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SECOND-ORDER DIFFERENTIAL EQUATIONS: ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS

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Abstract. In this work, we obtain necessary and sufficient conditions for the oscillation of all solutions of the second-order delay differential equation $\big(\pi(y')^\gamma\big)'(t)+p(t)f\big(y(\tau(t))\big)=0$, under the assumption $\int^\infty (\pi(\eta))^{-1/\gamma}d\eta=\infty$, we consider two cases: when $f(v)/v^\beta$ is non-increasing, and non-decreasing. In the final section, we provide examples illustrating the results and state an open problem.

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1. Introduction

The motivation to study the oscillation of differential equations comes from several systems in the real world, like species populations and neuronal populations that exhibit oscillatory behavior. Thus, equations having delayed, advanced or both delayed and advanced arguments have been used to model lossless transmission lines in engineering, the switching of data packets in high speed networks and several other natural or artificial processes, from celestial motion and bridge design to learning and memory formation in the synaptic contacts between neurons in the brain.

In 1978, Brands [6] showed that the solutions to

$$y''(t)+p(t)y(t-\tau(t))=0$$

are oscillatory if and only if the solutions to $y''(t)+p(t)y(t)=0$ are oscillatory. In 2018, Pinelas and Santra [13] have obtained necessary and sufficient conditions for the oscillations of the solutions of

$$(y(t)+b(t)y(t-\sigma))'+\sum_{i=1}^{m}p_i(t)f(y(t-\tau_i))=0,$$

for different ranges of the neutral coefficient b. Wong [18] studied necessary and sufficient conditions for the oscillation of the solutions to

$$\big(y(t)+by(t-\sigma)\big)''+p(t)f\big(y(t-\tau)\big)=0,$$

where the constant p satisfies $-1 < b < 0$. Santra [17] has obtained several sufficient conditions for the oscillation of the solutions for the equations

$$\left(\pi z'\right)'(t) + \sum_{i=1}^{m} p_i(t) f(y(t - \tau_i)) = 0, \quad z(t) = y(t) + b(t)y(t - \sigma).$$

Karpuz and Santra [9] have established sufficient conditions for the oscillation and asymptotic behavior of the solutions to the equation

$$\left(\pi z'\right)'(t) + \sum_{i=1}^{m} p_i(t) f_i(y(\tau_i(t))) = 0, \quad z(t) = y(t) + b(t)y(\sigma(t)).$$

Migda et al. have studied asymptotic behaviors of solutions of second order difference equations with deviating argument. For a more detailed account of the oscillatory behavior of the solutions to this type of equations, we refer the readers to [1–5, 7–17, 19]. Note that most publications consider only sufficient conditions, and merely a few consider necessary and sufficient conditions.

In this work, we establish necessary and sufficient conditions for the oscillation of all solutions to the second-order nonlinear delay differential equation

$$\left(\pi (y')^{\gamma}\right)'(t) + p(t) f\left(y(\tau(t))\right) = 0 \tag{1.1}$$

by considering two cases: when $f(v)/v^{\beta}$ is non-increasing, and non-decreasing.

We assume that the following conditions hold:

(A1) γ is the quotient of two odd positive integers, $\pi, p \in C(\mathbb{R}, \mathbb{R})$ with $\pi(t) > 0$ and p is not identically zero eventually, $\tau \in C([t_0, \infty), \mathbb{R})$ such that $\tau(t) \leq t$ for $t \geq t_0$, $\tau(t) \to \infty$ as $t \to \infty$.

(A2) $f \in C(\mathbb{R}, \mathbb{R})$, f is non-decreasing and $v f(v) > 0$ for $v \neq 0$. Moreover, we assume that $f(uv) = f(u) f(v)$, $\quad \forall u, v \in \mathbb{R}$.

(A3) $\pi(t) > 0$ and $\int_0^{\infty} (\pi(\eta))^{-1/\gamma} d\eta = \infty$. Letting $\Pi(t) = \int_0^t (\pi(\eta))^{-1/\gamma} d\eta$, we have $\lim_{t \to \infty} \Pi(t) = \infty$.

Initially, we consider a single delay. In a later section, we study for the several delays. As examples of functions satisfying (A2) and (A3), we have $f(u) = u^{\gamma}$ with γ the quotient of two odd positive integers and $\pi(t) = e^{-t}$ or $\pi(t) = 1$ respectively.

By a solution to equation (1.1), we mean a function $y \in \mathrm{C}([T_y, \infty), \mathbb{R})$, where $T_y \geq t_0$, such that $\pi y' \in \mathrm{C}^1([T_y, \infty), \mathbb{R})$, and satisfying (1.1) on the interval $[T_y, \infty)$. A solution y of (1.1) is said to be proper if y is not identically zero eventually, i.e., $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$. We assume that (1.1) possesses such solutions. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_y, \infty)$; otherwise, it is said to be non-oscillatory. (1.1) itself is said to be oscillatory if all of its solutions are oscillatory.

From [16], we know that (A2) implies f being odd. Indeed, $f(1)f(1) = f(1)$ and $f(1) > 0$ imply that $f(1) = 1$. Further, $\left(f(-1)\right)^2 = f(-1)f(-1) = f(1) = 1$. Since

$f(-1) < 0$, we conclude that $f(-1) = -1$. Hence,

$$f(-u) = f(-1)f(u) = -f(u).$$

On the other hand, $f(uv) = f(u)f(v)$ for $u > 0$ and $v > 0$ and $f(-u) = -f(u)$ imply that $f(xy) = f(x)f(y)$ for every $x, y \in \mathbb{R}$.

Also from [16], we have that under assumption (A2), if $y(t)$ is a solution of (1.1), then $-y(t)$ is also a solution of (1.1).

2. Preliminary results

Lemma 1. *Assume that (A1)–(A3) hold and y is an eventually positive solution of* (1.1)*. Then we have*

$$y'(t) > 0 \quad \textit{and} \quad (\pi(y')^{\gamma})'(t) < 0,$$

for sufficiently large t.

Proof. Suppose that there exists a $t_1 \geq t_0$ such that $y(t) > 0$ and $y(\tau(t)) > 0$ for $t \geq t_1$. From (1.1) and (A2), it follows that

$$(\pi(y')^{\gamma})'(t) = -p(t)f\big(y(\tau(t))\big) < 0 \quad \text{for } t \geq t_1. \tag{2.1}$$

Consequently, $\big(\pi(y')^{\gamma}\big)(t)$ is non-increasing on $[t_1, \infty)$. Since $\pi(t) > 0$, and thus either $y'(t) < 0$ or $y'(t) > 0$ for $t_2 \geq t_1$, where $t \geq t_2$.

We claim that $y'(t) > 0$ for $t \geq t_2$. To the contrary, assume that $y'(t) < 0$ for $t \geq t_2$, then there exists $\kappa_1 > 0$ such that $\big(\pi(y')^{\gamma}\big)(t) \leq -\kappa_1$ for $t \geq t_2$, which yields upon integration over $[t_2, t) \subset [t_2, \infty)$ after dividing through by π that

$$y(t) \leq y(t_2) - \kappa_1^{1/\gamma} \int_{t_2}^{t} \big(\pi(\eta)\big)^{-1/\gamma} d\eta \quad \text{for } t \geq t_2. \tag{2.2}$$

By virtue of condition (A3), $\lim_{t\to\infty} y(t) = -\infty$. This contradicts $y(t)$ being a positive solution. So, our claim is true. This completes the proof. □

3. Necessary and Sufficient Conditions for Oscillations

In this section, we study necessary and sufficient conditions for oscillations of solutions of (1.1) by considering the cases when $f(v)/v^{\beta}$ is non-increasing and non-decreasing.

3.1. *Non-increasing $f(v)/v^{\beta}$*

We assume that there exists a constant β such that $0 < \beta < \gamma$ and

$$\frac{f(v)}{v^{\beta}} \geq \frac{f(u)}{u^{\beta}}, \quad \text{for } 0 < v \leq u. \tag{3.1}$$

A typical example of a nonlinear function satisfying (3.1) is $f(y) = |y|^{\alpha} sgn(y)$ with $0 < \alpha < \beta$.

Lemma 2. *Assume that (A1)–(A3) hold and y is an eventually positive solution of* (1.1). *Then we have*

$$y(t) \leq \kappa^{1/\gamma}\Pi(t) \tag{3.2}$$

$$y(t) \geq \int_{t_3}^{t} \left[\frac{1}{\pi(\eta)} \int_{t}^{\infty} p(\zeta) \frac{f\left(\kappa^{1/\gamma}\Pi(\tau(\zeta))\right)}{\left(\kappa^{1/\gamma}\Pi(\tau(\zeta))\right)^{\beta}} y^{\beta}(\tau(\zeta))\, d\zeta\right]^{1/\gamma} d\eta\,, \tag{3.3}$$

for sufficiently large t.

Proof. Suppose that there exists $t_1 \geq t_0$ such that $y(t) > 0$ and $y(\tau(t)) > 0$ for $t \geq t_1$. Then, Lemma 1 holds true for $t \geq t_2$. Since $\left(\pi(y')^{\gamma}\right)(t)$ is positive and non-increasing, there exists $\kappa > 0$ and $t_3 \geq t_2$ such that $\left(\pi(y')^{\gamma}\right)(t) \leq \kappa$ for $t \geq t_3$. Integrating the inequality $y'(t) \leq (\kappa/\pi(t))^{1/\gamma}$, we have

$$y(t) \leq y(t_3) + \kappa^{1/\gamma}\left(\Pi(t) - \Pi(t_3)\right).$$

Since $\lim_{t\to\infty} \Pi(t) = \infty$, the last inequality becomes

$$y(t) \leq \kappa^{1/\gamma}\Pi(t) \quad \text{for } t \geq t_3\,.$$

Note that κ depends on y being evaluated at a time t_3. Thus, (3.2) must include all possible κ's.

By (3.1) and (3.2), we have

$$f\left(y(\tau(t))\right) = \frac{f\left(y(\tau(t))\right)}{y^{\beta}(\tau(t))} y^{\beta}(\tau(t)) \geq \frac{f\left(\kappa^{1/\gamma}\Pi(\tau(t))\right)}{\left(\kappa^{1/\gamma}\Pi(\tau(t))\right)^{\beta}} y^{\beta}(\tau(t))\,.$$

Integrating (1.1) from t to ∞, we have

$$\lim_{A\to\infty} \left[\left(\pi(y')^{\gamma}\right)(\eta)\right]_t^A + \int_t^{\infty} p(\eta) \frac{f\left(\kappa^{1/\gamma}\Pi(\tau(\eta))\right)}{\left(\kappa^{1/\gamma}\Pi(\tau(\eta))\right)^{\beta}} y^{\beta}(\tau(\eta))\, d\eta \leq 0.$$

Using that $\left(\pi(y')^{\gamma}\right)(t)$ is positive and non-increasing, we have

$$\int_t^{\infty} p(\eta) \frac{f\left(\kappa^{1/\gamma}\Pi(\tau(\eta))\right)}{\left(\kappa^{1/\gamma}\Pi(\tau(\eta))\right)^{\beta}} y^{\beta}(\tau(\eta))\, d\eta \leq \left(\pi(y')^{\gamma}\right)(t) \quad \text{for } t \geq t_3\,.$$

Therefore,

$$y'(t) \geq \left[\frac{1}{\pi(t)} \int_t^{\infty} p(\eta) \frac{f\left(\kappa^{1/\gamma}\Pi(\tau(\eta))\right)}{\left(\kappa^{1/\gamma}\Pi(\tau(\eta))\right)^{\beta}} y^{\beta}(\tau(\eta))\, d\eta\right]^{1/\gamma}. \tag{3.4}$$

Integrating (3.4) from t_3 to t, we obtain

$$y(t) \geq \int_{t_3}^{t} \left[\frac{1}{\pi(\eta)} \int_{\eta}^{\infty} p(\zeta) \frac{f\left(\kappa^{1/\gamma}\Pi(\tau(\zeta))\right)}{\left(\kappa^{1/\gamma}\Pi(\tau(\zeta))\right)^{\beta}} y^{\beta}(\tau(\zeta))\, d\zeta\right]^{1/\gamma} d\eta$$

$$\geq \int_{t_3}^{t} \left[\frac{1}{\pi(\eta)} \int_{t}^{\infty} p(\zeta) \frac{f\left(\kappa^{1/\gamma}\Pi(\tau(\zeta))\right)}{\left(\kappa^{1/\gamma}\Pi(\tau(\zeta))\right)^{\beta}} y^{\beta}(\tau(\zeta))\, d\zeta \right]^{1/\gamma} d\eta .$$

The proof of the lemma is complete. □

Theorem 1. *Assume that (A1)-(A3) hold. Then every solution of* (1.1) *is oscillatory if and only if*

$$\int_{0}^{\infty} p(\eta) f\left(\kappa^{1/\gamma}\Pi(\tau(\eta))\right) d\eta = +\infty \quad \forall \kappa > 0. \tag{3.5}$$

Proof. To prove sufficiency by contradiction, we assume that there exists a non-oscillatory solution $y(t)$ of (1.1). Since $-y(t)$ is also a solution of (1.1), we can confine our discussion only to the case where the solution $y(t)$ is eventually positive. Then there exists $t_1 \geq t_0$ such that $y(t) > 0$ and $y(\tau(t)) > 0$ for $t \geq t_1$. Since Lemmas 1 and 2 hold, (3.3) gives

$$y(t) > \left(\Pi(t) - \Pi(t_3)\right) \omega^{1/\gamma}(t) \quad \text{for} \quad t \geq t_3,$$

where

$$\omega(t) = \int_{t}^{\infty} p(\zeta) \frac{f\left(\kappa^{1/\gamma}\Pi(\tau(\zeta))\right)}{\left(\kappa^{1/\gamma}\Pi(\tau(\zeta))\right)^{\beta}} y^{\beta}(\tau(\zeta))\, d\zeta > 0.$$

Because $\lim_{t\to\infty} \Pi(t) = \infty$, there exists $t_4 \geq t_3$ such that $\Pi(t) - \Pi(t_3) \geq \frac{1}{2}\Pi(t)$ for $t \geq t_4$. Then

$$y(t) > \frac{1}{2}\Pi(t)\omega^{1/\gamma}(t) \quad \text{for } t \geq t_4 ,$$

and $y^{\beta}/(\kappa^{1/\gamma}\Pi)^{\beta} \geq \omega^{\beta/\gamma}/(2\kappa^{1/\gamma})^{\beta}$. Taking the derivative of ω we have

$$\begin{aligned} \omega'(t) &= -p(t) \frac{f\left(\kappa^{1/\gamma}\Pi(\tau(t))\right)}{\left(\kappa^{1/\gamma}\Pi(\tau(t))\right)^{\beta}} y^{\beta}(\tau(t)) \\ &\leq -p(t) f\left(\kappa^{1/\gamma}\Pi(\tau(t))\right) \omega^{\beta/\gamma}(\tau(t)) \left(2\kappa^{1/\gamma}\right)^{-\beta} \leq 0 . \end{aligned}$$

Therefore, $\omega(t)$ is non-increasing so $\omega^{\beta/\gamma}(\tau(t))/\omega^{\beta/\gamma}(t) \geq 1$, and

$$\left(\omega^{1-\beta/\gamma}(t)\right)\prime = (1-\beta/\gamma)\omega^{-\beta/\gamma}(t)\omega'(t) \leq -\frac{(1-\beta/\gamma)}{(2\kappa^{1/\gamma})^{\beta}} p(t) f\left(\kappa^{1/\gamma}\Pi(\tau(t))\right) .$$

Integrating this inequality form t_4 to t, we have

$$\left[\omega^{1-\beta/\gamma}(\eta)\right]_{t_4}^{t} \leq -\frac{(1-\beta/\gamma)}{(2\kappa^{1/\gamma})^{\beta}} \int_{t_4}^{t} p(\eta) f\left(\kappa^{1/\gamma}\Pi(\tau(\eta))\right) d\eta .$$

Since $\beta/\gamma < 1$ and $\omega(t)$ is positive and non-increasing, we have

$$\int_{t_4}^{t} p(\eta) f\left(\kappa^{1/\gamma}\Pi(\tau(\eta))\right) d\eta \leq \frac{(2\kappa^{1/\gamma})^{\beta}}{(1-\beta/\gamma)} \omega^{1-\beta/\gamma}(t_4) ,$$

which contradicts (3.5).

Next, we show that (3.5) is necessary. Suppose that (3.5) does not hold; so for some $\kappa > 0$ the integral in (3.5) is finite. Then there exists $T \geq t_0$ such that

$$\int_T^\infty p(\eta) f\big(\kappa^{1/\gamma}\Pi(\tau(\eta))\big) d\eta \leq \frac{\kappa}{2}. \tag{3.6}$$

Let us consider the closed subset of continuous functions

$$\begin{aligned} M = \big\{ & y \in C([t_0, +\infty), \mathbb{R}) : y(t) = 0 \text{ for } t_0 \leq t < T \text{ and} \\ & \big(\frac{\kappa}{2}\big)^{1/\gamma}[\Pi(t) - \Pi(T)] \leq y(t) \leq \kappa^{1/\gamma}[\Pi(t) - \Pi(T)] \text{ for } T \leq t \big\}. \end{aligned}$$

We define the operator $\Omega : M \to C([t_0, +\infty), \mathbb{R})$ by

$$(\Omega y)(t) = \begin{cases} 0, & t_0 \leq t < T \\ \int_T^t \big[\frac{1}{\pi(\eta)}\big[\frac{\kappa}{2} + \int_\eta^\infty p(\zeta) f\big(y(\tau(\zeta))\big) d\zeta\big]\big]^{1/\gamma} d\eta, & T \leq t. \end{cases}$$

For $y \in M$ and $t \geq T$, we have

$$(\Omega x)(t) \geq \int_T^t \big[\frac{1}{\pi(\eta)} \frac{\kappa}{2}\big]^{1/\gamma} d\eta = \big(\frac{\kappa}{2}\big)^{1/\gamma}[\Pi(t) - \Pi(T)].$$

For $y \in M$ and $t \geq T$, we have $y(t) \leq \kappa^{1/\gamma}\Pi(t)$ and $f(y) \leq f(\kappa^{1/\gamma}\Pi(t))$. Using (3.6), we have

$$(\Omega x)(t) \leq \int_T^t \big[\frac{1}{\pi(\eta)}\big(\frac{\kappa}{2} + \frac{\kappa}{2}\big)\big]^{1/\gamma} d\eta = \kappa^{1/\gamma}[\Pi(t) - \Pi(T)].$$

Thus, $\Omega x \in M$. Now, we define a sequence of continuous function $v_n : [t_0, +\infty) \to \mathbb{R}$ by the recursive formula

$$\begin{aligned} v_1(t) &= \begin{cases} 0, & t \in [t_0, T) \\ \big(\frac{\kappa}{2}\big)^{1/\gamma}[\Pi(t) - \Pi(T)], & t \geq T. \end{cases} \\ v_n(t) &= (\Omega v_{n-1})(t) \quad \text{for } n > 1. \end{aligned}$$

It is easy to verify that for $n > 1$,

$$\big(\frac{\kappa}{2}\big)^{1/\gamma}[\Pi(t) - \Pi(T)] \leq v_{n-1}(t) \leq v_n(t) \leq \kappa^{1/\gamma}[\Pi(t) - \Pi(T)].$$

Therefore, the pointwise limit of the sequence exists. Let $\lim_{n\to\infty} v_n(t) = v(t)$ for $t \geq t_0$. By Lebesgue's dominated convergence theorem $v \in M$ and $(\Omega v)(t) = v(t)$, where $v(t)$ is a solution of equation (1.1) on $[T, \infty)$. Hence, (3.5) is a necessary condition. This completes the proof. □

Example 1. Consider the delay differential equation

$$(e^{-t}(y'(t))^{3/5})' + \frac{1}{t+1}(y(t-2))^{1/3} = 0, \quad t \geq 0. \tag{3.7}$$

Here $\gamma=3/5$, $\pi(t)=e^{-t}$, $\tau(t)=t-2$, $\Pi(t)=\int_0^t e^{5s/3}\,ds=\frac{3}{5}\big(e^{5t/3}-1\big)$, $f(v)=v^{1/3}$. For $\beta=1/2$, we have $f(v)/v^\beta=v^{-1/6}$ which is a decreasing function. To check (3.5) we have

$$\int_0^\infty p(\eta)f\big(\kappa^{1/\gamma}\Pi(\tau(\eta))\big)d\eta=\int_0^\infty \frac{1}{\eta+1}\Big(\kappa^{5/3}\frac{3}{5}\big(e^{5(\eta-2)/3}-1\big)\Big)^{1/3}d\eta=\infty \quad \forall\kappa>0,$$

because the integral approaches $+\infty$ as $\eta\to+\infty$. So that all the conditions of Theorem 1 hold. Therefore, all solutions of (3.7) are oscillatory.

3.2. *Non-decreasing* $f(v)/v^\beta$

We assume that there exists $\beta>\gamma>0$ such that

$$\frac{f(v)}{v^\beta}\le\frac{f(u)}{u^\beta},\quad \text{for } 0<v\le u\,. \tag{3.8}$$

A typical example of a nonlinear function satisfying (3.8) is $f(y)=|y|^\alpha sgn(y)$ with $\gamma<\beta<\alpha$.

Theorem 2. *Assuming (A1)-(A3) and* $\tau'(t)\ge 1$, *every solution of* (1.1) *is oscillatory if and only if*

$$\int_0^\infty\Big[\frac{1}{\pi(\eta)}\int_\eta^\infty p(\zeta)\,d\zeta\Big]^{1/\gamma}d\eta=+\infty. \tag{3.9}$$

Proof. To prove sufficiency by contradiction, we assume that there exists a non-oscillatory solution $y(t)$ of (1.1). Since $-y(t)$ is also a solution of (1.1), we can confine our discussion only to the case where the solution $y(t)$ is eventually positive. Then there exists $t_1\ge t_0$ such that $y(t)>0$ and $y(\tau(t))>0$ for $t\ge t_1$. Then, Lemma 1 holds true for $t\ge t_3\ge t_2$. Since $y'(t)>0$, so y is increasing and $y(t)\ge y(t_3)$ for $t\ge t_3$. Therefore,

$$y(\tau(t))\ge y(\tau(t_3)):=\kappa>0\quad \text{for } t\ge t_3\,.$$

From (3.8), we have

$$f\big(y(\tau(t))\big)=\frac{f\big(y(\tau(t))\big)}{y^\beta(\tau(t))}y^\beta(\tau(t))\ge\frac{f(\kappa)}{\kappa^\beta}y^\beta(\tau(t))\,.$$

Integrating (1.1) from t to ∞, we have

$$\lim_{A\to\infty}\big[\big(\pi(y')^\gamma\big)(\eta)\big]_t^A+\frac{f(\kappa)}{\kappa^\beta}\int_t^\infty p(\eta)y^\beta(\tau(\eta))\,d\eta\le 0.$$

Using that $\big(\pi(y')^\gamma\big)(t)$ is positive and non-increasing, we have

$$\frac{f(\kappa)}{\kappa^\beta}\int_t^\infty p(\eta)y^\beta(\tau(\eta))\,d\eta\le\big(\pi(y')^\gamma\big)(t)\le\big(\pi(y')^\gamma\big)(\tau(t))\le\pi(t)\big((y')^\gamma\big)(\tau(t))$$

for all $t\ge t_3$. Therefore,

$$\Big(\frac{f(\kappa)}{\kappa^\beta}\Big)^{1/\gamma}\Big[\frac{1}{\pi(t)}\int_t^\infty p(\eta)y^\beta(\tau(\eta))\,d\eta\Big]^{1/\gamma}\le y'(\tau(t))$$

implies that

$$\left(\frac{f(\kappa)}{\kappa^{\beta}}\right)^{1/\gamma}\left[\frac{1}{\pi(t)}\int_{t}^{\infty}p(\eta)\,d\eta\right]^{1/\gamma}\leq\frac{y'(\tau(t))}{y^{\beta/\gamma}(\tau(t))}\leq\frac{y'(\tau(t))\tau'(t)}{y^{\beta/\gamma}(\tau(t))} \tag{3.10}$$

Integrating (3.10) from t_3 to ∞, we have

$$\left(\frac{f(\kappa)}{\kappa^{\beta}}\right)^{1/\gamma}\int_{t_3}^{\infty}\left[\frac{1}{\pi(\eta)}\int_{\eta}^{\infty}p(\zeta)\,d\zeta\right]^{1/\gamma}d\eta\leq\frac{y^{1-\beta/\gamma}(\tau(t_3))}{\beta/\gamma-1}<\infty,$$

which contradicts (3.9).

Next, we show that (3.9) is necessary. Suppose that (3.9) does not hold; so for each $\kappa>0$, there exists $T\geq t_0$ such that

$$\int_{T}^{\infty}\left[\frac{1}{\pi(\eta)}\int_{\eta}^{\infty}p(\zeta)d\zeta\right]^{1/\gamma}d\eta\leq\frac{\kappa}{2(f(\kappa))^{1/\gamma}} \tag{3.11}$$

Let us consider the closed subset of continuous functions

$$M=\left\{x\in C([t_0,+\infty),\mathbb{R}):x(t)=\frac{\kappa}{2}\text{ for }t\in[t_0,T)\text{ and }\frac{\kappa}{2}\leq x(t)\leq\kappa\text{ for }t\geq T\right\}.$$

We define the operator $\Omega:M\to C([t_0,+\infty),\mathbb{R})$ by

$$(\Omega y)(t)=\begin{cases}\kappa/2, & t_0\leq t<T\\ \kappa/2+\int_T^t\left[\frac{1}{\pi(\eta)}\int_\eta^\infty p(\zeta)f\big(y(\tau(\zeta))\big)d\zeta\right]^{1/\gamma}d\eta & T\leq t.\end{cases}$$

Note that for $y\in M$, we have $(\Omega y)(t)\geq\kappa/2$. Also for $y\in M$ and $t\geq T$, we have $y(t)\leq\kappa$ and by (3.11), $(\Omega y)(t)\leq\kappa$. Therefore, $\Omega y\in M$. As in the proof of Theorem 1, the mapping Ω has a fixed point $v\in M$; that is, $(\Omega v)(t)=v(t)$ for $t\geq t_0$. It can be easily verified that $v(t)$ is a solution of (1.1), such that $\kappa/2\leq v(t)\leq\kappa$ for $t\geq T$. Thus we have a non-oscillatory solution to (1.1). This completes the proof. □

Example 2. Consider the delay differential equation

$$((y'^{1/5})'^{5/3}=0,\quad t\geq0. \tag{3.12}$$

Here $\gamma=1/5$, $\pi(t)=1$, $\tau(t)=t-1$ and $f(u)=u^{5/3}$. For $\beta=4/3$, we have $f(v)/v^{\beta}=v^{1/3}$ which is an increasing function. Thus

$$\int_2^\infty\left[\int_\eta^\infty(\zeta+1)d\zeta\right]^5d\eta=\infty.$$

So, all the conditions of Theorem 2 are satisfied, and therefore all solution of (3.12) are oscillatory.

4. Conclusion

In this section, we conclude the paper by stating a Remark and presenting two examples.

Remark 1. The results of this paper also hold for equations of the form

$$\big(\pi(y'^{\gamma})'(t)+\sum_{i=1}^{m}p_i(t)f_i\big(y(\tau_i(t))\big)=0,$$

where π, p_i, f_i, τ_i $(i=1,2,\dots,m)$ satisfy the assumptions (A1)-(A3), (3.1) or (3.8). In order to extend Theorem 1 and Theorem 2, there exists an index i such that p_i, f_i, τ_i fulfill (3.5) and (3.9), respectively.

Next, we provide two examples, illustrating how Remark 3.1 can be applied.

Example 3. Consider the delay differential equation

$$(e^{-t}(y'^{3/5})'+\frac{1}{t+1}(y(t-2))^{1/3}+\frac{1}{t+2}(y(t-1))^{1/5}=0, \quad t\geq 0. \tag{4.1}$$

Here $\gamma=3/5$, $\pi(t)=e^{-t}$, $\tau_1(t)=t-2$, $\tau_2(t)=t-1$, $\Pi(t)=\int_0^t e^{5\eta/3}\,d\eta=\frac{3}{5}\big(e^{5t/3}-1\big)$, $f_1(v)=v^{1/3}$ and $f_2(v)=v^{1/5}$. For $\beta=1/2$, we have $f_1(v)/v^{\beta}=v^{-1/6}$ and $f_2(v)/v^{\beta}=v^{-3/10}$ which both are decreasing functions. To check (3.5) we have

$$\begin{aligned}\int_0^{\infty}\sum_{i=1}^{m}p_i(\eta)f_i\big(\kappa^{1/\gamma}\Pi(\tau_i(\eta))\big)d\eta &\geq \int_0^{\infty}p_1(\eta)f_1\big(\kappa^{1/\gamma}\Pi(\tau_1(\eta))\big)d\eta\\ &=\int_0^{\infty}\frac{1}{\eta+1}\Big(\kappa^{5/3}\frac{3}{5}\big(e^{5(\eta-2)/3}-1\big)\Big)^{1/3}d\eta=\infty \quad \forall\kappa>0,\end{aligned}$$

because the integral approaches $+\infty$ as $\eta\to+\infty$. So, all the conditions of Theorem 1 hold, and therefore all solution of (4.1) are oscillatory.

Example 4. Consider the delay differential equation

$$((y'^{3/5})'^{5/3}+(t+1)(y(t-1))^{7/3}=0, \quad t\geq 0. \tag{4.2}$$

Here $\gamma=3/5$, $\pi(t)=1$, $\tau_1(t)=t-2$, $\tau_2(t)=t-1$, $\Pi(t)=t$, $f_1(v)=v^{5/3}$ and $f_2(v)=v^{7/3}$. For $\beta=4/3$, we have $f_1(v)/v^{\beta}=v^{1/3}$ and $f_2(v)/v^{\beta}=v$ which both are increasing functions. Clearly, all the conditions of Theorem 2 hold. Thus, every solution of (4.2) oscillates.

Open problem

Based on this work and [6, 9, 13, 15, 17, 18] an open problem that arises is to establish necessary and sufficient conditions for the oscillation of the solutions of the second-order nonlinear neutral differential equation.

REFERENCES

[1] R. P. Agarwal, L. Berzansky, E. Braverman, and A. Domoshnitsky, *Nonoscillation theory of functional differential equations with applications*. Springer, 2012.

[2] B. Baculíková, T. Li, and J. Džurina, "Oscillation theorems for second order neutral differential equations," *Electron. J. Qual. Theory Differ. Equ.*, vol. 74, pp. 1–13, 2011, doi: 10.14232/ejqtde.2011.1.74.

[3] O. Bazighifan, "Kamenev and philos-types oscillation criteria for fourth-order neutral differential equations," *Advances in Difference Equations*, no. 201, pp. 1–12, 2020, doi: 10.1186/s13662-020-02661-6.

[4] O. Bazighifan, "On the oscillation of certain fourth-order differential equations with p-Laplacian like operator," *Applied Mathematics and Computation*, vol. 386, no. C, 2020, doi: 10.1016/j.amc.2020.125475.

[5] O. Bazighifan and H. Ramos, "On the asymptotic and oscillatory behavior of the solutions of a class of higher-order differential equations with middle term," *Applied Mathematics Letters*, vol. 107, p. 106431, 2020, doi: https://doi.org/10.1016/j.aml.2020.106431.

[6] J. Brands, "Oscillation theorems for second-order functional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 63, no. 1, pp. 54–64, 1978, doi: 10.1016/0022-247X(78)90104-X.

[7] C. Cesarano and O. Bazighifan, "Qualitative behavior of solutions of second order differential equations," *Symmetry*, vol. 11, no. 6, 2019, doi: 10.3390/sym11060777.

[8] J. Džurina, "Oscillation theorems for second order advanced neutral differential equations," *Tatra Mountains Mathematical Publications*, vol. 48, no. 1, pp. 61–71, 2012, doi: doi:10.2478/v10127-011-0006-4.

[9] B. Karpuz and S. S. Santra, "Oscillation theorems for second-order nonlinear delay differential equations of neutral type," *Hacet. J. Math. Stat.*, vol. 48, no. 3, pp. 633–643, 2019.

[10] Q. Li, R. Wang, F. Chen, and T. Li, "Oscillation of second-order nonlinear delay differential equations with nonpositive neutral coefficients," *Advances in Difference Equations*, no. 35, pp. 1–7, 2015, doi: 10.1186/s13662-015-0377-y.

[11] O. Moaaz, E. Elabbasy, and B. Qaraad, "An improved approach for studying oscillation of generalized Emden–Fowler neutral differential equation," *J Inequal Appl*, vol. 69, pp. 1–18, 2020, doi: https://doi.org/10.1186/s13660-020-02332-w.

[12] O. Moaaz, R. A. El-Nabulsi, W. Muhsin, and O. Bazighifan, "Improved oscillation criteria for 2nd-order neutral differential equations with distributed deviating arguments," *Mathematics*, vol. 8, no. 5, 2020, doi: 10.3390/math8050849.

[13] S. Pinelas and S. S. Santra, "Necessary and sufficient condition for oscillation of nonlinear neutral first-order differential equations with several delays," *J. Fixed Point Theory Appl.*, vol. 20, no. 1, p. 13, 2018, id/No 27.

[14] M. A. Ragusa, "Elliptic boundary value problem in Vanishing mean Oscillation hypothesis," *Commentat. Math. Univ. Carol.*, vol. 40, no. 4, pp. 651–663, 1999.

[15] M. A. Ragusa, "Homogeneous herz spaces and regularity results," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 12, pp. e1909–e1914, 2009, doi: https://doi.org/10.1016/j.na.2009.02.075.

[16] S. S. Santra, "Existence of positive solution and new oscillation criteria for nonlinear first-order neutral delay differential equations," *Differ. Equ. Appl.*, vol. 8, no. 1, pp. 33–51, 2016.

[17] S. S. Santra, "Oscillation analysis for nonlinear neutral differential equations of second order with several delays," *Mathematica*, vol. 59, no. 1-2, pp. 111–123, 2017.

[18] J. S. W. Wong, "Necessary and sufficient conditions for oscillation of second order neutral differential equations," *J. Math. Anal. Appl.*, vol. 252, no. 1, pp. 342–352, 2000.

[19] Q. Zhang and J. Yan, "Oscillation behavior of even order neutral differential equations with variable coefficients," *Appl. Math. Lett.*, vol. 19, no. 11, pp. 1202–1206, 2006.

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