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SOLUTION OF COMPLEX DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS BY USING REDUCED DIFFERENTIAL TRANSFORM

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Abstract. In this article, solution of complex partial derivative equations with variable coefficients from the first and second order have been investigated. For this solution, an iteration relation was obtained using the reduced differential transform method. This method also was been applied for ordinary complex differential equations which examined in the literature. The solution which has been obtained has been seen compatible with the literature.

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1. INTRODUCTION

Some equations that don't have a general solution in real space may have a general solution in complex space. For example, although the two-dimensional Laplace equation doesn't have a general solution in real space, the equation has general solution in complex space. Two basic sources of Complex variable differential equations theory are "Theory of Pseodo Analytic Functions" and "Generalized Analytic Functions", written by L. Bers and I. N. Vekua [3, 16]. Even today, the Pompeiu integral operator, an important operator of Complex variable differential equations theory, was discovered by D. Pompeiu, an important mathematician in this theory. The Pompeiu operator has been defined as

$$T_G : C^{(\alpha)}\left(\overline{G}\right) \longrightarrow C^{(\alpha)}\left(\overline{G}\right) \tag{1.1}$$
$$f \longrightarrow T_G f = -\frac{1}{\pi}\int_G\int \frac{f(\zeta)}{\zeta - z}d\xi d\eta$$

This T_G operator plays a crucial role for the existence and uniqueness of equations in following form.

$$w_{\bar{z}} = F(z, w, \frac{\partial w}{\partial z}). \tag{1.2}$$

Existence and uniqueness of solutions of Equation (1.2) are investigated in [1, 2, 15].

Generalized Beltrami equation which is defined by equality

$$\frac{\partial w}{\partial \bar{z}} + q_1 \frac{\partial w}{\partial z} + q_2 \overline{\left(\frac{\partial w}{\partial z}\right)} + aw + b\overline{w} + c = 0 \tag{1.3}$$

was reduced to singular integral equation by using Pompeiu integral operator.

In [16] linear complex differential equations have been reduced to a linear system of algebraic equations. Later approximation solution has been found by using appropriate iterative method. In [7] approximate solutions of equations of the same type have been found by using Taylor polynomial approximation. In [9], The Existence of solutions of the same type equations with analytical coefficients was investigated. In [8, 14]

$$f''(z) + A(z) f(z) = 0 \tag{1.4}$$

which $A(z)$ analytic equation has been investigated.

In this paper, it has been studied to find a special solution of the equations in the following form below that meets the given conditions.

$$A(z,\bar{z}) . \frac{\partial w}{\partial z} + B(z,\bar{z}) . \frac{\partial w}{\partial \bar{z}} + C(z,\bar{z}) .w = F(z,\bar{z}) \tag{1.5}$$

and

$$\begin{aligned} A(z,\bar{z}) \frac{\partial^2 w}{\partial z^2} + B(z,\bar{z}) \frac{\partial^2 w}{\partial z \partial \bar{z}} + C(z,\bar{z}) . \frac{\partial^2 w}{\partial \bar{z}^2} \\ + D(z,\bar{z}) \frac{\partial w}{\partial z} + E(z,\bar{z}) \frac{\partial w}{\partial \bar{z}} + F(z,\bar{z}) w = G(z,\bar{z}). \end{aligned} \tag{1.6}$$

Previously, such equations with constant coefficients were also examined by RDTM in [6]. Equations (1.5) and (1.6) were previously solved using the Adomian decomposition method in [4, 5]. In this study, RDTM is used for solution of Equations (1.5) and (1.6). With this RDTM, the solution directly is obtained without having to find the real and imaginary parts separately. RDTM was proposed firstly by Keskin in [11].

2. Basic Definitions and Theorems

Suppose $u(x,y)$ can be separated into variables as $f(x)g(y)$. Using a one-dimensional differential transformation, $u(x,y)$ can be written as follows.

$$u(x,y) = (\sum_{i=0}^{\infty} F_i x^i)(\sum_{j=0}^{\infty} G_j y^j) = (\sum_{k=0}^{\infty} U_k(x).y^k)$$

where $U_k(x)$ is called y dimensional spectrum function $u(x,y)$, and G_j, F_i. are differential transform of $g(y)$ and $f(x)$, respectively.

Definition 1. If reduced differential transform of $u(x,y)$ is $U_k(x)$ than

$$U_k(x) = \frac{1}{k!}\left(\frac{\partial^k}{\partial y^k}u(x,y)\right)_{y=0}. \tag{2.1}$$

Definition 2. The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,y) = \sum_{k=0}^{\infty} U_k(x)\, y^k. \tag{2.2}$$

From (2.1) and (2.2) ,we get

$$u(x,y) = \sum_{k=0}^{\infty} \frac{y^k}{k!}\left(\frac{\partial^k}{\partial y^k}u(x,y)\right)_{y=0} \tag{2.3}$$

Theorem 1. *[11,12] If $f(x,y) = a.g(x,y)+b.h(x,y)$ then $F_k(x) = a.G_k(x)+b.H_k(x)$.*

Theorem 2. *[11, 12] If $f(x,y) = x^m.y^n$, then $F_k(x) = x^m\delta(k-n)$.*

Theorem 3. *[11,12] If $f(x,y) = \frac{\partial^n g(x,y)}{\partial y^n}$, then $F_k(x) = (k+1)(k+2)...(k+n)G_{k+n}(x)$.*

Theorem 4. *[11, 12] If $f(x,y) = \frac{\partial^n g(x,y)}{\partial x^n}$, then $F_k(x) = \frac{\partial^n G_k(x)}{\partial x^n}$.*

Theorem 5. *[11, 12] If $f(x,y) = g(x,y).h(x,y)$, then*

$$F_k(x) = \sum_{r=0}^{k} G_r(x).H_{k-r}(x).$$

Now let's give the equivalent of complex derivatives in terms of real derivatives.

Definition 3. Partial derivatives of $w = w(z,\bar{z})$ function as $z = x+iy$ are defined as follows.

$$\frac{\partial w}{\partial z} = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) \tag{2.4}$$

$$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right) \tag{2.5}$$

$$\frac{\partial^2 w}{\partial z^2} = \frac{1}{4}\left[\frac{\partial^2 w}{\partial x^2} - 2i\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial^2 w}{\partial y^2}\right] \tag{2.6}$$

$$\frac{\partial^2 w}{\partial \bar{z}^2} = \frac{1}{4}\left[\frac{\partial^2 w}{\partial x^2} + 2i\frac{\partial^2 w}{\partial x\partial y} - \frac{\partial^2 w}{\partial y^2}\right] \tag{2.7}$$

$$\frac{\partial^2 w}{\partial z\partial \bar{z}} = \frac{1}{4}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] \tag{2.8}$$

Theorem 6. *Let the complex function w be a holomorph in the G region. In this case, for every $z_0 \in G$ following equalities are satisfied.*

$$\frac{\partial w}{\partial z}(z_0) = \frac{\partial w}{\partial x}(z_0) = -i\frac{\partial w}{\partial y}(z_0)$$

$$\frac{\partial w}{\partial z}(z_0) = \frac{dw}{dz}(z_0), \frac{\partial w}{\partial \bar{z}}(z_0) = 0$$

3. Solution of complex differential equations from first and second order which is variable coefficients.

In this section, a separate theorem is given for each of the equations (1.5) and (1.6) giving the solution satisfying the given conditions with an iteration relation. Then, the solution is found by applying the iteration relation obtained in these theorems on the examples.

Theorem 7. *A special solution of equation*

$$A(z,\bar{z})\frac{\partial w}{\partial z} + B(z,\bar{z})\frac{\partial w}{\partial \bar{z}} + C(z,\bar{z})w = F(z,\bar{z})$$

with condition

$$w(x,0) = f(x)$$

is

$$w(z,\bar{z}) = \sum_{k=0}^{\infty} W_k(x)y^k$$

where

$$\begin{aligned}
&[B^{**}(x,0) - A^{**}(x,0)](k+1)W_{k+1} \\
&\qquad = 2F^{**}(x,k) - \sum_{r=0}^{k}[A^{**}(x,r) + B^{**}(x,r)]\frac{\partial W_{k-r}}{\partial x} - 2\sum_{r=0}^{k} C^{**}(x,r)W_{k-r} \\
&\qquad - \sum_{r=1}^{k} i[B^{**}(x,r) - A^{**}(x,r)](k+1-r)W_{k+1-r}
\end{aligned}$$

and $W_0(x) = f(x)$. Here, $A^{**}(x,k), B^{**}(x,k), C^{**}(x,k), F^{**}(x,k)$ are the differential transformations of the equlities which are obtained when are written $x+iy$ instead of z and $x-iy$ instead of $\bar{z}$ in $A(z,\bar{z}), B(z,\bar{z}), C(z,\bar{z}), F(z,\bar{z})$ functions.

Proof. Let's rewrite equation in which theorem.

$$A(z,\bar{z})\frac{\partial w}{\partial z} + B(z,\bar{z})\frac{\partial w}{\partial \bar{z}} + C(z,\bar{z})w = F(z,\bar{z})$$

If, (2.4), (2.5) definition of complex derivatives are used, than following equality is obtained.

$$(A^*(x,y))\frac{1}{2}\left(\frac{\partial w}{\partial x}-i\frac{\partial w}{\partial y}\right)+(B^*(x,y))\frac{1}{2}\left(\frac{\partial w}{\partial x}+i\frac{\partial w}{\partial y}\right)+C^*(x,y)w=F^*(x,y) \tag{3.1}$$

The above equation (3.1) is equivalent to following equation.

$$[A^*(x,y)+B^*(x,y)]\frac{\partial w}{\partial x}+i[B^*(x,y)-A^*(x,y)]\frac{\partial w}{\partial y} +2C^*(x,y)w=2F^*(x,y) \tag{3.2}$$

If RDTM is applied to equation (3.2), the following equation is obtained

$$\sum_{r=0}^{k}[A^{**}(x,r)+B^{**}(x,r)]\frac{\partial W_{k-r}}{\partial x} +\sum_{r=0}^{k}i[B^{**}(x,r)-A^{**}(x,r)](k+1-r)W_{k+1-r} +2\sum_{r=0}^{k}C^{**}(x,r)W_{k-r}=2F^{**}(x,k) \tag{3.3}$$

If the term with the largest index is left alone in Equality (3.3), the following equality is obtained.

$$\sum_{r=0}^{k}i[B^{**}(x,r)-A^{**}(x,r)](k+1-r)W_{k+1-r} =2F^*(x,k)-\sum_{r=0}^{k}[A^{**}(x,r)+B^{**}(x,r)]\frac{\partial W_{k-r}}{\partial x} -2\sum_{r=0}^{k}C^{**}(x,r)W_{k-r} \tag{3.4}$$

Thus, the iteration relation required for the solution is obtained as follows from (3.4).

$$i[B^{**}(x,0)-A^{**}(x,0)](k+1)W_{k+1} =2F^*(x,k)-\sum_{r=0}^{k}[A^{**}(x,r)+B^{**}(x,r)]\frac{\partial W_{k-r}}{\partial x}-2\sum_{r=0}^{k}C^{**}(x,r)W_{k-r} \sum_{r=1}^{k}[B^{**}(x,r)-A^{**}(x,r)](k+1-r)W_{k+1-r}$$

Here $W_0(x)=f(x)$ due to condition. Thus the proof is completed. □

Theorem 8. *A special solution of equation*

$$A(z,\overline{z})\frac{\partial^2 w}{\partial z^2}+B(z,\overline{z})\frac{\partial^2 w}{\partial \overline{z}\partial \overline{z}}+C(z,\overline{z})\frac{\partial^2 w}{\partial \overline{z}^2}$$
$$+D(z,\overline{z})\frac{\partial w}{\partial z}+E(z,\overline{z})\frac{\partial w}{\partial \overline{z}}+F(z,\overline{z})w=G(z,\overline{z})$$

with conditions

$$w(x,0)=f(x)$$
$$\frac{\partial w}{\partial y}(x,0)=g(x).$$

is

$$w(z,\overline{z})=\sum_{k=0}^{\infty}W_k(x)y^k$$

where

$$\begin{aligned}
[-A^{**}(x,0)+B^{**}(x,0)-C^{**}(x,0)](k+1)(k+2)W_{k+2}&=4G^{**}(x,k)\\
&-2\sum_{r=0}^{k}[D^{**}(x,r)+E^{**}(x,r)]\frac{\partial W_{k-r}}{\partial x}\\
&-2i\sum_{r=0}^{k}[E^{**}(x,r)-D^{**}(x,r)](k+1-r)W_{k+1-r}-4\sum_{r=0}^{k}F^{**}(x,r)W_{k-r}\\
&-\sum_{r=1}^{k}[-A^{**}(x,r)+B^{**}(x,r)-C^{**}(x,r)](k+1-r)(k+2-r)W_{k+2-r}\\
&-\sum_{r=0}^{k}[A^{**}(x,r)+B^{**}(x,r)+C^{**}(x,r)]\frac{\partial^2 W_{k-r}}{\partial x^2}\\
&-2i\sum_{r=0}^{k}[C^{**}(x,r)-A^{**}(x,r)](k+1-r)\frac{\partial W_{k+1-r}}{\partial x}
\end{aligned}$$

and

$$W_0(x)=f(x).$$

Here, $A^{**}(x,k), B^{**}(x,k), C^{**}(x,k), D^{**}(x,k), E^{**}(x,k), F^{**}(x,k), G^{**}(x,k)$ *are the differential transformations of the equalities which are obtained when are written* $x+iy$ *instead of* z *and* $x-iy$ *instead of* $\overline{z}$ *in* $A(z,\overline{z})$, $B(z,\overline{z})$, $C(z,\overline{z})$, $D(z,\overline{z})$, $E(z,\overline{z})$, $F(z,\overline{z})$, $G(z,\overline{z})$ *functions.*

Proof. Let's rewrite the equation in which theorem .

$$A(z,\overline{z})\frac{\partial^2 w}{\partial z^2}+B(z,\overline{z})\frac{\partial^2 w}{\partial z\partial \overline{z}}+C(z,\overline{z}).\frac{\partial^2 w}{\partial \overline{z}^2}$$

$$+D(z,\bar{z})\frac{\partial w}{\partial z}+E(z,\bar{z})\frac{\partial w}{\partial \bar{z}}+F(z,\bar{z})w=G(z,\bar{z}).$$

If Equalities (2.4), (2.5), (2.6), (2.7), (2.8) is used above equality, than following equality is obtained.

$$A^*(x,y)\frac{1}{4}\left[\frac{\partial^2 w}{\partial x^2}-2i\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial^2 w}{\partial y^2}\right]+B^*(x,y)\frac{1}{4}\left[\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right]$$
$$+C^*(x,y)\frac{1}{4}\left[\frac{\partial^2 w}{\partial x^2}+2i\frac{\partial^2 w}{\partial x\partial y}-\frac{\partial^2 w}{\partial y^2}\right]+D^*(x,y)\frac{1}{2}\left(\frac{\partial w}{\partial x}-i\frac{\partial w}{\partial y}\right)$$
$$+E^*(x,y)\frac{1}{2}\left(\frac{\partial w}{\partial x}+i\frac{\partial w}{\partial y}\right)+F^*(x,y)w=G^*(x,y). \tag{3.5}$$

Where $A^*(x,y), B^*(x,y), C^*(x,y), D^*(x,y), E^*(x,y), F^*(x,y), G^*(x,y)$ expressions are obtained by writing $x+iy$ in place of z in $A(z,\bar{z}), B(z,\bar{z}), C(z,\bar{z})$ $D(z,\bar{z}), E(z,\bar{z})$, $F(z,\bar{z})$, $G(z,\bar{z})$ (3.5) equality is equivalent to following equality.

$$\left(\frac{A^*(x,y)+B^*(x,y)+C^*(x,y)}{4}\right)\frac{\partial^2 w}{\partial x^2}+2i\left(\frac{C^*(x,y)-A^*(x,y)}{4}\right)\frac{\partial^2 w}{\partial x\partial y}$$
$$\left(\frac{-A^*(x,y)+B^*(x,y)-C^*(x,y)}{4}\right)\frac{\partial^2 w}{\partial y^2}+\left(\frac{D^*(x,y)+E^*(x,y)}{2}\right)\frac{\partial w}{\partial x}$$
$$+i\left(\frac{E^*(x,y)-D^*(x,y)}{2}\right)\frac{\partial w}{\partial y}+F^*(x,y)w=G^*(x,y) \tag{3.6}$$

Let's apply RDTM to (3.6) equality.

$$4G^{**}(x,k)=\sum_{r=0}^{k}\left[A^{**}(x,r)+B^{**}(x,r)+C^{**}(x,r)\right]\frac{\partial^2 W_{k-r}}{\partial x^2}$$
$$+\sum_{r=0}^{k}2i\left[C^{**}(x,r)-A^{**}(x,r)\right](k+1-r)\frac{\partial W_{k+1-r}}{\partial x}$$
$$+\sum_{r=0}^{k}\left[-A^{**}(x,r)+B^{**}(x,r)-C^{**}(x,r)\right](k+1-r)(k+2-r)W_{k+2-r}$$
$$+2\sum_{r=0}^{k}\left[D^{**}(x,r)+E^{**}(x,r)\right]\frac{\partial W_{k-r}}{\partial x}$$
$$+2i\sum_{r=0}^{k}\left[E^{**}(x,r)-D^{**}(x,r)\right](k+1-r)W_{k+1-r}$$
$$+4\sum_{r=0}^{k}F^{**}(x,r)W_{k-r} \tag{3.7}$$

By writing as following the expression (3.7), iteration relation in the theorem has been obtained.

$$\begin{aligned}
&\left[-A^{**}(x,0)+B^{**}(x,0)-C^{**}(x,0)\right](k+1)(k+2)W_{k+2} \\
&\qquad = 4G^{**}(x,k)-2\sum_{r=0}^{k}\left[D^{**}(x,r)+E^{**}(x,r)\right]\frac{\partial W_{k-r}}{\partial x} \\
&\qquad -2i\sum_{r=0}^{k}\left[E^{**}(x,r)-D^{**}(x,r)\right](k+1-r)W_{k+1-r}-4\sum_{r=0}^{k}F^{**}(x,r)W_{k-r} \\
&\qquad -\sum_{r=1}^{k}\left[-A^{**}(x,r)+B^{**}(x,r)-C^{**}(x,r)\right](k+1-r)(k+2-r)W_{k+2-r} \\
&\qquad -\sum_{r=0}^{k}\left[A^{**}(x,r)+B^{**}(x,r)+C^{**}(x,r)\right]\frac{\partial^2 W_{k-r}}{\partial x^2} \\
&\qquad -\sum_{r=0}^{k}2i\left[C^{**}(x,r)-A^{**}(x,r)\right](k+1-r)\frac{\partial W_{k+1-r}}{\partial x} \qquad (3.8)
\end{aligned}$$

Thus the proof is completed. □

Example 1. Solve the following problem

$$z.w_z-\bar{z}.w_{\bar{z}}=2z^2+5\bar{z}$$

with the condition

$$w(x,0)=2x^2-5x.$$

Solution 1. The coefficients of the equation are $A=z, B=-z, C=0, F=2z^2+5\bar{z}$. If $x+iy, x-iy$ is written in place of $z,\bar{z}$ respectively than

$$A^*=x+iy, B^*=-x+iy, C^*=0, F^*=2x^2+5x-2y^2+i(4xy-5y)$$

Reduced differential transforms of the A^*,B^*,C^*,F^* are

$$\begin{aligned}
&A^{**}=x\delta(k)+i\delta(k-1), B^{**}=-x\delta(k)+i\delta(k-1), C^{**}=0,\\
&F^{**}=\left(2x^2+5x\right)\delta(k)+i(4x-5)\delta(k-1)-2\delta(k-2).
\end{aligned}$$

Following equality has been obtained from iteration relation from theorem.

$$\begin{aligned}
-2ix(k+1)W_{k+1} &= \left(4x^2+10x\right)\delta(k)+i(8x-10)\delta(k-1)-4\delta(k-2)\\
&\quad -\sum_{r=0}^{k}2i\delta(r-1)\frac{\partial W_{k-r}}{\partial x}-\sum_{r=1}^{k}-2ix\delta(r)(k+1-r)W_{k+1-r}
\end{aligned}$$

Because of condition in the example

$$W_0(x)=2x^2-5x. \qquad (3.9)$$

If $k=0$ than

$$-2ixW_1=4x^2+10x, W_1(x)=i(2x+5). \qquad (3.10)$$

If $k=1$ than
$$2i\frac{\partial W_0}{\partial x}-2ixW_2=8ix-10i, W_2(x)=0. \tag{3.11}$$
If $k\geq 2$ than
$$W_k(x)=0. \tag{3.12}$$
Solution is obtained from (3.9),(3.10),(3.11),(3.12) as following.
$$w(x,y)=\sum_{k=0}^{\infty}W_k(x)y^k=W_0(x)+W_1(x)y+W_2(x)y^2 \tag{3.13}$$
$$=2x^2-5x+i(2x+5)y. \tag{3.14}$$

Then,
$$w(z,\bar{z})=z^2+z\bar{z}-5\bar{z}.$$

Example 2. Let's solve the second order variable coefficient problem below.
$$z^2\frac{\partial^2 w}{\partial z^2}-2\bar{z}^2\frac{\partial^2 w}{\partial \bar{z}^2}+5\frac{\partial w}{\partial z}+3z\frac{\partial w}{\partial \bar{z}}+4w=10z^3-8z\bar{z}-25\bar{z}$$
$$w(x,0)=x^3-3x^2$$
$$\frac{\partial w}{\partial y}(x,0)=i\left(3x^2-4x\right)$$

Solution 2. Coefficients of the equation are $A=z^2, B=0, C=-2\bar{z}^2, D=5, E=3z, F=4, G=10z^3-8z\bar{z}-25\bar{z}$.

If $x+iy$ in place of z, $x-iy$ in place of $\bar{z}$ is written than the coefficients are obtained as following.

$A^*=x^2+2ixy-y^2, B^*=0, C^*=-2x^2+4ixy+2y^2, D^*=5, E^*=3x+3iy, F^*=4, G^*=10x^3+30x^2iy-30xy^2-10iy^3-8x^2-8y^2-25x+25iy.$

Let's RDTM of above equalities are

$$A^{**}(x,k)=x^2\delta(k)+2ix\delta(k-1)-\delta(k-2),$$
$$B^{**}(x,k)=0,$$
$$C^{**}(x,k)=-2x^2\delta(k)+4ix\delta(k-1)+2\delta(k-2),$$
$$D^{**}(x,k)=5\delta(k),$$
$$E^{**}(x,k)=3x\delta(k)+3i\delta(k-1)$$

From conditions in which are given
$$W_0(x)=x^3-3x^2, W_1(x)=i(3x^2-4x) \tag{3.15}$$

From theorem

$$2x^2W_2=40x^3-32x^2-100x-18x^3+36x^2-30x^2+60x+18x^3-24x^2$$

$$-30x^2+40x-16x^3+48x^2+6x^3-6x^2-36x^3+24x^2$$

$$W_2=-3x-7 \tag{3.16}$$

$$6x^2W_3=i\left(120x^2+100\right)-2i\,(5+3x)\,(6x-4)-6i\left(3x^2-6x\right)$$
$$-2i\,(3x-5)\,(-3x-7)+6i\left(3x^2-4x\right)-16i\left(3x^2-4x\right)$$
$$+12ix\,(-3x-7)+6ix^2-6ix\,(6x-6)-18ix^2+4ix\,(6x-4)$$

$$W_3=-i \tag{3.17}$$

For $n>3$

$$W_n=0 \tag{3.18}$$

Thus solution is obtained by using (3.15), (3.16), (3.17), (3.18) as that.

$$W_0(x)+W_1\,(x)\,y+W_2\,(x)\,y^2+W_3\,(x)\,y^3$$
$$=x^3-3x^2+i(3x^2-4x)y+(-3x-7)y^2-iy^3$$
$$=x^3+3ix^2y-3xy^2-iy^3-5x^2-5y^2+2(x^2-2ixy-y^2)$$

If $\frac{z+\bar{z}}{2},\frac{z-\bar{z}}{2i}$ is written in place of x,y respectively than solution is obtained as that.

$$w(z,\bar{z})=z^3-5z\bar{z}+2\bar{z}^2.$$

Studies in the literature are generally about ordinary complex differential equations. The two theorems and solved differential equations given in the article so far were partial differential equations. Now let's solve two ordinary complex differential equations, which have been studied by other methods in the literature, using the reduced differential transformation method.

Example 3. [7,13] Let's solve the second order variable coefficient problem below.

$$f''(z)+zf(z)=e^z+ze^z \tag{3.19}$$

$$f(0)=f'(0)=1 \tag{3.20}$$

Solution 3. This equationis a ordinary differential equation. The f function in the equation depends on only the variable z. This indicates that the function f is holomorph. From (2.2) it can be written that

$$f(z)=f(x+iy)=F_0(x)+F_1(x)y+F_2(x)y^2+\ldots=\left(\sum_{k=0}^{\infty}F_k\,(x)\,y^k\right)$$

Due to from $f(0)=1$ condition

$$f(0)=F_0(0)=1. \tag{3.21}$$

Due to from $f'(0)=1$ condition

$$f'(0)=\frac{\partial f}{\partial x}(0,0)=F_0'(0)=1. \tag{3.22}$$

Let's rewrite the given equation.

$$f''(z) + zf(z) = e^z + ze^z.$$

Since f function is analytic $f''(z) = \frac{\partial^2 f}{\partial x^2} = -\frac{\partial^2 f}{\partial y^2}$. If $x+iy, \frac{\partial^2 f}{\partial x^2}$ is written in place of $z, f''(z)$ respectively the following equation is obtained.

$$\frac{\partial^2 f}{\partial x^2} + (x+iy)f(x+iy) = (1+x+iy)e^x e^{iy}. \tag{3.23}$$

Let's apply the RDTM to the above equation.

$$\begin{aligned} \frac{d^2F_k(x)}{dx^2} &+ xF_k(x) + i\sum_{r=0}^{k}\delta(r-1)F_{k-r}(x) \\ &= (1+x)e^x\frac{i^k}{k!} + ie^x\sum_{r=0}^{k}\delta(r-1)\frac{i^{k-r}}{(k-r)!}. \end{aligned} \tag{3.24}$$

Similarly, if $x+iy, -\frac{\partial^2 f}{\partial y^2}$ is written in place of $z, f''(z)$ respectively, than the following equality is obtained.

$$-\frac{\partial^2 f}{\partial y^2} + (x+iy)f(x+iy) = (1+x+iy)e^x e^{iy} \tag{3.25}$$

Let's apply the RDTM to the above (3.25)equation

$$\begin{aligned} - &\quad (k+1)(k+2)F_{k+2}(x) + xF_k(x) + i\sum_{r=0}^{k}\delta(r-1)F_{k-r}(x) \\ = &\quad (1+x)e^x\frac{i^k}{k!} + ie^x\sum_{r=0}^{k}\delta(r-1)\frac{i^{k-r}}{(k-r)!}. \end{aligned} \tag{3.26}$$

If $k=0$ is written in equality (3.24), than following equality is obtained.

$$\frac{d^2F_0(x)}{dx^2} + xF_0(x) = (1+x)e^x. \tag{3.27}$$

It is clear that

$$F_0(x) = e^x \tag{3.28}$$

from $F_0(0) = F_0'(0) = 1$ and (3.27). If $k=0$ is written in the equation (3.26) , following equality is obtained.

$$-2F_2(x) + xF_0(x) = (1+x)e^x \tag{3.29}$$

Therefore

$$F_2(x) = -\frac{e^x}{2}. \tag{3.30}$$

If $k=2$ is written in the equation (3.24), following equality is obtained

$$\frac{d^2F_2(x)}{dx^2} + xF_2(x) + iF_1(x) = -\frac{(1+x)e^x}{2} - e^x \tag{3.31}$$

From (3.30), (3.31) it is seen that

$$F_1(x) = ie^x \tag{3.32}$$

If $k = 1$ is written in the equation (3.26) , following equality is obtained

$$-6F_3(x) + xF_1(x) + iF_0(x) = i(1+x)e^x + ie^x \tag{3.33}$$

Therefore

$$F_3(x) = -i\frac{e^x}{6} \tag{3.34}$$

Similarly it can be seen that

$$F_4(x) = \frac{e^x}{24}, F_5(x) = i\frac{e(x)}{120}, ..., F_n(x) = i^n\frac{e^x}{n!} \tag{3.35}$$

Thus, from (3.28), (3.30), (3.32), (3.34), (3.35) solution of the problem is that:

$$\begin{aligned} f(z) &= f(x+iy) = F_0(x) + F_1(x)y + F_2(x)y^2 + ... \\ &= e^x + ie^x y + i^2 e^x \frac{y^2}{2} + ... = e^x\left(1 + iy - \frac{y^2}{2} - i\frac{y^3}{6} + ...\right) \\ &= e^x e^{iy} = e^{x+iy} = e^z \end{aligned}$$

Example 4. [13] Let's solve the second order variable coefficient problem below.

$$f''(z) + zf(z) = \frac{z^5}{12} - \frac{z^4}{6} \tag{3.36}$$

$$f(0) = 1, f'(0) = -1 \tag{3.37}$$

Solution 4. This equation is an ordinary complex differential equation like the equation in the previous example. The f function in the equation depends on only the variable z. This indicates that the function f is holomorph. From (2.2) it can be written that

$$f(z) = f(x+iy) = F_0(x) + F_1(x)y + F_2(x)y^2 + ... = \left(\sum_{k=0}^{\infty} F_k(x)y^k\right)$$

Due to from $f(0) = 1$ condition

$$f(0) = F_0(0) = 1. \tag{3.38}$$

Due to from $f'(0) = 1$ condition

$$f'(0) = \frac{\partial f}{\partial x}(0,0) = F_0'(0) = -1. \tag{3.39}$$

Let's rewrite the given equation.

$$f''(z) + zf(z) = \frac{z^5}{12} - \frac{z^4}{6}.$$

Since f function is analytic $f''(z) = \frac{\partial^2 f}{\partial x^2} = -\frac{\partial^2 f}{\partial y^2}$. If $x+iy, \frac{\partial^2 f}{\partial x^2}$ is written in place of $z, f''(z)$ respectively the following equation is obtained.

$$\frac{\partial^2 f}{\partial x^2} + (x+iy)f(x+iy) = \frac{(x+iy)^5}{12} - \frac{(x+iy)^4}{6}. \tag{3.40}$$

Let's apply the RDTM to the above equation.

$$\begin{aligned} &\tfrac{d^2F_k(x)}{dx^2} + xF_k(x) + i\sum_{r=0}^{k}\delta(r-1)F_{k-r}(x) \\ &= (\tfrac{x^5}{12} - \tfrac{x^4}{6})\delta(k) + (\tfrac{5x^4}{12} - i\tfrac{4x^3}{6})\delta(k-1) + (-\tfrac{10x^3}{12} + \tfrac{6x^2}{6})\delta(k-2) \\ &+(\tfrac{-10x^2}{12} + i\tfrac{4x}{6})\delta(k-3) + (\tfrac{5x}{12} - \tfrac{1}{6})\delta(k-4) + \tfrac{i}{12}\delta(k-5). \end{aligned} \tag{3.41}$$

Similarly, if $x+iy, -\frac{\partial^2 f}{\partial y^2}$ is written in place of $z, f''(z)$ respectively, than the following equality is obtained.

$$-\frac{\partial^2 f}{\partial y^2} + (x+iy)f(x+iy) = \frac{(x+iy)^5}{12} - \frac{(x+iy)^4}{6} \tag{3.42}$$

Let's apply the RDTM to the above (3.42)equation

$$\begin{aligned} &-(k+1)(k+2)F_{k+2}(x) + xF_k(x) + i\sum_{r=0}^{k}\delta(r-1)F_{k-r}(x) \\ &= (\tfrac{x^5}{12} - \tfrac{x^4}{6})\delta(k) + (\tfrac{5x^4}{12} - i\tfrac{4x^3}{6})\delta(k-1) + (-\tfrac{10x^3}{12} + \tfrac{6x^2}{6})\delta(k-2) \\ &+(\tfrac{-10x^2}{12} + i\tfrac{4x}{6})\delta(k-3) + (\tfrac{5x}{12} - \tfrac{1}{6})\delta(k-4) + \tfrac{i}{12}\delta(k-5). \end{aligned} \tag{3.43}$$

If $k=0$ is written in equality (3.41), than following equality is obtained.

$$\frac{d^2F_0(x)}{dx^2} + xF_0(x) = \left(\frac{x^5}{12} - \frac{x^4}{6}\right). \tag{3.44}$$

It is clear that $F_0(x)$ is selectable as $F_0(x) = ax^4 + bx^3 + cx^2 - x + 1$ since $F_0(0) = 1, F_0'(0) = -1$. If $F_0(x)$ is written in the equation (3.44), than a,b,c are obtained as $a = \frac{1}{12}, b = \frac{-1}{6}, c = 0$. Thus $F_0(x)$ is obtained $F_0(x) = \frac{x^4}{12} - \frac{x^3}{6} - x + 1$. If $k=0$ is written in the equation (3.43) , following equality is obtained.

$$-2F_2(x) + xF_0(x) = \frac{x^5}{12} - \frac{x^4}{6} \tag{3.45}$$

Therefore

$$F_2(x) = -\frac{x^2}{2} + \frac{x}{2}. \tag{3.46}$$

If $k=2$ is written in the equation (3.41), following equality is obtained

$$\frac{d^2F_2(x)}{dx^2} + xF_2(x) + iF_1(x) = -\frac{10x^3}{12} + x^2 \tag{3.47}$$

From (3.46), (3.47) it is seen that

$$F_1(x) = \frac{ix^3}{3} - \frac{ix^2}{2} - i \tag{3.48}$$

If $k = 1$ is written in the equation (3.43) , following equality is obtained

$$-6F_3(x) + xF_1(x) + iF_0(x) = i\left(\frac{5x^4}{12} - \frac{4x^3}{6}\right) \tag{3.49}$$

Therefore

$$F_3(x) = i\left(\frac{1-2x}{6}\right) \tag{3.50}$$

Similarly it can be seen that

$$F_4(x) = \frac{1}{12}, F_5(x) = F_6(x) = F_7(x) = ... = 0 \tag{3.51}$$

Thus, solution of the problem is that:

$$\begin{aligned} f(z) &= f(x+iy) = F_0(x) + F_1(x)y + F_2(x)y^2 + ... \\ &= \frac{x^4}{12} - \frac{x^3}{6} - x + 1 + \left(\frac{ix^3}{3} - \frac{ix^2}{2} - i\right)y + \left(-\frac{x^2}{2} + \frac{x}{2}\right)y^2 + i\left(\frac{1-2x}{6}\right)y^3 + \frac{1}{12}y^4 \\ &= \frac{x^4 + 4ix^3y - 6x^2y^2 - 4ix^3y^3 + y^4}{12} - \frac{x^3 + 3ix^2y - 3xy^2 - iy^3}{6} - (x+iy) + 1 \\ &= \frac{z^4}{12} - \frac{z^3}{6} - z + 1 \end{aligned}$$

Conclusion 1. In this study, first and second order complex partial derivative equations with variable coefficients were solved by the reduced differential conversion method. The results were seen to be consistent.In addition, two ordinary derivative complex differential equations were solved by the method of reduced differential transformation. Using the method in [10], converting complex partial differential equations into ordinary differential equations and solving them can be considered as another study.

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