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GALERKIN METHODS FOR THE NUMERICAL SOLUTION OF THE SCHRÖDINGER EQUATION BY USING TRIGONOMETRIC B-SPLINES

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Abstract. This paper includes four finite element methods which are based on quadratic, cubic, quartic and quintic trigonometric B-spline functions for space discretization and Crank-Nicolson method for time discretization, to be achieved the numerical solution of the Schrödinger equation (SE). The algorithms obtained by different degrees trigonometric B-spline Galerkin methods are new for getting numerical solution of the SE. To see the accuracy of the proposed methods, two numerical experiments are investigated and the comparison of the methods are given in the test problem section.

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1. INTRODUCTION

The SE with the conditions is

$$iu_t + u_{xx} + q\,|u|^2 u = 0,\ i = \sqrt{-1}, \tag{1.1}$$

$$u(a,t) = 0,\ u(b,t) = 0,\ t \geq 0, \tag{1.2}$$

$$u_x(a,t) = 0,\ u_x(b,t) = 0, \tag{1.3}$$

$$u(x,0) = f(x), \tag{1.4}$$

where u is a complex function, q is a real number corresponds to a focusing $(q > 0)$ or defocusing $(q < 0)$ effect of the cubic nonlinearity and $f(x)$ will be defined later. The SE has application to problems in nonlinear physical phenomena such as the propagation of optical pulses, waves in water and plasmas, superconductivity, self-focussing effects in laser pulses, quantum hydro dynamics and Bose-Einstein condensates. The lack of an analytical solution for the general initial conditions of

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the SE has led to the search for numerical solutions. Although the studies [3, 5, 6, 8, 9, 12, 16, 18, 19], which obtain the numerical solutions of the SE by using Galerkin method, have been included in the literature before, the use of trigonometric B-splines base for the Galerkin method to solve the SE is new.

The trigonometric B-spline function was worked out by Schoenberg in 1964 [17]. Lyche and Winther showed that a recurrence relation similar to the one for polynomial splines was satisfied by trigonometric B-splines [13]. The some properties of the trigonometric B-spline were studied by Walz in 1997 [20]. Cubic trigonometric B-spline was used for numerical solution of an important equation system in celestial mechanics [15]. Abbas et al. obtained the approximate solution of nonclassical diffusion problem using the two-time level implicit technique based on cubic trigonometric B-spline function [2]. The wave equation was solved using the cubic trigonometric B-spline collocation method in study [1]. Dag et al. solved the regularized long wave equation by using the trigonometric B-spline collocation method [7]. The studies [4] included the different finite element method based on trigonometric B-spline function to solve numerically the Burgers' equation. The cubic and quadratic trigonometric B-spline Galerkin finite element methods were introduced by Irk and Keskin to achieve the numerical solution of regularized long wave equation [10, 11]. Cubic trigonometric B-spline collocation method was applied to have the numerical solution of the fractional sub-diffusion equation and second-order hyperbolic telegraph equation [14, 21]. Zhu et al. suggested the cubic trigonometric B-spline based differential quadrature method for the numerical solution of fractional advection-diffusion equation [23]. Zahra interested the numerical solutions of the PHI-Four and Allen-Cahn equations by applying the trigonometric B-spline collocation method in study [22]. In this paper, we develop four finite element methods which are based on quadratic, cubic, quartic and quintic trigonometric B-spline functions. In the first section, the trigonometric B-spline functions are introduced. In the next section, the space and time discretization of the equation are described to obtain a fully discretized form of the SE for both of the methods. In the test problems section, the accuracy of the proposed methods are discussed with the help of the results obtained from the two test problems, and the obtained results are compared with each others.

Consider $\Omega = [a,b] \times [0,T]$ be smooth region with the grid points (x_m, t_n), where

$$x_m = a + mh,\ m = 0,1,2,\ldots,N,\ t_n = n\Delta t,\ n = 0,1,2,\ldots,$$

h and Δt are mesh size in the space and time direction respectively.

2. TRIGONOMETRIC B-SPLINES

The trigonometric B-spline which is of order 0 is indicated as:

$$T_i^0(x) = \begin{cases} 1, & x \in [x_i, x_{i+1}) \\ 0 & \text{otherwise.} \end{cases}$$

The others which are of orders $k = 1,2,3,\ldots$ can be obtained by using the following recursive equation

$$T_i^k(x) = \sin\left(\frac{x - x_i}{2}\right) B_i^{k-1}(x) + \sin\left(\frac{x_{i+k} - x}{2}\right) B_{i+1}^{k-1}(x),$$

where

$$B_j^{k-1}(x) = \begin{cases} \dfrac{T_j^{k-1}(x)}{\sin\left(\dfrac{x_{j+k-1} - x_j}{2}\right)}, & x_j < x_{j+k-1}, \\ 0, & x_j = x_{j+k-1}, \end{cases}$$

for $j = i$ or $j = i+1$. Accordingly, for $k = 2$ the quadratic trigonometric B-spline is obtained as:

$$T_i^2(x) = \frac{1}{\psi} \begin{cases} \varphi^2(x_{i-1}), & x \in [x_{i-1}, x_i), \\ -\varphi(x_{i-1})\varphi(x_{i+1}) - \varphi(x_{i+2})\varphi(x_i), & x \in [x_i, x_{i+1}), \\ \varphi^2(x_{i+2}), & x \in [x_{i+1}, x_{i+2}), \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

and for $k = 3$ the cubic trigonometric B-spline is obtained as:

$$T_i^3(x) = \frac{1}{\phi} \begin{cases} \varphi^3(x_{i-2}), & x \in [x_{i-2}, x_{i-1}), \\ -\varphi^2(x_{i-2})\varphi(x_i) - \varphi(x_{i-2})\varphi(x_{i-1})\varphi(x_{i+1}) & \\ -\varphi(x_{i+2})\varphi^2(x_{i-1}), & x \in [x_{i-1}, x_i), \\ \varphi(x_{i-2})\varphi^2(x_{i+1}) + \varphi(x_{i+2})\varphi(x_{i+1})\varphi(x_{i-1}) & \\ +\varphi^2(x_{i+2})\varphi(x_i), & x \in [x_i, x_{i+1}), \\ -\varphi^3(x_{i+2}), & x \in [x_{i+1}, x_{i+2}), \\ 0 & \text{otherwise,} \end{cases} \tag{2.2}$$

and for $k = 4$ the quartic trigonometric B-spline is obtained as:

$$T_i^4(x) = \frac{1}{\vartheta} \begin{cases} \varphi^4(x_{i-2}), & x \in [x_{i-2}, x_{i-1}), \\ -\varphi^3(x_{i-2})\varphi(x_i) - \varphi^2(x_{i-2})\varphi(x_{i+1})\varphi(x_{i-1}) \\ -\varphi(x_{i+2})\varphi(x_{i-2})\varphi^2(x_{i-1}) - \varphi(x_{i+3})\varphi^3(x_{i-1}), & x \in [x_{i-1}, x_i), \\ \varphi^2(x_{i-2})\varphi^2(x_{i+1}) + \varphi(x_{i+2})\varphi(x_{i+1})\varphi(x_{i-1})\varphi(x_{i-2}) \\ +\varphi^2(x_{i+2})\varphi(x_i)\varphi(x_{i-2}) + \varphi(x_{i+3})\varphi(x_{i+1})\varphi^2(x_{i-1}) & x \in [x_i, x_{i+1}), \\ +\varphi(x_{i+3})\varphi(x_{i+2})\varphi(x_i)\varphi(x_{i-1}) + \varphi^2(x_{i+3})\varphi^2(x_i), \\ -\varphi(x_{i-2})\varphi^3(x_{i+2}) - \varphi(x_{i+3})\varphi^2(x_{i+2})\varphi(x_{i-1}) \\ -\varphi^2(x_{i+3})\varphi(x_{i+2})\varphi(x_i) - \varphi^3(x_{i+3})\varphi(x_{i+1}), & x \in [x_{i+1}, x_{i+2}), \\ \varphi^4(x_{i+3}), & x \in [x_{i+2}, x_{i+3}), \\ 0 & \text{otherwise}, \end{cases} \tag{2.3}$$

and for $k = 5$ the quintic trigonometric B-spline is obtained as:

$$T_m^5(x) = \frac{1}{\tau} \begin{cases} \varphi^5(x_{m-3}), & x_{m-3} \leq x < x_{m-2} \\ -\varphi^4(x_{m-3})\varphi(x_{m-1}) - \varphi^3(x_{m-3})\varphi(x_m)\varphi(x_{m-2}) \\ -\varphi^2(x_{m-3})\varphi(x_{m+1})\varphi^2(x_{m-2}) & x_{m-2} \leq x < x_{m-1} \\ -\varphi(x_{m-3})\varphi(x_{m+2})\varphi^3(x_{m-2}) - \varphi(x_{m+3})\varphi^4(x_{m-2}), \\ \varphi^3(x_{m-3})\varphi^2(x_m) + \varphi^2(x_{m-3})\varphi(x_{m+1})\varphi(x_{m-2})\varphi(x_m) \\ +\varphi^2(x_{m-3})\varphi^2(x_{m+1})\varphi(x_{m-1}) \\ +\varphi(x_{m-3})\varphi(x_{m+2})\varphi^2(x_{m-2})\varphi(x_m) \\ +\varphi(x_{m-3})\varphi(x_{m+2})\varphi(x_{m-2})\varphi(x_{m+1})\varphi(x_{m-1}) \\ +\varphi(x_{m-3})\varphi^2(x_{m+2})\varphi^2(x_{m-1}) + \varphi(x_{m+3})\varphi^3(x_{m-2})\varphi(x_m) & x_{m-1} \leq x < x_m \\ +\varphi(x_{m+3})\varphi^2(x_{m-2})\varphi(x_{m+1})\varphi(x_{m-1}) \\ +\varphi(x_{m+3})\varphi(x_{m-2})\varphi(x_{m+2})\varphi^2(x_{m-1}) \\ +\varphi^2(x_{m+3})\varphi^3(x_{m-1}), \\ -\varphi^2(x_{m-3})\varphi^3(x_{m+1}) - \varphi(x_{m-3})\varphi(x_{m+2})\varphi(x_{m-2})\varphi^2(x_{m+1}) \\ -\varphi(x_{m-3})\varphi^2(x_{m+2})\varphi(x_{m-1})\varphi(x_{m+1}) \\ -\varphi(x_{m-3})\varphi^3(x_{m+2})\varphi(x_m) - \varphi(x_{m+3})\varphi^2(x_{m-2})\varphi^2(x_{m+1}) \\ -\varphi(x_{m+3})\varphi(x_{m-2})\varphi(x_{m+2})\varphi(x_{m-1})\varphi(x_{m+1}) & x_m \leq x < x_{m+1} \\ -\varphi(x_{m+3})\varphi(x_{m-2})\varphi^2(x_{m+2})\varphi(x_m) \\ -\varphi^2(x_{m+3})\varphi^2(x_{m-1})\varphi(x_{m+1}) \\ -\varphi^2(x_{m+3})\varphi(x_{m-1})\varphi(x_{m+2})\varphi(x_m) - \varphi^3(x_{m+3})\varphi^2(x_m), \\ \varphi(x_{m-3})\varphi^4(x_{m+2}) + \varphi(x_{m+3})\varphi(x_{m-2})\varphi^3(x_{m+2}) \\ +\varphi^2(x_{m+3})\varphi(x_{m-1})\varphi^2(x_{m+2}) & x_{m+1} \leq x < x_{m+2} \\ +\varphi^3(x_{m+3})\varphi(x_m)\varphi(x_{m+2}) + \varphi^4(x_{m+3})\varphi(x_{m+1}), \\ -\varphi^5(x_{m+3}), & x_{m+2} \leq x < x_{m+3} \\ 0 & \text{otherwise}, \end{cases} \tag{2.4}$$

where

$$h = x_i - x_{i-1}, \varphi(x_i) = \sin\left(\frac{x - x_i}{2}\right), \psi = \sin\left(\frac{h}{2}\right)\sin(h), \phi = \sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right),$$

$$\vartheta = \sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right)\sin(2h), \tau = \sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right)\sin(2h)\sin\left(\frac{5h}{2}\right).$$

3. Finite element solutions of SE

Let take the complex function u as

$$u(x,t) = r(x,t) + is(x,t), \tag{3.1}$$

where $r(x,t)$ and $s(x,t)$ are real functions. With substituting (3.1) into (1.1), the real and imaginary parts of the obtained function can be rewritten as two real differential equations:

$$r_t + s_{xx} + q(r^2 + s^2)s = 0, \tag{3.2}$$

$$s_t - r_{xx} - q(r^2 + s^2)r = 0. \tag{3.3}$$

Using the Crank-Nicolson method the time discretized forms of (3.2) and (3.3) are obtained as:

$$r^{n+1} + \frac{\Delta t}{2}(s_{xx})^{n+1} + q\frac{\Delta t}{2}\left(\left(r^2+s^2\right)s\right)^{n+1} = r^n - \frac{\Delta t}{2}(s_{xx})^n - q\frac{\Delta t}{2}\left(\left(r^2+s^2\right)s\right)^n, \tag{3.4}$$

$$s^{n+1} - \frac{\Delta t}{2}(r_{xx})^{n+1} - q\frac{\Delta t}{2}\left(\left(r^2+s^2\right)r\right)^{n+1} = s^n + \frac{\Delta t}{2}(r_{xx})^n + q\frac{\Delta t}{2}\left(\left(r^2+s^2\right)r\right)^n. \tag{3.5}$$

3.1. *Quadratic trigonometric B-spline Galerkin method (Method 1)*

By applying Galerkin finite element method to (3.4) and (3.5) and using integration by parts, we get

$$\int_a^b \left[W(x)\left(r^{n+1} + q\frac{\Delta t}{2}\left(\left(r^2+s^2\right)s\right)^{n+1}\right) - \frac{\Delta t}{2}W_x(x)(s_x)^{n+1}\right]dx$$
$$= \int_a^b \left[W(x)\left(r^n - q\frac{\Delta t}{2}\left(\left(r^2+s^2\right)s\right)^n\right) + \frac{\Delta t}{2}W_x(x)(s_x)^n\right]dx, \tag{3.6}$$

$$\int_a^b \left[W(x)\left(s^{n+1} - q\frac{\Delta t}{2}\left(\left(r^2+s^2\right)r\right)^{n+1}\right) + \frac{\Delta t}{2}W_x(x)(r_x)^{n+1}\right]dx$$
$$= \int_a^b \left[W(x)\left(s^n + q\frac{\Delta t}{2}\left(\left(r^2+s^2\right)r\right)^n\right) - \frac{\Delta t}{2}W_x(x)(r_x)^n\right]dx, \tag{3.7}$$

where $W(x)$ is a weight function. Let divide $[a,b]$ by N equally subintervals with the knots $x_j, j=1,\dots,N$. The approximate function U over all elements $[x_m,x_{m+1}]$ is

$$u(x,t) \simeq U(x,t) = \sum_{j=m-1}^{m+1} \delta_j(t) T_j^2(x), \tag{3.8}$$

where $\delta_j = \alpha_j + i\beta_j$. The approximate function and its first derivative can be written by using the quadratic trigonometric B-spline as

$$U(x_m,t) = \frac{\sin^2(\frac{h}{2})}{\psi}(\delta_{m-1}+\delta_m),\ U'(x_m,t) = \frac{\sin(h)}{\psi}(-\delta_{m-1}+\delta_m).$$

Let choose the weight function $W(x)$ as the quadratic trigonometric B-spline shape function (2.1) and by substituting approximate function (3.8) into (3.6) and (3.7), we have the following approximation over the subinterval $[x_m,x_{m+1}]$:

$$\begin{aligned}
&\sum_{j=m-1}^{m+1}\left\{\left(\int_{x_m}^{x_{m+1}} T_iT_j dx\right)\alpha_j^{n+1} - \frac{1}{2}\Delta t\left(\int_{x_m}^{x_{m+1}} T_i'T_j' dx\right)\beta_j^{n+1}\right.\\
&\left.+\frac{1}{2}q\Delta t\sum_{j=m-1}^{m+1}\sum_{l=m-1}^{m+1}\left(\int_{x_m}^{x_{m+1}} T_i\left[T_k\beta_k^{n+1}T_l\beta_l^{n+1} + T_k\alpha_k^{n+1}T_l\alpha_l^{n+1}\right]T_j dx\right)\beta_j^{n+1}\right\}\\
&-\sum_{j=m-1}^{m+1}\left\{\left(\int_{x_m}^{x_{m+1}} T_iT_j dx\right)\alpha_j^{n} + \frac{1}{2}\Delta t\left(\int_{x_m}^{x_{m+1}} T_i'T_j' dx\right)\beta_j^{n}\right.\\
&\left.-\frac{1}{2}q\Delta t\sum_{k=m-1}^{m+1}\sum_{l=m-1}^{m+1}\left(\int_{x_m}^{x_{m+1}} T_i\left[T_k\beta_k^{n}T_l\beta_l^{n} + T_k\alpha_k^{n}T_l\alpha_l^{n}\right]T_j dx\right)\beta_j^{n}\right\}
\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}
&\sum_{j=m-1}^{m+1}\left\{\left(\int_{x_m}^{x_{m+1}} T_iT_j dx\right)\beta_j^{n+1} + \frac{1}{2}\Delta t\left(\int_{x_m}^{x_{m+1}} T_i'T_j' dx\right)\alpha_j^{n+1}\right.\\
&\left.-\frac{1}{2}q\Delta t\sum_{k=m-1}^{m+1}\sum_{l=m-1}^{m+1}\left(\int_{x_m}^{x_{m+1}} T_i\left[T_k\beta_k^{n+1}T_l\beta_l^{n+1} + T_k\alpha_k^{n+1}T_l\alpha_l^{n+1}\right]T_j dx\right)\alpha_j^{n+1}\right\}\\
&-\sum_{j=m-1}^{m+1}\left\{\left(\int_{x_m}^{x_{m+1}} T_iT_j dx\right)\beta_j^{n} - \frac{1}{2}\Delta t\left(\int_{x_m}^{x_{m+1}} T_i'T_j' dx\right)\alpha_j^{n}\right.\\
&\left.+\frac{1}{2}q\Delta t\sum_{k=m-1}^{m+1}\sum_{l=m-1}^{m+1}\left(\int_{x_m}^{x_{m+1}} T_i\left[T_k\beta_k^{n}T_l\beta_l^{n} + T_k\alpha_k^{n}T_l\alpha_l^{n}\right]T_j dx\right)\alpha_j^{n}\right\}.
\end{aligned}\tag{3.10}$$

Denoting the integrals in the systems (3.9) and (3.10) by

$$A_{ij}^{e} = \int_{x_m}^{x_{m+1}} T_i T_j dx, \qquad B_{ij}^{e} = \int_{x_m}^{x_{m+1}} T_i^{'} T_j^{'} dx,$$

$$C_{ij}^{e} = \int_{x_m}^{x_{m+1}} T_i \left[T_k \beta_k^{n+1} T_l \beta_l^{n+1} + T_k \alpha_k^{n+1} T_l \alpha_l^{n+1} \right] T_j dx,$$

$$D_{ij}^{e} = \int_{x_m}^{x_{m+1}} T_i \left[T_k \beta_k^{n} T_l \beta_l^{n} + T_k \alpha_k^{n} T_l \alpha_l^{n} \right] T_j dx$$

and the systems (3.9), (3.10) are gathered over all elements, the matrix form of (3.9) and (3.10) can be written as

$$\mathbf{A}\alpha^{n+1} - \frac{\Delta t}{2}\mathbf{B}\beta^{n+1} + \frac{\Delta tq}{2}\mathbf{C}\beta^{n+1} = \mathbf{A}\alpha^{n} + \frac{\Delta t}{2}\mathbf{B}\beta^{n} - \frac{\Delta tq}{2}\mathbf{D}\beta^{n}, \tag{3.11}$$

$$\mathbf{A}\beta^{n+1} + \frac{\Delta t}{2}\mathbf{B}\alpha^{n+1} - \frac{\Delta tq}{2}\mathbf{C}\alpha^{n+1} = \mathbf{A}\beta^{n} - \frac{\Delta t}{2}\mathbf{B}\alpha^{n} + \frac{\Delta tq}{2}\mathbf{D}\alpha^{n}, \tag{3.12}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are the element matrices, $\alpha = (\alpha_{-1}, \alpha_0, \ldots, \alpha_{N-1}, \alpha_N)^T$ and $\beta = (\beta_{-1}, \beta_0, \ldots, \beta_{N-1}, \beta_N)^T$. The set of equations consists of $2N+4$ equations with $2N+4$ unknown parameters. For the nonlinear terms in these systems, we apply an inner iteration procedure. Before starting the procedure, boundary conditions must be adapted into the system. For this purpose, we delete first and last equations from the systems (3.11) and (3.12), and eliminate the terms α_{-1}, β_{-1} and α_N, β_N from the remaining systems (3.11) and (3.12) by using boundary conditions (1.2). So, we obtain a new matrix system with the dimension $(2N) \times (2N)$.

Using the boundary conditions (1.3) and initial condition (1.4), the initial parameters $(\alpha_{-1}^0, \beta_{-1}^0, \alpha_0^0, \beta_0^0, \ldots, \alpha_{N-1}^0, \beta_{N-1}^0, \alpha_N^0, \beta_N^0)$ can be obtained. Once the initial parameters α^0 and β^0 are determined, the unknown parameters α^n and β^n at time $t^n = n\Delta t, n = 1, 2, 3, \ldots$ are obtained. Thus the approximate solution (3.8) can be determined by using these values.

3.2. *Cubic trigonometric B-spline Galerkin method (Method 2)*

Let take the approximate function U over all elements $[x_m, x_{m+1}]$ as

$$u(x,t) \simeq U(x,t) = \sum_{j=m-1}^{m+2} \delta_j(t) T_j^3(x) \tag{3.13}$$

where $\delta_j = \alpha_j + i\beta_j$. The approximate function U and its first two derivatives can be obtained by using the cubic trigonometric B-spline functions (2.2). When we repeat the same procedure with selecting the cubic trigonometric B-spline shape function (2.2) as weight function $W(x)$ in the Galerkin discretized form of SE (3.6) and (3.7), and using the approximate function (3.13) and its derivatives in (3.6) and (3.7), we have the following approximation over the subinterval $[x_m, x_{m+1}]$:

$$\sum_{j=m-1}^{m+2}\left\{\left(\int_{x_m}^{x_{m+1}} T_iT_j dx\right)\alpha_j^{n+1}+\frac{1}{2}\Delta t\left(\int_{x_m}^{x_{m+1}} T_iT_j^{''} dx\right)\beta_j^{n+1}\right.$$
$$\left.+\frac{1}{2}q\Delta t\sum_{k=m-1}^{m+2}\sum_{l=m-1}^{m+2}\left(\int_{x_m}^{x_{m+1}} T_i\left[T_k\left(\beta_k^{n+1}\right)T_l\left(\beta_l^{n+1}\right)+T_k\left(\alpha_k^{n+1}\right)T_l\left(\alpha_l^{n+1}\right)\right]T_j dx\right)\beta_j^{n+1}\right\}$$
$$-\sum_{j=m-1}^{m+2}\left\{\left(\int_{x_m}^{x_{m+1}} T_iT_j dx\right)\alpha_j^{n}-\frac{1}{2}\Delta t\left(\int_{x_m}^{x_{m+1}} T_iT_j^{''} dx\right)\beta_j^{n}\right.$$
$$\left.-\frac{1}{2}q\Delta t\sum_{k=m-1}^{m+2}\sum_{l=m-1}^{m+2}\left(\int_{x_m}^{x_{m+1}} T_i\left[T_k\left(\beta_k^{n}\right)T_l\left(\beta_l^{n}\right)+T_k\left(\alpha_k^{n}\right)T_l\left(\alpha_l^{n}\right)\right]T_j dx\right)\beta_j^{n}\right\}. \tag{3.14}$$

$$\sum_{j=m-1}^{m+2}\left\{\left(\int_{x_m}^{x_{m+1}} T_iT_j dx\right)\beta_j^{n+1}-\frac{1}{2}\Delta t\left(\int_{x_m}^{x_{m+1}} T_iT_j^{''} dx\right)\alpha_j^{n+1}\right.$$
$$\left.-\frac{1}{2}q\Delta t\sum_{k=m-1}^{m+2}\sum_{l=m-1}^{m+2}\left(\int_{x_m}^{x_{m+1}} T_i\left[T_k\left(\beta_k^{n+1}\right)T_l\left(\beta_l^{n+1}\right)+T_k\left(\alpha_k^{n+1}\right)T_l\left(\alpha_l^{n+1}\right)\right]T_j dx\right)\alpha_j^{n+1}\right\}$$
$$-\sum_{j=m-1}^{m+2}\left\{\left(\int_{x_m}^{x_{m+1}} T_iT_j dx\right)\beta_j^{n}+\frac{1}{2}\Delta t\left(\int_{x_m}^{x_{m+1}} T_iT_j^{''} dx\right)\alpha_j^{n}\right.$$
$$\left.+\frac{1}{2}q\Delta t\sum_{k=m-1}^{m+2}\sum_{l=m-1}^{m+2}\left(\int_{x_m}^{x_{m+1}} T_i\left[T_k\left(\beta_k^{n}\right)T_l\left(\beta_l^{n}\right)+T_k\left(\alpha_k^{n}\right)T_l\left(\alpha_l^{n}\right)\right]T_j dx\right)\alpha_j^{n}\right\}. \tag{3.15}$$

Denoting the integrals in (3.14) and (3.15) by

$$A_{ij}^e=\int_{x_m}^{x_{m+1}} T_iT_j dx, \qquad B_{ij}^e=\int_{x_m}^{x_{m+1}} T_iT_j^{''} dx,$$
$$C_{ij}^e=\int_{x_m}^{x_{m+1}} T_i\left[T_k\left(\beta_k^{n+1}\right)T_l\left(\beta_l^{n+1}\right)+T_k\left(\alpha_k^{n+1}\right)T_l\left(\alpha_l^{n+1}\right)\right]T_j dx,$$
$$D_{ij}^e=\int_{x_m}^{x_{m+1}} T_i\left[T_k\left(\beta_k^{n}\right)T_l\left(\beta_l^{n}\right)+T_k\left(\alpha_k^{n}\right)T_l\left(\alpha_l^{n}\right)\right]T_j dx$$

and collecting the element matrices over all elements $[x_m, x_{m+1}]$, we get the following matrix form of (3.14) and (3.15):

$$\mathbf{A}\alpha^{n+1} + \frac{\Delta t}{2}\mathbf{B}\beta^{n+1} + \frac{\Delta tq}{2}\mathbf{C}\beta^{n+1} = \mathbf{A}\alpha^{n} - \frac{\Delta t}{2}\mathbf{B}\beta^{n} - \frac{\Delta tq}{2}\mathbf{D}\beta^{n}, \tag{3.16}$$

$$\mathbf{A}\beta^{n+1} - \frac{\Delta t}{2}\mathbf{B}\alpha^{n+1} - \frac{\Delta tq}{2}\mathbf{C}\alpha^{n+1} = \mathbf{A}\beta^{n} + \frac{\Delta t}{2}\mathbf{B}\alpha^{n} + \frac{\Delta tq}{2}\mathbf{D}\alpha^{n}, \tag{3.17}$$

where $\alpha = (\alpha_{-1}, \alpha_0, \ldots, \alpha_N, \alpha_{N+1})^T$ and $\beta = (\beta_{-1}, \beta_0, \ldots, \beta_N, \beta_{N+1})^T$. The linearization of the obtained systems are done in the same way as in method 1. Similarly, by eliminating the terms α_{-1}, β_{-1} and $\alpha_{N+1}, \beta_{N+1}$ from the systems (3.16) and (3.17), the obtained matrix system with dimension $(2N+2) \times (2N+2)$ is solved by Gauss elimination method. Again, firstly we attain the initial parameters α^0 and β^0, then the value of the approximate function U becomes computable.

3.3. *Quartic trigonometric B-spline Galerkin method (Method 3)*

In this method, the approximate function U over all elements $[x_m, x_{m+1}]$ with the quartic trigonometric B-spline functions is taken as

$$u(x,t) \simeq U(x,t) = \sum_{j=m-2}^{m+2} \delta_j(t) T_j^4(x) \tag{3.18}$$

where $\delta_j = \alpha_j + i\beta_j$. This approximate function and first and second derivatives of this can be obtained by the quartic trigonometric B-spline functions (2.3). With a similar procedure applied in the previous methods, by choosing the weight function as the quartic trigonometric B-spline functions in (3.6) and (3.7), the following expressions are obtained over the subinterval $[x_m, x_{m+1}]$:

$$\begin{aligned}
&\sum_{j=m-2}^{m+2} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \alpha_j^{n+1} + \frac{1}{2}\Delta t \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \beta_j^{n+1} \right. \\
&\left. + \frac{1}{2} q \Delta t \sum_{k=m-2}^{m+2} \sum_{l=m-2}^{m+2} \left(\int_{x_m}^{x_{m+1}} T_i \left[T_k \left(\beta_k^{n+1}\right) T_l \left(\beta_l^{n+1}\right) + T_k \left(\alpha_k^{n+1}\right) T_l \left(\alpha_l^{n+1}\right) \right] T_j dx \right) \beta_j^{n+1} \right\} \\
&- \sum_{j=m-2}^{m+2} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \alpha_j^{n} - \frac{1}{2}\Delta t \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \beta_j^{n} \right. \\
&\left. - \frac{1}{2} q \Delta t \sum_{k=m-2}^{m+2} \sum_{l=m-2}^{m+2} \left(\int_{x_m}^{x_{m+1}} T_i \left[T_k \left(\beta_k^{n}\right) T_l \left(\beta_l^{n}\right) + T_k \left(\alpha_k^{n}\right) T_l \left(\alpha_l^{n}\right) \right] T_j dx \right) \beta_j^{n} \right\},
\end{aligned} \tag{3.19}$$

$$\sum_{j=m-2}^{m+2}\left\{\left(\int_{x_m}^{x_{m+1}} T_iT_jdx\right)\beta_j^{n+1}-\frac{1}{2}\Delta t\left(\int_{x_m}^{x_{m+1}} T_iT_j^{''}dx\right)\alpha_j^{n+1}\right.$$
$$\left.-\frac{1}{2}q\Delta t\sum_{k=m-2}^{m+2}\sum_{l=m-2}^{m+2}\left(\int_{x_m}^{x_{m+1}} T_i\left[T_k\left(\beta_k^{n+1}\right)T_l\left(\beta_l^{n+1}\right)+T_k\left(\alpha_k^{n+1}\right)T_l\left(\alpha_l^{n+1}\right)\right]T_jdx\right)\alpha_j^{n+1}\right\}$$
$$-\sum_{j=m-2}^{m+2}\left\{\left(\int_{x_m}^{x_{m+1}} T_iT_jdx\right)\beta_j^{n}+\frac{1}{2}\Delta t\left(\int_{x_m}^{x_{m+1}} T_iT_j^{''}dx\right)\alpha_j^{n}\right.$$
$$\left.+\frac{1}{2}q\Delta t\sum_{k=m-2}^{m+2}\sum_{l=m-2}^{m+2}\left(\int_{x_m}^{x_{m+1}} T_i\left[T_k\left(\beta_k^{n}\right)T_l\left(\beta_l^{n}\right)+T_k\left(\alpha_k^{n}\right)T_l\left(\alpha_l^{n}\right)\right]T_jdx\right)\alpha_j^{n}\right\}. \tag{3.20}$$

Denoting the above integrals by

$$A_{ij}^e=\int_{x_m}^{x_{m+1}} T_iT_jdx, \qquad B_{ij}^e=\int_{x_m}^{x_{m+1}} T_iT_j^{''}dx,$$
$$C_{ij}^e=\int_{x_m}^{x_{m+1}} T_i\left[T_k\left(\beta_k^{n+1}\right)T_l\left(\beta_l^{n+1}\right)+T_k\left(\alpha_k^{n+1}\right)T_l\left(\alpha_l^{n+1}\right)\right]T_jdx,$$
$$D_{ij}^e=\int_{x_m}^{x_{m+1}} T_i\left[T_k\left(\beta_k^{n}\right)T_l\left(\beta_l^{n}\right)+T_k\left(\alpha_k^{n}\right)T_l\left(\alpha_l^{n}\right)\right]T_jdx$$

and collecting the element matrices over all elements $[x_m, x_{m+1}]$, we get the following matrix forms of (3.19) and (3.20):

$$\mathbf{A}\alpha^{n+1}+\frac{\Delta t}{2}\mathbf{B}\beta^{n+1}+\frac{\Delta tq}{2}\mathbf{C}\beta^{n+1}=\mathbf{A}\alpha^{n}-\frac{\Delta t}{2}\mathbf{B}\beta^{n}-\frac{\Delta tq}{2}\mathbf{D}\beta^{n}, \tag{3.21}$$
$$\mathbf{A}\beta^{n+1}-\frac{\Delta t}{2}\mathbf{B}\alpha^{n+1}-\frac{\Delta tq}{2}\mathbf{C}\alpha^{n+1}=\mathbf{A}\beta^{n}+\frac{\Delta t}{2}\mathbf{B}\alpha^{n}+\frac{\Delta tq}{2}\mathbf{D}\alpha^{n}, \tag{3.22}$$

where $\alpha=(\alpha_{-2},\alpha_{-1},\alpha_0,\dots,\alpha_N,\alpha_{N+1})^T$ and $\beta=(\beta_{-2},\beta_{-1},\beta_0,\dots,\beta_N,\beta_{N+1})^T$. By the same linearization procedure as shown in Methods 1 and 2, and the elimination of the terms α_{-2},β_{-2} and α_{N+1},β_{N+1} from the systems (3.21) and (3.22), a matrix system with dimension $(2N+4)\times(2N+4)$ is obtained. Then the value of the approximate function (3.18) is found similarly to the Methods 1 and 2.

3.4. *Quintic trigonometric B-spline Galerkin method (Method 4)*

By the fourth method, the approximate function U over all elements $[x_m, x_{m+1}]$ is choosen as

$$u(x,t) \simeq U(x,t) = \sum_{j=m-2}^{m+3} \delta_j(t) T_j^5(x) \tag{3.23}$$

where $\delta_j = \alpha_j + i\beta_j$. The approximate function, first and second derivatives of this can be obtained by the quintic trigonometric B-spline function (2.4). By choosing the weight function as the quintic trigonometric B-spline functions in (3.6) and (3.7), over the subinterval $[x_m, x_{m+1}]$ the following expressions are obtained:

$$\begin{aligned}
&\sum_{j=m-2}^{m+3} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \alpha_j^{n+1} + \frac{1}{2}\Delta t \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \beta_j^{n+1} \right. \\
&\left. + \frac{1}{2} q \Delta t \sum_{k=m-2}^{m+3} \sum_{l=m-2}^{m+3} \left(\int_{x_m}^{x_{m+1}} T_i \left[T_k \left(\beta_k^{n+1}\right) T_l \left(\beta_l^{n+1}\right) + T_k \left(\alpha_k^{n+1}\right) T_l \left(\alpha_l^{n+1}\right) \right] T_j dx \right) \beta_j^{n+1} \right\} \\
&- \sum_{j=m-2}^{m+3} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \alpha_j^{n} - \frac{1}{2}\Delta t \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \beta_j^{n} \right. \\
&\left. - \frac{1}{2} q \Delta t \sum_{k=m-2}^{m+3} \sum_{l=m-2}^{m+3} \left(\int_{x_m}^{x_{m+1}} T_i \left[T_k \left(\beta_k^{n}\right) T_l \left(\beta_l^{n}\right) + T_k \left(\alpha_k^{n}\right) T_l \left(\alpha_l^{n}\right) \right] T_j dx \right) \beta_j^{n} \right\},
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
&\sum_{j=m-2}^{m+3} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \beta_j^{n+1} - \frac{1}{2}\Delta t \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \alpha_j^{n+1} \right. \\
&\left. - \frac{1}{2} q \Delta t \sum_{k=m-2}^{m+3} \sum_{l=m-2}^{m+3} \left(\int_{x_m}^{x_{m+1}} T_i \left[T_k \left(\beta_k^{n+1}\right) T_l \left(\beta_l^{n+1}\right) + T_k \left(\alpha_k^{n+1}\right) T_l \left(\alpha_l^{n+1}\right) \right] T_j dx \right) \alpha_j^{n+1} \right\} \\
&- \sum_{j=m-2}^{m+3} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \beta_j^{n} + \frac{1}{2}\Delta t \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \alpha_j^{n} \right. \\
&\left. + \frac{1}{2} q \Delta t \sum_{k=m-2}^{m+3} \sum_{l=m-2}^{m+3} \left(\int_{x_m}^{x_{m+1}} T_i \left[T_k \left(\beta_k^{n}\right) T_l \left(\beta_l^{n}\right) + T_k \left(\alpha_k^{n}\right) T_l \left(\alpha_l^{n}\right) \right] T_j dx \right) \alpha_j^{n} \right\}.
\end{aligned} \tag{3.25}$$

Denoting the above integrals with

$$A_{ij}^{e}=\int_{x_m}^{x_{m+1}} T_iT_j dx, \qquad B_{ij}^{e}=\int_{x_m}^{x_{m+1}} T_iT_j^{''} dx,$$

$$C_{ij}^{e}=\int_{x_m}^{x_{m+1}} T_i\left[T_k\left(\beta_k^{n+1}\right)T_l\left(\beta_l^{n+1}\right)+T_k\left(\alpha_k^{n+1}\right)T_l\left(\alpha_l^{n+1}\right)\right]T_j dx,$$

$$D_{ij}^{e}=\int_{x_m}^{x_{m+1}} T_i\left[T_k\left(\beta_k^{n}\right)T_l\left(\beta_l^{n}\right)+T_k\left(\alpha_k^{n}\right)T_l\left(\alpha_l^{n}\right)\right]T_j dx$$

and collecting the element matrices over all elements $[x_m, x_{m+1}]$, we get the following matrix forms of (3.24) and (3.25):

$$\mathbf{A}\alpha^{n+1}+\frac{\Delta t}{2}\mathbf{B}\beta^{n+1}+\frac{\Delta tq}{2}\mathbf{C}\beta^{n+1}=\mathbf{A}\alpha^{n}-\frac{\Delta t}{2}\mathbf{B}\beta^{n}-\frac{\Delta tq}{2}\mathbf{D}\beta^{n}, \tag{3.26}$$

$$\mathbf{A}\beta^{n+1}-\frac{\Delta t}{2}\mathbf{B}\alpha^{n+1}-\frac{\Delta tq}{2}\mathbf{C}\alpha^{n+1}=\mathbf{A}\beta^{n}+\frac{\Delta t}{2}\mathbf{B}\alpha^{n}+\frac{\Delta tq}{2}\mathbf{D}\alpha^{n}, \tag{3.27}$$

where $\alpha=(\alpha_{-2},\alpha_{-1},\alpha_0,\ldots,\alpha_{N+1},\alpha_{N+2})^T$ and $\beta=(\beta_{-2},\beta_{-1},\beta_0,\ldots,\beta_{N+1},\beta_{N+2})^T$. Then, by eliminating the terms α_{-2},β_{-2} and α_{N+2},β_{N+2} from the matrix systems (3.26) and (3.27), the obtained matrix system with dimension $(2N+6)\times(2N+6)$ is solved similarly to the previous methods and the approximate function (3.23) can be computed.

4. Test Problems

In this section, we investigate the propagation of single soliton and the interaction of two solitons to demonstrate the efficiency of the given algorithms. Accuracy of the methods are measured by the error norm

$$L_\infty=\|u-U_M\|_\infty=\max_{0\le j\le N}\left|\|(u)_j\|-\|(U_M)_j\|\right|.$$

The conservation laws which are verified by the SE are

$$C_1=\int_a^b |U|^2 dx,\; C_2=\int_a^b\left[|U_x|^2-\frac{1}{2}q|U|^4\right]dx,$$

which are calculated by using the trapezoidal rule. The order of convergence for space and time steps are computed by the formulas

$$\text{order}=\frac{\log\left|(L_\infty)_{h_i}/(L_\infty)_{h_i+1}\right|}{\log|h_i/h_{i+1}|},\quad \text{order}=\frac{\log\left|(L_\infty)_{\Delta t_i}/(L_\infty)_{\Delta t_i+1}\right|}{\log|\Delta t_i/\Delta t_{i+1}|},$$

where $(L_\infty)_{h_i}$ is the error norm for space step and $(L_\infty)_{\Delta t_i}$ is the error norm for time step.

4.1. *The propagation of single soliton*

The single soliton solution of the SE, which is formed by the equilibrium between nonlinearity and dispersion is

$$U(x,t) = \lambda\sqrt{\frac{2}{q}}e^{i\eta}\mathrm{sech}\lambda(x-\xi t),$$

where $\eta = \frac{\xi}{2}x - \frac{1}{4}\left(\xi^2-\lambda^2\right)t$. This soliton, whose magnitude depends on the parameters ξ, λ and q, travels with the velocity ξ. In this experiment, we choose $\xi = 4$, $\lambda = 1$, $q = 2$, $\Delta t = 0.00001$ and $h = 0.05$ with range $[-30,30]$, the simulation is done up to $t = 1$, the initial condition is obtained from the solution for $t = 0$ and the conditions $U(-30,t)$, $U(30,t)$, $U_x(-30,t)$ and $U_x(30,t)$ are selected as zero. The values of the error norm L_∞ and the conservation invariants are listed in Table 1. Method 3 has the smallest value of the error norm. The simulation of the travelling wave for Method 3 is plotted in Figure 1 and it is observed that there is no change in its form throughout the simulation period. The absolute errors calculated for the wave up to $t = 1$ can be seen in Figure 2 for all of the proposed methods.

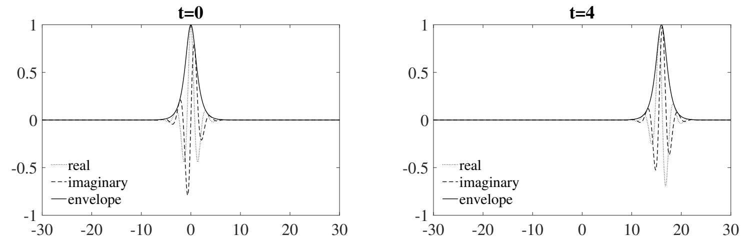

Fig. 1 The simulation of the travelling wave for Method 3

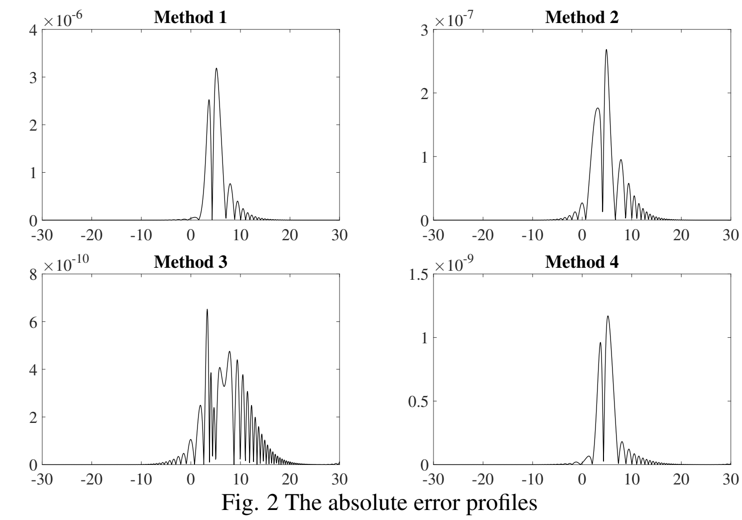

Fig. 2 The absolute error profiles

The rate of convergence of the proposed methods for space and time are shown in Tables 2 and 3. For all of the methods the rate of convergence for time is almost same and 2, and the rate of convergence for space is over 4 for Methods 1 and 2, over 6 for Methods 3 and 4. In addition, it can be said that all of the proposed methods give more accurate results as the space and time steps become smaller in Tables 2 and 3.

TABLE 1. The error norm L_∞ and the conservation invariants with range $[-30,30]$, $h = 0.05$ and $\Delta t = 0.00001$ at time $t = 1$.

Method	C_1	C_2	L_∞
Method 1	2.0000000000	7.3501541167	3.19e−06
Method 2	2.0000000000	7.3333173459	2.69e−07
Method 3	2.0000000000	7.3333357360	6.52e−10
Method 4	2.0000000000	7.3333333281	1.17e−09
Exact	2.0000000000	7.3333333333	0

TABLE 2. The error norm L_∞ and the order of convergence with range $[-30,30]$, $h = 0.05$ at time $t = 1$.

	Method 1		Method 2		Method 3		Method 4	
Δt	L_∞	order	L_∞	order	L_∞	order	L_∞	order
1e−1	1.12e−1		1.12e−1		1.12e−1		1.12e−1	
5e−2	2.72e−2	2.04	2.72e−2	2.04	2.72e−2	2.04	2.72e−2	2.04
2e−2	4.22e−3	2.03	4.22e−3	2.04	4.22e−3	2.03	4.22e−3	2.03
1e−1	1.05e−3	2.01	1.05e−3	2.01	1.05e−3	2.01	1.05e−3	2.01
5e−3	2.62e−4	2.00	2.62e−4	2.00	2.62e−4	2.00	2.62e−4	2.00
2e−3	4.20e−5	2.00	4.20e−5	2.00	4.19e−5	2.00	4.19e−5	2.00
1e−3	1.05e−5	2.00	1.05e−5	2.00	1.05e−5	2.00	1.05e−5	2.00

TABLE 3. The error norm L_∞ and the order of convergence with range $[-30,30]$, $\Delta t = 0.00001$ at time $t = 1$.

	Method 1		Method 2		Method 3		Method 4	
h	L_∞	order	L_∞	order	L_∞	order	L_∞	order
1	5.93e−1		1.94e−1		3.80e−1		1.54e−1	
0.5	4.90e−2	3.60	4.95e−3	5.29	5.07e−3	6.23	8.98e−4	7.42
0.2	8.96e−4	4.37	7.16e−5	4.62	5.06e−6	7.54	1.27e−6	7.16
0.1	5.19e−5	4.11	4.33e−6	4.05	6.71e−8	6.24	1.30e−8	6.61

4.2. *The interaction of two solitons*

As a second test problem, let take the initial condition

$$U(x,0) = U_1(x,0) + U_2(x,0),\ U_j(x,0) = \lambda_j \sqrt{\frac{2}{q}} e^{i\eta_j} \operatorname{sech} \lambda_j (x - x_j),$$
$$\eta_j = \frac{\xi_j}{2}(x - x_j).$$

By choosing the parameters $q = 2, h = 0.1, \Delta t = 0.01, \lambda_1 = \lambda_2 = 1, \xi_1 = 4, x_1 = -10, \xi_2 = -4, x_2 = 10$ over the interval $[-30, 30]$, two solitons appear and they move in opposite directions, so a collision is occurred. These interacted profiles obtained by Method 3 are shown in Figure 3 at various times. It can be seen in Figure 3 that two waves moving in the opposite direction collide, then separate and they appear to maintain their first forms.

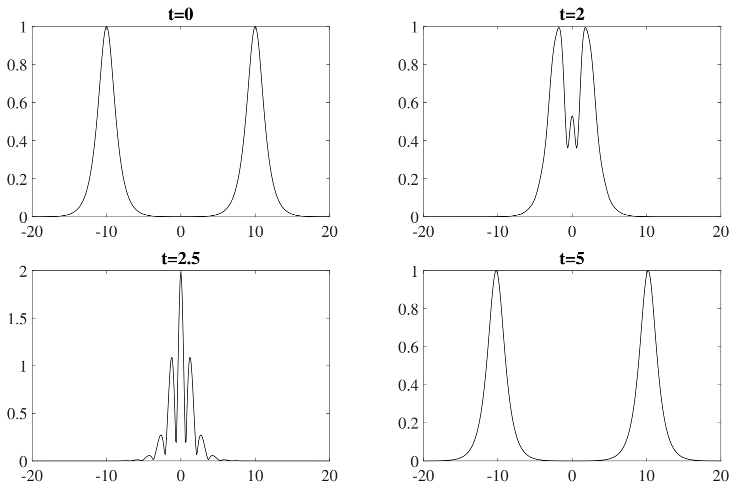

Fig. 3 The interaction of two solitons for Method 3

TABLE 4. The conservation invariants with $\Delta t = 0.01$ at time $t = 5$.

	$h = 0.1$		$h = 0.01$	
Method	C_1	C_2	C_1	C_2
Method 1	3.9999994111	14.8023658179	3.9999994314	14.6679992477
Method 2	3.9999994469	14.6661385046	3.9999994314	14.6666558174
Method 3	3.9999994311	14.6667354303	3.9999994314	14.6666558762
Method 4	3.9999994314	14.6666551815	3.9999994314	14.6666558685
Exact	4	14.6666666667	4	14.6666666667

The calculated values of the conservation constants are tabulated in Table 4. The best approximation value to the conservation constants has been obtained by Method 3 when $h = 0.01$ is chosen and Method 4 when $h = 0.1$ is chosen.

5. Conclusion

In this study, the fully discretized form of the SE has been obtained by the help of the four various trigonometric B-spline Galerkin methods and the accuracy of the proposed methods are investigated by comparing with each others. According to the results obtained in the first test problem in which the motion of soliton wave is examined, it is seen that by the too small selection for the time step Δt, the errors due to time discretization are minimized. Also it is seen that the errors decreases when the degree of spline function used in the calculations increases. By the use of high degree spline functions the better results have been obtained, although the cost of the process increases. From the second test problem in which the interaction of two solitary wave is examined, it can be said that the conservation conditions are nearly the same by selecting the various values of the h for all of the methods. In conclusion, the methods give the considerable results for both of the test problems.

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