mu e^1}{4} > 0. \tag{3.38}$$

which concludes the claim. So we may define $c_1 = \inf\{J(u) : u \in K\}$. According to (3.38), we get $c_1 > 0$. Let $\{u_n\} \subseteq K$ be such that $J(u_n) \to c_1$, as $n \to \infty$. By Lemma 5, we know that $\{u_n\}$ is bounded. Therefore, there exists a subsequence (still denote $\{u_n\}$) and $u_n \to u_0$ in $H_0^1(\Omega)$. By the continuation of J', we imply that $J(u_0) = c_1$ and $J'(u_0) = 0$. Then, u_0 is a ground state solution of equation (1.1). This completes the proof of Theorem 3. □

Proof of Remark 3. Define $K_1 = \{u \in H_0^s(\Omega)\setminus\{0\} : J'(u) = 0\}$. Similarly to proof of Lemma 3 and Lemma 5, we know that K_1 is nonempty. So for any $u \in K_1$, according to the condition (H_1), fractional logarithmic Sobolev inequality and similarly to proof of Lemma 2, we have

$$\begin{aligned}
0 &= \langle J'(u), u\rangle \\
&\geqslant \delta_1 \int_\Omega |(-\Delta)^{s/2}u|^2 \mathrm{d}x - \int_\Omega Hu^2 \log\frac{|u|}{|u|_2}\mathrm{d}x \\
&\quad - \int_\Omega Hu^2 \log|u|_2 \mathrm{d}x + \int_\Omega Hu^2 \mathrm{d}x - \int_\Omega Hu^2 \mathrm{d}x \\
&\geqslant \left(\delta_1 - \frac{a^2 M}{\pi^s}\right)\int_\Omega (-\Delta)^{s/2}u|^2 \mathrm{d}x + M(3 + \frac{3}{s}\log a \\
&\quad + \log\frac{s\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2s})} - 2|\Omega|^{\frac{1}{2}} - 1)|u|_2^2 + (1 - \log|u|_2)\int_\Omega Hu^2 \mathrm{d}x,
\end{aligned} \tag{3.39}$$

where $\delta_1 = 1 - S^{-1}|V^-|_{3/2s}$. Take $a = \left(\frac{\Gamma(\frac{3}{2s})}{(\frac{3}{2})} e^{\frac{2|\Omega|^{1/2}-2}{3}}\right)^{3/s}$, according to (3.39), we deduce

$$1 - \log|u|_2 \leqslant 0. \tag{3.40}$$

It follows from (3.40) that $e^1 \leqslant \int_\Omega u^2$. Next, we claim that J is bounded from below on K. For any $u \in K$, we obtain

$$J(u) - \frac{1}{2}\langle J'(u), u\rangle = \frac{1}{4}\int_\Omega Hu^2 \mathrm{d}x \geqslant \frac{\mu e^1}{4} > 0. \tag{3.41}$$

which concludes the claim. So we may define $c_1 = \inf\{J(u) : u \in K\}$. According to (3.41), we get $c_1 > 0$. Let $\{u_n\} \subseteq K$ be such that $J(u_n) \to c_1$, as $n \to \infty$. Similarly to proof of Lemma 5, we know that $\{u_n\}$ is bounded. Therefore, there exists a subsequence (still denote $\{u_n\}$) and $u_n \to u_0$ in $H_0^s(\Omega)$. By the continuation of J', we imply that $J(u_0) = c_1$ and $J'(u_0) = 0$. Then, u_0 is a ground state solution. This completes the proof of Remark 3. □

4. Conclusion

This paper considers the existence of ground state solutions for Choquard equation with logarithmic nonlinearity and the limit of behaviors of ground solution to (1.1) as either $\alpha \to 0$ or $\alpha \to N$ by variational method and logarithmic inequality. In addition, we obtain a new existence theorem through the property of logarithmic nonlinearity. Note that fractional Choquard has been widely studied in recent years. In the forthcoming paper, we will consider the existence of ground state solutions for fractional Choquard equation with logarithmic nonlinearity and sign-change potential function on the whole space by variational method and fractional logarithmic inequality. Lacking compactness will be the biggest difficulty.

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