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SOME PROPERTIES OF COFINITELY WEAK ESSENTIAL SUPPLEMENTED MODULES

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Abstract. Let M be an $R-$module. If every cofinite essential submodule of M has a weak supplement in M, then M is called a cofinitely weak essential supplemented (or briefly cwe-supplemented) module. In this work, some properties of these modules are investigated. It is proved that any sum of cwe-supplemented modules is cwe-supplemented. It is also proved that every factor module and every homomorphic image of a cwe-supplemented module are cwe-supplemented.

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1. Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an $R-$module. We will denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* (or *superfluous*) submodule of M and denoted by $N \ll M$. A submodule N of an R -module M is called an *essential* submodule and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equvalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. A submodule K of M is called a *cofinite* submodule of M if M/K is finitely generated. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is called a *supplemented* module if every submodule of M has a supplement in M. M is called an *essential supplemented* module if every essential submodule of M has a supplement in M. M is called a *cofinitely supplemented* module if every cofinite submodule of M has a supplement in M. M is called a *cofinitely essential supplemented* module if every cofinite essential submodule of M has a supplement in M. Let M be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a supplement V' with $V' \leq V$, we say U has *ample supplements* in M. If every submodule of M has ample supplements in M, then M is called an *amply supplemented* module. If every essential submodule of M has ample supplements in M, then M is called an *amply essential supplemented* module. If every cofinite submodule of M has ample supplements in M, then M is called an *amply cofinitely supplemented* module. If every cofinite essential submodule of M has ample supplements in M, then M is called an *amply cofinitely essential supplemented* module. Let M be an $R-$module and $U,V \leq M$. If $M = U + V$ and $U \cap V \ll M$, then V is called a *weak supplement* of U in M. M is said to be *weakly supplemented* if every submodule of M has a weak supplement in M. M is said to be *cofinitely weak supplemented* if every cofinite submodule of M has a weak supplement in M. M is called a *weakly essential supplemented* module if every essential submodule of M has a weak supplement in M. The intersection of all maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. Let M be an $R-$module and $U,K \leq M$. We say U *lies above* K in M if $K \leq U$ and $U/K \ll M/K$.

More informations about (amply) supplemented modules are in [4, 12–14]. The definitions of (amply) essential supplemented modules and some properties of them are in [8, 10, 11]. The definitions of (amply) cofinitely supplemented modules and some properties of them are in [1]. The definitions of (amply) cofinitely essential supplemented modules and some details of them are in [6, 7]. Some details about weakly supplemented and cofinitely weak supplemented modules are in [2, 4]. The definition of weakly essential supplemented modules and some properties of these modules are in [9].

2. Cofinitely weak essential supplemented modules

Definition 1. Let M be an $R-$module. If every cofinite essential submodule of M has a weak supplement in M, then M is called a cofinitely weak essential supplemented (or briefly cwe-supplemented) module. (See also [5])

Lemma 1. *Let M be a finitely generated $R-$module. Then M is weakly essential supplemented if and only if M is cwe-supplemented.*

Proof. Clear from definitions. □

Proposition 1. *Let M be a cwe-supplemented module. Then $M/RadM$ have no proper cofinite essential submodules.*

Proof. Let $\frac{K}{RadM}$ be any cofinite essential submodule of $\frac{M}{RadM}$. By $\frac{M}{K} \cong \frac{M/RadM}{K/RadM}$, K is a cofinite submodule of M. Since $\frac{K}{RadM} \trianglelefteq \frac{M}{RadM}$, then $K \trianglelefteq M$ and since M is cwe-supplemented, K has a weak supplement V in M. Here $M = K + V$ and $K \cap V \ll M$. Since $M = K + V$, $\frac{M}{RadM} = \frac{K}{RadM} + \frac{V+RadM}{RadM}$. Since $K \cap V \ll M$, by [13, 21.5], $K \cap V \leq RadM$. Then $\frac{K}{RadM} \cap \frac{V+RadM}{RadM} = \frac{K\cap V+RadM}{RadM} = 0$ and $\frac{M}{RadM} = \frac{K}{RadM} \oplus \frac{V+RadM}{RadM}$.

Since $\frac{M}{RadM} = \frac{K}{RadM} \oplus \frac{V+RadM}{RadM}$ and $\frac{K}{RadM} \trianglelefteq \frac{M}{RadM}$, $\frac{K}{RadM} = \frac{M}{RadM}$. Hence $\frac{M}{RadM}$ have no proper cofinite essential submodules. □

Proposition 2. *Let M be a cwe-supplemented module. If K is a proper cofinite essential submodule of M and $RadM \leq K$, then $K/RadM$ is not essential in $M/RadM$.*

Proof. Since $RadM \leq K$ and $K \neq M$, $K/RadM \neq M/RadM$. Since M is cwe-supplemented, K has a weak supplement V in M. Here $M = K+V$ and $K \cap V \ll M$. Since $M = K+V$, $\frac{M}{RadM} = \frac{K}{RadM} + \frac{V+RadM}{RadM}$. By $K \cap V \leq RadM$, $\frac{K}{RadM} \cap \frac{V+RadM}{RadM} = \frac{K\cap V+RadM}{RadM} = 0$ and $\frac{M}{RadM} = \frac{K}{RadM} \oplus \frac{V+RadM}{RadM}$. Following these we have $\frac{V+RadM}{RadM} \neq 0$ and since $\frac{K}{RadM} \cap \frac{V+RadM}{RadM} = 0$, $K/RadM$ is not essential in $M/RadM$. □

Lemma 2. *Let M be an $R-$module, U be a cofinite essential submodule of M and $M_1 \leq M$. If M_1 is cwe-supplemented and $U+M_1$ has a weak supplement in M, then U has a weak supplement in M.*

Proof. Let X be a weak supplement of $U+M_1$ in M. Then $M = U+M_1+X$ and $X \cap (U+M_1) \ll M$. Since U is a cofinite submodule of M and $\frac{M/U}{(U+X)/U} \cong \frac{M}{U+X} = \frac{M_1+U+X}{U+X} \cong \frac{M_1}{M_1\cap(U+X)}$, $M_1 \cap (U+X)$ is a cofinite submodule of M_1. Since $U \trianglelefteq M$, $(U+X) \trianglelefteq M$ and $(U+X)\cap M_1 \trianglelefteq M_1$. Then by M_1 being cwe-supplemented, $(U+X)\cap M_1$ has a weak supplement Y in M_1. Here $M_1 = (U+X)\cap M_1 + Y$ and $(U+X)\cap Y = (U+X)\cap M_1 \cap Y \ll M_1 \leq M$. Then $M = U+M_1+X = U+X+(U+X)\cap M_1 + Y = U+X+Y$ and $U \cap (X+Y) \leq (U+X)\cap Y + (U+Y)\cap X \leq (U+M_1)\cap X + (U+X)\cap Y \ll M$. Hence $X+Y$ is a weak supplement of U in M. □

Corollary 1. *Let U be a cofinite essential submodule of M and $M_i \leq M$ for $i = 1,2,...,n$. If M_i is cwe-supplemented for every $i = 1,2,...,n$ and $U+M_1+M_2+...+M_n$ has a weak supplement in M, then U has a weak supplement in M.*

Proof. Clear from Lemma 2. □

Lemma 3. *Any sum of cwe-supplemented modules is cwe-supplemented.*

Proof. Let U be a cofinite essential submodule of M and $M = \sum_{\lambda\in\Lambda} M_\lambda$ for $M_\lambda \leq M$ and M_λ be cwe-supplemented for every $\lambda \in \Lambda$. Since U is a cofinite submodule of M, then there exist $\lambda_1, \lambda_2, ..., \lambda_n \in \Lambda$ such that $M = U + M_{\lambda_1} + M_{\lambda_2} + ... + M_{\lambda_n}$. Then 0 is a weak supplement of $U + M_{\lambda_1} + M_{\lambda_2} + ... + M_{\lambda_n}$ in M. Since M_{λ_i} is cwe-supplemented for every $i = 1,2,...,n$, by Corollary 1, U has a weak supplement in M. Hence M is cwe-supplemented. □

Corollary 2. *Let M be a cwe-supplemented $R-$module. Then $M^{(\Lambda)}$ is cwe-supplemented for every index set Λ.*

Proof. Clear from Lemma 3. □

Lemma 4. *Every factor module of a cwe-supplemented module is cwe-supplemented.*

Proof. Let M be a cwe-supplemented $R-$module and $\frac{M}{K}$ be any factor module of M. Let $\frac{U}{K}$ be a cofinite essential submodule of $\frac{M}{K}$. Then U is a cofinite essential submodule of M and since M is cwe-supplemented, U has a weak supplement V in M. Here $M = U + V$ and $U \cap V \ll M$. Following we have $\frac{M}{K} = \frac{U}{K} + \frac{V+K}{K}$ and $\frac{U}{K} \cap \frac{V+K}{K} = \frac{U \cap V+K}{K} \ll \frac{V+K}{K}$. Hence $\frac{V+K}{K}$ is a weak supplement of $\frac{U}{K}$ in $\frac{M}{K}$ and $\frac{M}{K}$ is cwe-supplemented. □

Corollary 3. *Every homomorphic image of a cwe-supplemented module is cwe-supplemented.*

Proof. Clear from Lemma 4. □

Lemma 5. *Let M be a cwe-supplemented module. Then every $M-$generated $R-$module is cwe-supplemented.*

Proof. Let N be a $M-$generated $R-$module. Then there exist an index set Λ and an $R-$module epimorphism $f : M^{(\Lambda)} \longrightarrow N$. Since M is cwe-supplemented, by Corollary 2, $M^{(\Lambda)}$ is cwe-supplemented. Then by Corollary 3, N is cwe-supplemented. □

Lemma 6. *Let M be an $R-$module, $K \ll M$ and $\frac{U+K}{K} \trianglelefteq \frac{M}{K}$ for every $U \trianglelefteq M$. If M/K is cwe-supplemented, then M is also cwe-supplemented.*

Proof. Let U be any cofinite essential submodule of M. Since U is a cofinite submodule of M, we clearly see that $U + K$ is a cofinite submodule of M. By $\frac{M/K}{(U+K)/K} \cong \frac{M}{U+K}$, $(U+K)/K$ is a cofinite submodule of M/K. By hypothesis, $\frac{U+K}{K} \trianglelefteq \frac{M}{K}$ and since M/K is cwe-supplemented, $\frac{U+K}{K}$ has a weak supplement V/K in M/K. Here $\frac{M}{K} = \frac{U+K}{K} + \frac{V}{K} = \frac{U+V}{K}$ and $\frac{U \cap V+K}{K} = \frac{U+K}{K} \cap \frac{V}{K} \ll \frac{M}{K}$. Since $\frac{M}{K} = \frac{U+V}{K}$, then $M = U + V$. Let $U \cap V + T = M$ with $T \leq M$. Then $\frac{U \cap V+K}{K} + \frac{T+K}{K} = \frac{M}{K}$ and since $\frac{U \cap V+K}{K} \ll \frac{M}{K}$, $\frac{T+K}{K} = \frac{M}{K}$ and we have $T + K = M$. Since $K \ll M$, we have $T = M$. Hence $U \cap V \ll M$ and V is a weak supplement of U in M. Therefore, M is cwe-supplemented. □

Corollary 4. *Let $f : M \longrightarrow N$ be an $R-$module epimorphism, $Ker(f) \ll M$ and $f(U) \trianglelefteq N$ for every $U \trianglelefteq M$. If N is cwe-supplemented, then M is also cwe-supplemented.*

Proof. Clear from Lemma 6. □

Proposition 3. *Let R be a ring. The following assertions are equivalent.*
(i) *${}_RR$ is weakly essential supplemented*
(ii) *${}_RR$ is cwe-supplemented.*
(iii) *$R^{(\Lambda)}$ is cwe-supplemented for every index set Λ.*
(iv) *Every $R-$module is cwe-supplemented.*

Proof. $(i) \Longleftrightarrow (ii)$ Clear from Lemma 1, since ${}_RR$ is finitely generated.
$(ii) \Longleftrightarrow (iii)$ Clear from Corollary 2.

$(iii) \Longrightarrow (iv)$ Let M be an $R-$module. Then there exist an index set Λ and an $R-$module epimorphism $f : R^{(\Lambda)} \longrightarrow M$. Since $R^{(\Lambda)}$ is cwe-supplemented, by Corollary 3, M is also cwe-supplemented.
$(iv) \Longrightarrow (ii)$ Clear. □

Let M be an $R-$module. We say submodules X and Y of M are β^* equivalent, $X\beta^*Y$, if and only if $(X+Y)/X \ll M/X$ and $(X+Y)/Y \ll M/Y$. More details about β^* relation are in [3].

Lemma 7. *Let M be an $R-$module. If every cofinite essential submodule of M is β^* equivalent to a weak supplement submodule in M, then M is cwe-supplemented.*

Proof. Let U be any cofinite essential submodule of M. By hypothesis, there exists a weak supplement submodule X such that $U\beta^*X$ in M. Since X is a weak supplement submodule in M, there exists $V \leq M$ such that X is a weak supplement of V in M. Then V is a weak supplement of X in M. Since $U\beta^*X$, by [3, Theorem 2.6 (ii)], V is a weak supplement of U in M. Hence M is cwe-supplemented. □

Corollary 5. *Let M be an $R-$module. If every cofinite essential submodule lies above a weak supplement submodule in M, then M is cwe-supplemented.*

Proof. Let U be any cofinite essential submodule of M. By hypothesis, there exists a weak supplement submodule X such that U lies above X in M. Since U lies above X in M, we clearly see that $U\beta^*X$ in M. Hence every cofinite essential submodule of M is β^* equivalent to a weak supplement submodule in M and by Lemma 7, M is cwe-supplemented. □

Corollary 6. *Let M be an $R-$module. If every cofinite essential submodule of M is β^* equivalent to a supplement submodule in M, then M is cwe-supplemented.*

Proof. Clear from Lemma 7, since every supplement submodule is a weak supplement submodule in M. □

Corollary 7. *Let M be an $R-$module. If every cofinite essential submodule lies above a supplement submodule in M, then M is cwe-supplemented.*

Proof. Clear from Corollary 5, since every supplement submodule is a weak supplement submodule in M. □

Lemma 8. *Let M be a cwe-supplemented $R-$module. If every weak supplement of any cofinite essential submodule of M is a supplement in M, then M is cofinitely essential supplemented.*

Proof. Let U be any cofinite essential submodule of M. Since M is cwe-supplemented, U has a weak supplement V in M. Here $M = U + V$ and $U \cap V \ll M$. By hypothesis, V is a supplement in M. Since $U \cap V \ll M$, by [12, Lemma 2.5], $U \cap V \ll V$. Hence V is a supplement of U in M. Therefore, M is cofinitely essential supplemented. □

Corollary 8. *Let M be a finitely generated cwe-supplemented $R-$module. If every weak supplement submodule in M is a supplement in M, then M is essential supplemented.*

Proof. Clear from Lemma 8, since every submodule of M is cofinite. □

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