Miskolc Mathematical Notes
Vol. 23 (2022), No. 1, pp. 253–269
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2022.3415

POSITIVE SOLUTIONS FOR SECOND ORDER IMPULSIVE DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS ON AN INFINITE INTERVAL

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Received 07 September, 2020

Abstract. This paper is concerned with the existence of positive solutions of second-order impulsive differential equations with integral boundary conditions on an infinite interval. As an application, an example is given to demonstrate our main results.

2010 *Mathematics Subject Classification:* 34B37;34B15;34B18;34B40

Keywords: impulsive differential equations, positive solutions, fixed point theorems in cones, impulsive boundary value problems, integral boundary conditions

1. INTRODUCTION

Consider the following second-order impulsive integral boundary value problem (IBVP) with integral boundary conditions,

$$\begin{cases} \dfrac{1}{p(t)}(p(t)x'(t))' + f(t,x(t),x'(t)) = 0, \quad \forall t \in J_+^{'} \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \qquad k = 1,2,\ldots,n \\ \Delta x'|_{t=t_k} = -\bar{I}_k(x(t_k)), \quad k = 1,2,\ldots,n \\ a_1 x(0) - b_1 \lim\limits_{t\to 0^+} p(t)x'(t) = \displaystyle\int_0^\infty g_1(x(s))\psi(s)ds, \\ a_2 \lim\limits_{t\to\infty} x(t) + b_2 \lim\limits_{t\to\infty} p(t)x'(t) = \displaystyle\int_0^\infty g_2(x(s)\psi(s)ds, \end{cases} \tag{1.1}$$

where $J = [0,\infty)$, $J_+ = (0,\infty)$, $J'_+ = J_+\backslash\{t_1,\ldots,t_n\}$, $0 < t_1 < t_2 < \ldots < t_n$, and note $J_0 = [0,t_1)$, $J_i = (t_i,t_{i+1}]$, $(i = 1,2,\ldots,n)$, $\Delta x|_{t=t_k}$ and $\Delta x'|_{t=t_k}$ denote the jump of $x(t)$ and $x^{'}(t)$ at $t = t_k$, i.e.,

$$\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-), \qquad \Delta x'|_{t=t_k} = p(t_k)[x'(t_k^+) - x'(t_k^-)],$$

where $x(t_k^+)$, $x^{'}(t_k^+)$ and $x(t_k^-)$, $x^{'}(t_k^-)$ denote the right-hand limit and left-hand limit of $x(t)$ and $x^{'}(t)$ at $t = t_k$, $k = 1,2,\ldots,n$, respectively.

Throughout this paper, we assume that the following conditions hold;

$(H1)$ $a_1, a_2, b_1, b_2 \in J$ with $D = a_2 b_1 + a_1 b_2 + a_1 a_2 B(0,\infty) > 0$ in which

$$B(t,s) = \int_t^s \frac{d\sigma}{p(\sigma)}.$$

$(H2)$ $f \in C(J_+ \times J_+ \times \mathbb{R}, J_+)$ and also, $f(t,x,y) \le k(t)h(x,y)$, $(t \in J_+)$, where $h \in C(J_+ \times \mathbb{R}, J_+)$ and for $t \in J_+$, x, y in a bounded set, $h(x,y)$ is bounded and $k \in C(J_+, J_+)$.

$(H3)$ $g_1, g_2 : J_+ \to J_+$ are continuous, nondecreasing functions, and for $t \in J_+$, x in a bounded set, $g_1(x), g_2(x)$ are bounded.

$(H4)$ $I_k, \bar{I}_k \in C(J_+, J_+)$ are bounded functions where

$$[b_2 + a_2 B(t_k,\infty)]\bar{I}_k(x(t_k)) - \frac{a_2}{p(t_k)} I_k(x(t_k)) > 0, \quad (k = 1,2,...,n).$$

$(H5)$ $\psi : J \to J$ is a continuous function with $\int_0^\infty \psi(s)ds < \infty$.

$(H6)$ $p \in C(J,J_+) \cap C^1(J_+,J_+)$ with $p > 0$ on J_+, and $\int_0^\infty \frac{ds}{p(s)} < \infty$.

Impulsive differential equations encountered in physics, chemical technology, population dynamics, biotecnology, economics etc. (see [2] and the references there in) have become more important in recent years due to the appearance of some important in recent years due to the appearance of some mathematical models of the actual processes. A significant development was observed in theory of impulsive differential equations with fixed time of pulses; see the monographs by Bainov and Simeorov [5], Lakshmikantham, et al. [1], Samoilenko and Prestyuk [18], Benchohra, et al. [4] and the papers [3, 6, 7, 9–11, 13–17, 19].

The existence and multiplicity of positive solutions for linear and nonlinear second-order impulsive dynamic equations have been extensively studied, see [11,12,21–23]. Due to the fact that an infinite interval is noncompact, the discussion about boundary value problems on the half-line more complicated, in particular, for impulsive IBVP on an infinite interval, few works were done, see [8,25]. There is not work on positive solutions for double impulsive IBVP on an infinite interval expect that in [24, 27].

In [27], Zhang, Yang and Feng studied the following double impulsive IBVP:

$$\begin{cases} -x''(t) = f(t,x(t),x'(t)), \quad t \in J, \quad t \ne t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \qquad k = 1,2,... \\ \Delta x'|_{t=t_k} = \bar{I}_k(x(t_k)), \qquad k = 1,2,... \\ x(0) = \int_0^\infty g(t)x(t)dt, \qquad x'(\infty) = 0. \end{cases}$$

Using the fixed point theorem in cones, they obtained criteria for existence of the multiple positive solutions.

In [24], Yu, Wang and Guo discussed the existence and multiple positive solutions for the following nonlinear second-order double impulsive integral boundary value problems:

$$\begin{cases} (\phi_p(x^{'}(t)))^{'} + a(t)f(t,x(t),x^{'}(t)) = 0, \ \ t \in J, \ \ t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \qquad k = 1,2,... \\ \Delta \phi_p(x^{'})|_{t=t_k} = \bar{I}_k(x(t_k)), \quad k = 1,2,... \\ x(0) = \int_0^{\infty} g(t)x(t)dt, \qquad x^{'}(\infty) = 0. \end{cases}$$

Motivated by the above works, in this study, we consider the existence of two positive solutions for the second-order double impulsive integral boundary value problem (1.1). Our boundary conditions are more general. Hence, these results can be considered as a contribution to this field.

The present paper is organized as follows. In Section 2, we present some preliminaries and lemmas which are key tools for our main results. We give and prove our main results in Section 3. Finally, in Section 4, we give an example to demonstrate our results.

2. Preliminaries and Lemmas

In this section, we will employ several lemmas to prove the main results in this paper. Set

$$PC(J) = \{x : J \to \mathbb{R} : \ x \in C(J^{'}), \ x(t_k^{+}) \text{ and } x(t_k^{-}) \text{ exist and } x(t_k^{-}) = x(t_k), \ \ 1 \le k \le n\}.$$

$$PC^1(J) = \{x \in PC(J) : \ x^{'} \in C(J^{'}), \ x^{'}(t_k^{+}) \text{ and } x^{'}(t_k^{-}) \text{ exist and } x^{'}(t_k^{-}) = x^{'}(t_k)\}.$$

$$BPC^1(J) = \{x \in PC^1(J) : \lim_{t\to\infty} x(t) \text{ exists, and } \sup_{t\in J} |x^{'}(t)| < \infty\}.$$

It is easy to see that $BPC^1(J)$ is a Banach space with the norm

$$\|x\| = \sup_{t\in J} \{|x(t)| + |x^{'}(t)|\}.$$

A function $x \in PC^1(J) \cap C^2(J^{'}_{+})$ is called a positive solution of the impulsive IBVP (1.1) if $x(t) > 0$ for all $t \in J$ and $x(t)$ satisfies (1.1).

We define a cone $K \subset BPC^1(J)$ as follows:

$$K = \{x \in BPC^1(J) : x(t) > 0, \ t \in J_+\}.$$

K is a positive cone in $BPC^1(J)$.

By θ and φ we denote the solutions of the corresponding homogeneous equation

$$\frac{1}{p(t)}(p(t)x'(t))' = 0, \ t \in (0,\infty), \tag{2.1}$$

under the initial conditions,

$$\begin{aligned} \theta(0) = b_1, \ \ &\lim_{t\to 0^+} p(t)\theta'(t) = a_1, \\ \lim_{t\to\infty} \varphi(t) = b_2, \ \ &\lim_{t\to\infty} p(t)\varphi'(t) = -a_2. \end{aligned} \tag{2.2}$$

Using the initial conditions (2.2), we can deduce from equation (2.1) for $\theta(t)$ and $\varphi(t)$, the following equations:

$$\theta(t) = b_1 + a_1 \int_0^t \frac{ds}{p(s)}, \tag{2.3}$$

$$\varphi(t) = b_2 + a_2 \int_t^\infty \frac{ds}{p(s)}. \tag{2.4}$$

Let $G(t,s)$ be the Green Function for (1.1) is given by

$$G(t,s) = \frac{1}{D} \begin{cases} \theta(t)\varphi(s), & 0 \le t \le s < \infty, \\ \theta(s)\varphi(t), & 0 \le s \le t < \infty. \end{cases} \tag{2.5}$$

where $\theta(t)$ and $\varphi(t)$ are given in (2.3) and (2.4) respectively.

Lemma 1. *Suppose that $(H1)-(H6)$ are satisfied. Then $x \in PC^1(J) \cap C^2(J_+^{'})$ is a solution of the impulsive IBVP (1.1) if and only if $x(t)$ is a solution of the following integral equation*

$$x(t) = \int_0^\infty G(t,s)p(s)f(s,x(s),x^{'}(s))ds + \frac{\varphi(t)}{D}\int_0^\infty g_1(x(s))\psi(s)ds$$
$$+ \frac{\theta(t)}{D}\int_0^\infty g_2(x(s))\psi(s)ds + \sum_{k=1}^n G(t,t_k)\bar{I}_k(x(t_k)) + \sum_{k=1}^n p(t_k)G_s(t,s)|_{s=t_k} I_k(x(t_k)),$$

where G(t,s) is given by (2.5).

Remark 1. Under the conditions $(H1)$ and $(H6)$, the Green function $G(t,s)$ in equation (2.5) possesses the following properties:

(1) $G(t,s)$ is continuous on $J_+ \times J_+$,
(2) for each $s \in J_+$, G(t,s) is continuously differentiable on J_+ except $t = s$,
(3) $\dfrac{\partial G(t,s)}{\partial t}|_{t=s^+} - \dfrac{\partial G(t,s)}{\partial t}|_{t=s^-} = \dfrac{1}{p(s)}$,
(4) $G(t,s) \le G(s,s) < \infty$, and $G_s(t,s) \le G_s(t,s)\big|_{t=s} < \infty$,
(5) $|G_t(t,s)| \le \dfrac{c}{p(t)} G(s,s)$, and $|G_{st}(t,s)| \le \dfrac{c}{p(t)} G_s(t,s)\big|_{t=s}$, where

$$c = \frac{\max\{a_1,a_2\}}{\min\{b_1,b_2\}}, \tag{2.6}$$

(6) $\overline{G}(s) = \lim\limits_{t\to\infty} G(t,s) = \dfrac{b_2}{D}\,\theta(s) \le G(s,s) < \infty$,
(7) $\overline{G}^{'}(s) = \lim\limits_{t\to\infty} G_s(t,s) = \dfrac{b_2}{D}\,\theta^{'}(s) \le G_s(t,s)\big|_{t=s} < \infty$,
(8) for any $t \in [a,b] \subset (0,\infty)$ and $s \in [0,\infty)$, we have

$$G(t,s) \ge wG(s,s),$$

where

$$w = \min\Big\{\frac{b_1 + a_1 B(0,a)}{b_1 + a_1 B(0,\infty)}, \frac{b_2 + a_2 B(b,\infty)}{b_2 + a_2 B(0,\infty)}\Big\}. \tag{2.7}$$

Obviously, $0 < w < 1$.

Define

$$\begin{aligned}(Tx)(t) = &\int_0^\infty G(t,s)p(s)f(s,x(s),x'(s))ds + \frac{\varphi(t)}{D}\int_0^\infty g_1(x(s))\psi(s)ds \\ &+ \frac{\theta(t)}{D}\int_0^\infty g_2(x(s))\psi(s)ds + \sum_{k=1}^n G(t,t_k)\bar{I}_k(x(t_k)) \\ &+ \sum_{k=1}^n p(t_k)G_s(t,s)|_{s=t_k} I_k(x(t_k)),\end{aligned} \tag{2.8}$$

where G is defined by as in (2.5).

Obviously, the impulsive IBVP (1.1) has a solution x if and only if $x \in K$ is a fixed point of the operator T defined by (2.8).

It is convenient to list the following condition which is to be used in our theorems:

$(H7)$ $0 < \int_0^\infty G(s,s)p(s)k(s)ds < \infty.$

As we know that the Ascoli-Arzela Theorem does not hold in infinite interval J, we need the following compactness criterion:

Lemma 2 ([20]). *Let $M \subset BPC^1(J)$. Then M is relatively compact in $BPC^1(J)$ if the following conditions hold.*

- (i) *M is uniformly bounded in $BPC^1(J)$.*
- (ii) *The function belonging to M are equicontinuous on any compact interval of $[0,\infty)$.*
- (iii) *The functions from M are equiconvergent, that is, for any given $\varepsilon > 0$, there exist a $T = T(\varepsilon) > 0$ such that $|f(t) - f(\infty)| < \varepsilon$ for any $t > T$, $f \in M$.*

The main tool of this work is a fixed point theorem in cones.

Lemma 3 ([26]). *Let X be an Banach space and K is a positive cone in X. Assume that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Let $T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ be a completely continuous operator such that*

- (i) *$\|Tx\| \le \|x\|$ for all $x \in K \cap \partial\Omega_1$.*
- (ii) *There exists a $\Phi \in K$ such that $x \neq Tx + \lambda\Phi$, for all $x \in K \cap \partial\Omega_2$ and $\lambda > 0$.*

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Lemma 4. *If (H1)-(H7) are satisfied, then for any bounded open set $\Omega \subset BPC^1(J)$, $T : K \cap \overline{\Omega} \to K$ is a completely continuous operator.*

Proof. For any bounded open set $\Omega \subset BPC^1(J)$, there exists a constant $M > 0$ such that $\|x\| \leq M$ for any $x \in \overline{\Omega}$.

First, we show $T : K \cap \overline{\Omega} \to K$ is well defined. Let $x \in K \cap \overline{\Omega}$. From $(H2)$, $(H3)$ and $(H4)$, we have

$$S_M = sup\{S_1, S_2, S_3, S_4, S_5\}, \tag{2.9}$$

where

$$\begin{aligned} &S_1 = sup\{h(x,y) : |x| + |y| \leq M\} < \infty, && S_2 = sup\{I_k(x) : \ 0 \leq x \leq M\}, \\ &S_3 = sup\{\bar{I}_k(x) : \ 0 \leq x \leq M\}, && S_4 = sup\{g_1(x) : \ 0 \leq x \leq M\}, \\ &S_5 = sup\{g_2(x) : \ 0 \leq x \leq M\}. \end{aligned}$$

Let $t_1, t_2 \in J, \ t_1 < t_2,$ then

$$\int_0^\infty \big|G(t_1,s) - G(t_2,s)\big| p(s)k(s)ds \leq 2\int_0^\infty G(s,s)p(s)k(s)ds < \infty. \tag{2.10}$$

Hence, by the Lebesgue dominated convergence theorem, we have for any $t_1, t_2 \in J$, $x \in K \cap \overline{\Omega}$, we have

$$\begin{aligned} &\big|(Tx)(t_1) - (Tx)(t_2)\big| \\ &\leq \int_0^\infty \big|G(t_1,s) - G(t_2,s)\big| p(s) f(s, x(s), x^{'}(s))ds \\ &\quad + \frac{|\varphi(t_1) - \varphi(t_2)|}{D}\int_0^\infty g_1(x(s))\psi(s)ds + \frac{|\theta(t_1) - \theta(t_2)|}{D}\int_0^\infty g_2(x(s))\psi(s)ds \\ &\quad + \sum_{k=1}^n \big|G(t_1,t_k) - G(t_2,t_k)\big| \bar{I}_k(x(t_k)) \\ &\quad + \sum_{k=1}^n \big|p(t_k)G_s(t,s)|_{\substack{t=t_1 \\ s=t_k}} - p(t_k)G_s(t,s)|_{\substack{t=t_2 \\ s=t_k}}\big| I_k(x(t_k)) \\ &\leq S_M \bigg\{ \int_0^\infty \big|G(t_1,s) - G(t_2,s)\big| p(s)k(s)ds \\ &\quad + \left[\frac{|\varphi(t_1) - \varphi(t_2)|}{D} + \frac{|\theta(t_1) - \theta(t_2)|}{D}\right]\int_0^\infty \psi(s)ds + \sum_{k=1}^n \big|G(t_1,t_k) - G(t_2,t_k)\big| \\ &\quad + \frac{1}{D}\sum_{t_k \leq t_1} p(t_k)\theta^{'}(t_k)\big|\varphi(t_1) - \varphi(t_2)\big| \bigg\} + \frac{1}{D}\sum_{t_2 \leq t_k} p(t_k)\big|\varphi^{'}(t_k)\big|\big|\theta(t_1) - \theta(t_2)\big| \\ &\quad + \frac{1}{D}\sum_{t_1 \leq t_k \leq t_2} p(t_k)\big|\theta^{'}(t_1)\varphi^{'}(t_k) - \theta^{'}(t_k)\varphi^{'}(t_2)\big| \bigg\} \\ &\to 0 \ \text{ as } \ t_1 \to t_2, \end{aligned} \tag{2.11}$$

$$\big|(Tx)^{'}(t_1) - (Tx)^{'}(t_2)\big|$$

$$
\begin{aligned}
&\leq S_M\Big\{\frac{a_2}{D}\Big|\frac{1}{p(t_1)}-\frac{1}{p(t_2)}\Big|\int_0^{t_1}\theta(s)p(s)k(s)ds+\frac{a_1}{Dp(t_1)}\int_{t_1}^{t_2}\varphi(s)p(s)k(s)ds\\
&\quad+\frac{a_1}{D}\Big|\frac{1}{p(t_1)}-\frac{1}{p(t_2)}\Big|\int_{t_2}^{\infty}\varphi(s)p(s)k(s)ds+\frac{a_2}{Dp(t_2)}\int_{t_1}^{t_2}\theta(s)p(s)k(s)ds\\
&\quad+\frac{a_2}{D}\Big|\frac{1}{p(t_2)}-\frac{1}{p(t_1)}\Big|\int_0^{\infty}\psi(s)ds+\frac{a_1}{D}\Big|\frac{1}{p(t_1)}-\frac{1}{p(t_2)}\Big|\int_0^{\infty}\psi(s)ds\\
&\quad+\frac{a_1}{D}\Big|\frac{1}{p(t_1)}-\frac{1}{p(t_2)}\Big|\sum_{t_2\leq t_k}\big[\varphi(t_k)+p(t_k)|\varphi'(t_k)|\big]\\
&\quad+\frac{a_2}{D}\Big|\frac{1}{p(t_1)}-\frac{1}{p(t_2)}\Big|\sum_{t_k\leq t_1}\big[\theta(t_k)+p(t_k)\theta'(t_k)\big]\\
&\quad+\frac{a_1}{Dp(t_1)}\sum_{t_1\leq t_k\leq t_2}\big[\varphi(t_k)+p(t_k)|\varphi'(t_k)|\big]+\frac{a_2}{Dp(t_2)}\sum_{t_1\leq t_k\leq t_2}\big[\theta(t_k)+p(t_k)\theta'(t_k)\big]\Big\}\\
&\to 0 \text{ as } t_1\to t_2. \qquad (2.12)
\end{aligned}
$$

Thus, $Tx\in PC^1(J)$. We can show that $Tx\in BPC^1(J)$.

Then by $(H5)$, $(H7)$, the properties (5), (6), (7) of Remark 1 and the Lebesgue dominated convergence theorem, we have

$$
\begin{aligned}
\lim_{t\to\infty}(Tx)(t)&=\int_0^{\infty}\overline{G}(s)p(s)f(s,x(s),x'(s))ds+\frac{\varphi(\infty)}{D}\int_0^{\infty}g_1(x(s))\psi(s)ds\\
&\quad+\frac{\theta(\infty)}{D}\int_0^{\infty}g_2(x(s))\psi(s)ds+\sum_{k=1}^{n}\overline{G}(t_k)\overline{I}_k(x(t_k)) \qquad (2.13)\\
&\quad+\sum_{k=1}^{n}p(t_k)\overline{G}'(t_k)I_k(x(t_k))<\infty
\end{aligned}
$$

and

$$
\begin{aligned}
\big|(Tx)'(t)\big|\leq S_M\Big\{&\frac{c}{p(t)}\int_0^{\infty}G(s,s)p(s)k(s)ds+\frac{\max\{a_1,a_2\}}{Dp(t)}\int_0^{\infty}\psi(s)ds\\
&+\frac{c}{p(t)}\sum_{k=1}^{n}G(t_k,t_k)+\frac{c}{p(t)}\sum_{k=1}^{n}p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}}\Big\}<\infty. \qquad (2.14)
\end{aligned}
$$

Therefore, $\sup_{t\in J}|(Tx)'(t)|<\infty$. Hence $T:K\cap\overline{\Omega}\to K$ is well defined.

Next, we prove that T is continuous. Let $x_n\to x$ in $K\cap\overline{\Omega}$, then $\|x_n\|\leq M$, $(n=1,2,\ldots)$. We will show that $Tx_n\to Tx$. For any $\varepsilon>0$, by $(H7)$ there exists a constant $A_0>0$ such that

$$
S_M\int_{A_0}^{\infty}G(s,s)p(s)k(s)ds\leq\frac{\varepsilon}{12}. \qquad (2.15)
$$

On the other hand, by the continuity of $f(t,u,v)$ on $(0,A_0]\times J_+\times\mathbb{R}$, the continuities of g_1, g_2 on J_+ and the continuities of I_k, $\bar{I}_k$ on J_+, for the above $\varepsilon>0$, there exists a $\delta>0$ such that, for any u, v, u_1, v_1, satisfying $|u|+|v|<M$, and $|u_1|+|v_1|<M$, $\left|u-u_1\right|+\left|v-v_1\right|<\delta$,

$$\begin{aligned}
\left|f(s,u,v)-f(s,u_1,v_1)\right| &< \frac{\varepsilon}{6}\left(\int_0^{A_0} G(s,s)p(s)ds\right)^{-1}, \\
\left|g_1(u)-g_1(u_1)\right| &< \frac{\varepsilon}{6}\left(\frac{\varphi(0)}{D}\int_0^{\infty}\psi(s)ds\right)^{-1}, \\
\left|g_2(u)-g_2(u_1)\right| &< \frac{\varepsilon}{6}\left(\frac{\theta(\infty)}{D}\int_0^{\infty}\psi(s)ds\right)^{-1}, \\
\left|I_k(u(t_k))-I_k(u_1(t_k))\right| &< \frac{\varepsilon}{6}\left(\sum_{k=1}^{n} p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}}\right)^{-1}, \\
\left|\bar{I}_k(u(t_k))-\bar{I}_k(u_1(t_k))\right| &< \frac{\varepsilon}{6}\left(\sum_{k=1}^{n} G(t_k,t_k)\right)^{-1}.
\end{aligned} \tag{2.16}$$

From the fact that $\|x_n-x\|\to 0$ as $n\to\infty$, for above δ, there exists a sufficiently large number N such that, when $n>N$, we have, for $t\in(0,A_0]$,

$$|x_n(t)-x(t)|+|x_n^{'}(t)-x^{'}(t)|\le\|x_n-x\|<\delta. \tag{2.17}$$

By (2.15)-(2.16), we have, for $n>N$,

$$\begin{aligned}
\left|(Tx_n)(t)-(Tx)(t)\right| \le \Bigg| &\int_0^{\infty} G(s,s)p(s)\left[f(s,x_n(s),x_n^{'}(s))-f(s,x(s),x^{'}(s))\right]ds \\
&+\frac{\varphi(0)}{D}\int_0^{\infty}\left[g_1(x_n(s))-g_1(x(s))\right]\psi(s)ds \\
&+\frac{\theta(\infty)}{D}\int_0^{\infty}\left[g_2(x_n(s))-g_2(x(s))\right]\psi(s)ds \\
&+\sum_{k=1}^{n} G(t_k,t_k)\left[\bar{I}_k(x_n(t_k))-\bar{I}_k(x(t_k))\right] \\
&+\sum_{k=1}^{n} p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}}\left[I_k(x_n(t_k))-I_k(x(t_k))\right]\Bigg| \\
\le &\int_0^{A_0} G(s,s)p(s)\left|f(s,x_n(s),y_n(s))-f(s,x(s),y(s))\right|ds \\
&+2S_M\int_{A_0}^{\infty} G(s,s)p(s)k(s)ds \\
&+\frac{\varphi(0)}{D}\int_0^{\infty}\left|g_1(x_n(s))-g_1(x(s))\right|\psi(s)ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\theta(\infty)}{D}\int_0^\infty \left|g_2(x_n(s))-g_2(x(s))\right|\psi(s)ds\\
&+\sum_{k=1}^{n} G(t_k,t_k)\left|\bar{I}_k(x_n(t_k))-\bar{I}_k(x(t_k))\right|\\
&+\sum_{k=1}^{n} p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}}\left|I_k(x_n(t_k))-I_k(x(t_k))\right|\\
<&\ \frac{\varepsilon}{6}+\frac{\varepsilon}{6}+\frac{\varepsilon}{6}+\frac{\varepsilon}{6}+\frac{\varepsilon}{6}+\frac{\varepsilon}{6}=\varepsilon.
\end{aligned}
$$

Similarly, we can see that when $\|x_n-x\|\to 0$ as $n\to\infty$, $|(Tx_n)^{'}(t)-(Tx)^{'}(t)|\to 0$ as $n\to\infty$. This implies that T is a continuous operator.

Finally we show that $T: K\cap\overline{\Omega}\to K$ is a compact operator. In fact for any bounded set $D\subset\overline{\Omega}$, there exists a constant $R>0$ such that $\|x\|\le R$ for any $x\in K\cap D$. Hence, we have

$$
\begin{aligned}
|(Tx)(t)|\le&\ \Bigg|\int_0^\infty G(s,s)p(s)f(s,x(s),x^{'}(s))ds+\frac{\varphi(0)}{D}\int_0^\infty g_1(x(s))\psi(s)ds\\
&+\frac{\theta(\infty)}{D}\int_0^\infty g_2(x(s))\psi(s)ds\\
&+\sum_{k=1}^{n} G(t_k,t_k)\bar{I}_k(x(t_k))+\sum_{k=1}^{n} p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}}I_k(x(t_k))\Bigg|\\
\le&\ S_R\Bigg(\int_0^\infty G(s,s)p(s)k(s)ds+\frac{\varphi(0)}{D}\int_0^\infty \psi(s)ds\\
&+\frac{\theta(\infty)}{D}\int_0^\infty \psi(s)ds+\sum_{k=1}^{n} G(t_k,t_k)+\sum_{k=1}^{n} p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}}\Bigg)<\infty.
\end{aligned}
$$

From (2.14), we get $|(Tx)^{'}(t)|<\infty$, for $t\in J$. Therefore, $T(K\cap D)$ is uniformly bounded in $BPC^1(J)$.

Given $r>0$, for any $t_1,t_2\in J$, $x\in K\cap D$, as the proof of (2.11),(2.12), we can get $|(Tx)(t_1)-(Tx)(t_2)|\to 0$ and $|(Tx)^{'}(t_1)-(Tx)^{'}(t_2)|\to 0$ as $t_1\to t_2$, i.e., $\|(Tx)(t_1)-(Tx)(t_2)\|\to 0$ as $t_1\to t_2$. Thus $F=\{Tx: x\in K\cap D\}$ is equicontinuous on $[0,r]$. Since $r>0$ arbitrary, F is locally equicontinuous on J_+. By $(H5)$, $(H7)$ the properties (5), (6), (7) and the Lebesgue dominated converges theorem, we get

$$
\begin{aligned}
|(Tx)(t)-(Tx)(\infty)|\le S_R\Bigg(&\int_0^\infty \left|G(t,s)-\overline{G}(s)\right|p(s)k(s)ds\\
&+\frac{|\varphi(t)-\varphi(\infty)|}{D}\int_0^\infty \psi(s)ds+\frac{|\theta(t)-\theta(\infty)|}{D}\int_0^\infty \psi(s)ds\\
&+\sum_{k=1}^{n}\left|G(t,t_k)-\overline{G}(t_k)\right|+\frac{1}{D}\sum_{t_k\le t} p(t_k)\theta^{'}(t_k)\left|\varphi(t)-\varphi(\infty)\right|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{D}\sum_{t\le t_k} p(t_k)\left|\theta(t)\varphi'(t_k)-\theta(t_k)\varphi'(\infty)\right|\Big)\\
&\to 0 \ \text{ as } \ t\to\infty
\end{aligned}
$$

and

$$
\begin{aligned}
|(Tx)'(t)-(Tx)'(\infty)| \le S_R\Big\{ & \frac{a_2}{D}\left|\frac{1}{p(t)}-\frac{1}{p(\infty)}\right|\int_0^t \theta(s)p(s)k(s)ds\\
&+\frac{a_1}{D}\left|\frac{1}{p(t)}-\frac{1}{p(\infty)}\right|\int_t^\infty \varphi(s)p(s)k(s)ds\\
&+\frac{a_2}{D}\left|\frac{1}{p(t)}-\frac{1}{p(\infty)}\right|\int_0^\infty \psi(s)ds\\
&+\frac{a_1}{D}\left|\frac{1}{p(t)}-\frac{1}{p(\infty)}\right|\int_0^\infty \psi(s)ds\\
&+\frac{a_1}{Dp(t)}\sum_{t\le t_k}\left[\varphi(t_k)+p(t_k)|\varphi'(t_k)|\right]\\
&+\frac{a_2}{D}\left|\frac{1}{p(t)}-\frac{1}{p(\infty)}\right|\sum_{t_k\le t}\left[\theta(t_k)+p(t_k)\theta'(t_k)\right]\\
&+\frac{a_2}{Dp(\infty)}\sum_{t\le t_k}p(t_k)\left[\theta(t_k)+p(t_k)\theta'(t_k)\right]\Big\}\\
\to 0 \ \text{ as } \ t\to\infty. &
\end{aligned} \tag{2.18}
$$

Hence $T(K\cap D)$ is equiconvergent in $BPC^1(J)$. By Lemma 2, we have that F is relatively compact in $BPC^1(J)$. Therefore, $T: K\cap\overline{\Omega}\to K$ is completely continuous. □

3. Main Results

For convenience and simplicity in the following discussion, we use following notations:

$$
\begin{aligned}
f_0 &= \lim_{|x|+|y|\to 0} inf \min_{t\in[a,b]}\frac{f(t,x,y)}{|x|+|y|}, & f_\infty &= \lim_{|x|+|y|\to\infty} inf \min_{t\in[a,b]}\frac{f(t,x,y)}{|x|+|y|},\\
g_{i_0} &= \lim_{x\to 0} inf\frac{g_i(x)}{x} \ \ (1\le i\le 2), & g_{i_\infty} &= \lim_{x\to\infty} inf\frac{g_i(x)}{x} \ \ (1\le i\le 2),\\
h^q &= \lim_{|x|+|y|\to q} sup\frac{h(x,y)}{|x|+|y|}, & g_i^q &= \lim_{x\to q} sup\frac{g_i(x)}{x} \ \ (1\le i\le 2),\\
I_0(k) &= \lim_{x\to 0} inf\frac{I_k(x)}{x}, & I_\infty(k) &= \lim_{x\to\infty} inf\frac{I_k(x)}{x},\\
\bar{I}_0(k) &= \lim_{x\to 0} inf\frac{\bar{I}_k(x)}{x}, & \bar{I}_\infty(k) &= \lim_{x\to\infty} inf\frac{\bar{I}_k(x)}{x},
\end{aligned}
$$

$$I^q(k) = \lim_{x\to q} sup \frac{I_k(x)}{x}, \qquad \bar{I}^q(k) = \lim_{x\to q} sup \frac{\bar{I}_k(x)}{x}.$$

Theorem 1. *Assume that the conditions (H1)-(H7) are satisfied. Then the impulsive IBVP (1.1) has at least two positive solutions satisfying* $0 < \|x_1\| < q < \|x_2\|$ *if for* $[a,b] \subset (0,\infty)$*, the following conditions hold:*

(A1)
$$w\bigg(f_0 \int_a^b G(s,s)p(s)ds + \frac{\min\{\varphi(\infty),\theta(0)\}}{D}(g_{1_0}+g_{2_0})\int_a^b \psi(s)ds + \sum_{k=1}^n G(t_k,t_k)\bar{I}_0(k) + \sum_{k=1}^n p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}} I_0(k)\bigg) > 1,$$
$$w\bigg(f_\infty \int_a^b G(s,s)p(s)ds + \frac{\min\{\varphi(\infty),\theta(0)\}}{D}(g_{1_\infty}+g_{2_\infty})\int_a^b \psi(s)ds + \sum_{k=1}^n G(t_k,t_k)\bar{I}_\infty(k) + \sum_{k=1}^n p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}} I_\infty(k)\bigg) > 1,$$

(A2) *There exists a* $q > 0$ *such that*
$$\bigg[1 + c\sup_{t\in J}\frac{1}{p(t)}\bigg]\bigg\{h^q\int_0^\infty G(s,s)p(s)k(s)ds + \frac{\max\{\varphi(0),\theta(\infty)\}}{D}(g_1^q+g_2^q) \times \int_0^\infty \psi(s)ds + \sum_{k=1}^n G(t_k,t_k)\bar{I}^q(k) + \sum_{k=1}^n p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}} I^q(k)\bigg\} < 1,$$
for all $0 < |x| + |x'| \le q$, *a.e.* $t \in [0,\infty)$.

Proof. By the definition of $f_0, I_0, \bar{I}_0, g_{1_0}$ and g_{2_0} for any $\varepsilon > 0$, there exist $r \in (0,q)$ such that,

$$\begin{aligned} f(t,x,y) &\ge (1-\varepsilon)f_0(|x|+|y|), & (|x|+|y| \le r,\quad t\in[a,b])\\ g_1(x) &\ge (1-\varepsilon)g_{1_0}x, & g_2(x) \ge (1-\varepsilon)g_{2_0}x,\\ I_k(x) &\ge (1-\varepsilon)I_0(k)x, & \bar{I}_k(x) \ge (1-\varepsilon)\bar{I}_0(k)x, \end{aligned}$$

$$(1-\varepsilon)w\bigg(f_0\int_a^b G(s,s)p(s)ds + \frac{\min\{\varphi(\infty),\theta(0)\}}{D}(g_{1_0}+g_{2_0})\int_a^b \psi(s)ds + \sum_{k=1}^n G(t_k,t_k)\bar{I}_0(k) + \sum_{k=1}^n p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}} I_0(k)\bigg) \ge 1.$$

Define the open sets

$$\Omega_r = \{x \in BPC^1(J) : \|x\| < r\}.$$

Let $\Phi = 1$ then, $\Phi \in K$. Now we prove that

$$x \ne Tx + \lambda\Phi, \quad \forall x \in K \cap \partial\Omega_r, \quad \lambda > 0. \tag{3.1}$$

We assume that $x_0 = Tx_0 + \lambda_0\Phi$ where $x_0 \in K \cap \partial\Omega_r$ and $\lambda_0 > 0$. Let $\mu = \min_{t\in[a,b]} x_0(t)$, then for any $t \in [a,b]$, we have

$$\begin{aligned}
x_0(t) &= Tx_0(t) + \lambda_0\Phi \\
&= \int_0^\infty G(t,s)p(s)f(s,x_0(s),x_0'(s))ds \\
&\quad + \frac{\varphi(t)}{D}\int_0^\infty g_1(x_0(s))\psi(s)ds + \frac{\theta(t)}{D}\int_0^\infty g_2(x_0(s))\psi(s)ds \\
&\quad + \sum_{k=1}^n G(t,t_k)\bar{I}_k(x_0(t_k)) + \sum_{k=1}^n p(t_k)G_s(t,s)|_{s=t_k} I_k(x_0(t_k)) + \lambda_0 \\
&> w\mu(1-\varepsilon)\Big\{ f_0\int_a^b G(s,s)p(s)ds + \frac{\min\{\varphi(\infty),\theta(0)\}}{D}(g_{1_0}+g_{2_0}) \\
&\quad \times \int_a^b \psi(s)ds + \sum_{k=1}^n G(t_k,t_k)\bar{I}_0(k) + \sum_{k=1}^n p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}} I_0(k)\Big\} + \lambda_0 \\
&\geq \mu + \lambda_0.
\end{aligned}$$

This implies $\mu > \mu + \lambda_0$ a contradiction. Therefore (3.1) holds.

By the definition of $f_\infty, I_\infty, \bar{I}_\infty, g_{1_\infty}$ and g_{2_∞} for any $\varepsilon > 0$, there exist $R > q$ such that,

$$\begin{aligned}
&f(t,x,y) \geq (1-\varepsilon)f_\infty(|x|+|y|), && (|x|+|y| \geq R, \quad t\in[a,b]), \\
&g_1(x) \geq (1-\varepsilon)g_{1_\infty}x, \quad (\forall |x| \geq R), && g_2(x) \geq (1-\varepsilon)g_{2_\infty}x, \\
&I_k(x) \geq (1-\varepsilon)I_\infty(k)x, && \bar{I}_k(x) \geq (1-\varepsilon)\bar{I}_\infty(k)x,
\end{aligned}$$

$$\begin{aligned}
(1-\varepsilon)w\Big(f_\infty\int_0^\infty G(s,s)p(s)ds + \frac{\min\{\varphi(\infty),\theta(0)\}}{D}(g_{1_\infty}+g_{2_\infty})\int_0^\infty \psi(s)ds \\
+ \sum_{k=1}^n G(t_k,t_k)\bar{I}_\infty(k) + \sum_{k=1}^n p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}} I_\infty(k)\Big) \geq 1.
\end{aligned}$$

Define the open sets

$$\Omega_R = \{x \in BPC^1(J) : \|x\| < R\}. \tag{3.2}$$

As the proof of (3.1), we can get that

$$x \neq Tx + \lambda\Phi, \ \ \forall x \in K\cap\partial\Omega_R, \quad \lambda > 0. \tag{3.3}$$

On the other hand, for any $\varepsilon > 0$, choose q in $(A2)$ such that

$$(1+\varepsilon)\left[1 + c\sup_{t\in J}\frac{1}{p(t)}\right]\Big\{ h^q\int_0^\infty G(s,s)p(s)k(s)ds + \frac{\max\{\varphi(0),\theta(\infty)\}}{D}(g_1^q+g_2^q)$$

$$\times \int_0^\infty \psi(s)ds + \sum_{k=1}^n G(t_k,t_k)\bar{I}^q(k) + \sum_{k=1}^n p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}} I^q(k) \Big\} \le 1. \quad (3.4)$$

By the definition of h^q, I^q, $\bar{I}^q$, g_1^q and g_2^q, for the above $\varepsilon > 0$, there exists $\delta > 0$, when $|x|, |x|+|y| \in (q-\delta, q+\delta)$; thus, we get

$$h(x,y) \le (1+\varepsilon)h^q(|x|+|y|), \qquad g_i(x) \le (1+\varepsilon)g_i^q x, \quad 1 \le i \le 2$$
$$I_k(x) \le (1+\varepsilon)I^q(k)x, \qquad \bar{I}_k(x) \le (1+\varepsilon)\bar{I}^q(k)x.$$

Define the open sets

$$\Omega_q = \left\{x \in BPC^1(J) : \|x\| < q\right\}. \quad (3.5)$$

Then for any $x \in K \cap \partial\Omega_q$ and $t \in J$ we obtain that

$$\begin{aligned}
|(Tx)(t)| + |(Tx)'(t)| \le & \int_0^\infty G(s,s)p(s)f(s,x(s),x'(s))ds \\
& + \frac{\max\{\varphi(0),\theta(\infty)\}}{D}\int_0^\infty [g_1(x(s))+g_2(x(s))]\psi(s)ds \\
& + \sum_{k=1}^n G(t_k,t_k)\bar{I}_k(x(t_k)) + \sum_{k=1}^n p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}} I_k(x(t_k)) \\
& + c\sup_{t\in J}\frac{1}{p(t)}\int_0^\infty G(s,s)p(s)f(s,x(s),y(s))ds \\
& + \frac{\max\{a_1,a_2\}}{D}\sup_{t\in J}\frac{1}{p(t)}\int_0^\infty [g_1(x(s))+g_2(x(s))]\psi(s)ds \\
& + c\sup_{t\in J}\frac{1}{p(t)}[\sum_{k=1}^n G(t_k,t_k)\bar{I}_k(x(t_k)) \\
& + \sum_{k=1}^n p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}} I_k(x(t_k))] \\
\le & \left[1 + c\sup_{t\in J}\frac{1}{p(t)}\right]\int_0^\infty G(s,s)p(s)k(s)h(x(s),x'(s))ds \\
& + \left[\frac{\max\{\varphi(0),\theta(\infty)\}}{D} + \frac{\max\{a_1,a_2\}}{D}\sup_{t\in J}\frac{1}{p(t)}\right] \\
& \times \int_0^\infty [g_1(x(s))+g_2(x(s))]\psi(s)ds \\
& + \left[1 + c\sup_{t\in J}\frac{1}{p(t)}\right]\left[\sum_{k=1}^n G(t_k,t_k)\bar{I}_k(x(t_k))\right. \\
& \left. + \sum_{k=1}^n p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}} I_k(x(t_k))\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq (1+\varepsilon)\left[1+c\sup_{t\in J}\frac{1}{p(t)}\right]\left[h^q\int_0^\infty G(s,s)p(s)k(s)ds\right.\\
&\quad+\frac{\max\{\varphi(0),\theta(\infty)\}}{D}\times(g_1^q+g_2^q)\int_0^\infty \psi(s)ds\\
&\quad\left.+\sum_{k=1}^n G(t_k,t_k)\bar{I}^q(k)+\sum_{k=1}^n p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}}I^q(k)\right]\|x\|\leq\|x\|.
\end{aligned}
$$

Therefore $\|Tx\|\leq\|x\|$.

So, we can get the existence of two positive solutions x_1 and x_2 satisfying $0<\|x_1\|<q<\|x_2\|$ by using Lemma 3. $\square$

Using a similar proof of Theorem 1, we can get the following theorem.

Theorem 2. *Assume that the conditions (H1)-(H7) are satisfied. Then the impulsive IBVP* (1.1) *has at least two positive solutions satisfying* $0<\|x_1\|<q<\|x_2\|$ *if for* $[a,b]\subset(0,\infty)$*, the following conditions hold;*

$(A3)$
$$
\begin{aligned}
&\left[1+c\sup_{t\in J}\frac{1}{p(t)}\right]\left\{h^0\int_0^\infty G(s,s)p(s)k(s)ds+\frac{\max\{\varphi(0),\theta(\infty)\}}{D}(g_1^0+g_2^0)\right.\\
&\quad\left.\times\int_0^\infty\psi(s)ds+\sum_{k=1}^n G(t_k,t_k)\bar{I}^0(k)+\sum_{k=1}^n p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}}I^0(k)\right\}<1,\\
&\left[1+c\sup_{t\in J}\frac{1}{p(t)}\right]\left\{h^\infty\int_0^\infty G(s,s)p(s)k(s)ds+\frac{\max\{\varphi(0),\theta(\infty)\}}{D}(g_1^\infty+g_2^\infty)\right.\\
&\quad\left.\times\int_0^\infty\psi(s)ds+\sum_{k=1}^n G(t_k,t_k)\bar{I}^\infty(k)+\sum_{k=1}^n p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}}I^\infty(k)\right\}<1,
\end{aligned}
$$

$(A4)$ *There exists a* $q>0$ *such that*
$$
\begin{aligned}
&w\left\{f_q\int_a^b G(s,s)p(s)ds+\frac{\min\{\varphi(\infty),\theta(0)\}}{D}(g_{1_q}+g_{2_q})\int_a^b\psi(s)ds\right.\\
&\quad\left.+\sum_{k=1}^n G(t_k,t_k)\bar{I}_q(k)+\sum_{k=1}^n p(t_k)G_s(t,s)|_{\substack{t=t_k\\ s=t_k}}I_q(k)\right\}>1,
\end{aligned}
$$
for all $0<|x|+|x^{'}|\leq q,$ *a.e.* $t\in[0,\infty)$.

4. Example

To illustrate how our main results can be used in practise we present the following example.

Consider the following boundary value problem:

$$\begin{cases} e^{-t}(e^t x'(t))' + f(t,x(t),x^{'}(t)) = 0, \ \ t \in J_+, \ t \neq \frac{1}{2}, \\ \Delta x|_{t=\frac{1}{2}} = \dfrac{1}{100} x(\dfrac{1}{2}), \\ \Delta x'|_{t=\frac{1}{2}} = \dfrac{1}{2-e^{-\frac{1}{2}}} x^{-1}(\dfrac{1}{2}), \\ x(0) = \dfrac{1}{40\pi} \displaystyle\int_0^\infty \dfrac{x^2(s)}{1+s^2} ds, \\ \lim\limits_{t\to\infty} e^t x'(t) = \dfrac{1}{8000\pi} \displaystyle\int_0^\infty \dfrac{x^2(s)}{1+s^2} ds, \end{cases} \tag{4.1}$$

where

$$f(t,x(t),x^{'}(t)) = \frac{e^{-t}}{e^t - 2}, \ a_1 = 1, a_2 = 0, \ b_1 = 1, \ b_2 = 1, \ p(t) = e^t, \ \psi(t) = \frac{1}{1+t^2},$$

$$I_k(x(t)) = \frac{x(t)}{100}, \ \bar{I}_k(x(t)) = \frac{x^{-1}(t)}{2-e^{-\frac{1}{2}}}, \ g_1(x(s)) = \frac{x^2(s)}{40\pi}, \ g_2(x(s)) = \frac{x^4(s)}{8000\pi}.$$

Set $k(t) = \dfrac{e^{-t}}{e^t - 2}$, $h(x(t), \ y(t)) = 1$ and $q = 10$. It follows from a direct calculation that

$$g_1^q = \frac{1}{4\pi}, \ g_2^q = \frac{1}{8\pi}, \ g_{1_\infty} = \infty, \ g_{2_\infty} = \infty, \ g_{1_0} = 0, \ g_{2_0} = 0, \ I^q(k) = \frac{1}{100},$$

$$\bar{I}^q(k) = \frac{1}{100(1-e^{-\frac{1}{2}})}, I_0(k) = \frac{1}{100}, \ \bar{I}_0(k) = \infty, \ I_\infty(k) = \frac{1}{100}, \ \bar{I}_\infty(k) = 0.$$

Furthermore, $f_0 = \infty, \ f_\infty = 1, \ \int_0^\infty G(s,s)p(s)k(s)ds = 1$ and $\int_0^\infty \psi(s)ds = \dfrac{\pi}{2}$. Thus $(A1)$ and $(A2)$ are satisfied. Therefore, by Theorem 1, the impulsive IBVP (4.1) has at least two positive solutions x_1, x_2 satisfying $0 < \|x_1\| < 10 < \|x_2\|$.

ACKNOWLEDGEMENT

The second author was supported in part by Ege University Scientific Research Project, Grant No.18-FEN-033.

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