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L^p ESTIMATES FOR ROUGH SEMICLASSICAL FOURIER INTEGRAL OPERATORS

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Abstract. In this paper the author investigates the global boundedness in Banach L^p spaces of rough semiclassical Fourier integral operators defined by generalized rough Hörmander class amplitudes and rough class phase functions which behave in the spatial variable like L^p functions.

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1. INTRODUCTION

A Fourier integral operator or FIO in short is a singular operator defined by

$$\left(I_{a,\phi}f\right)(x)=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{i\phi(x,\xi)}a(x,\xi)\,\widehat{f}(\xi)\,d\xi,\ f\in C_0^\infty(\mathbb{R}^n),$$

where $a(x,\xi)$ and $\phi(x,\xi)$ are smooth functions called respectively the amplitude and the phase function.

The study of the theory of FIOs has a long history, many efforts have been made to study the regularity of these operators in functional spaces and there is a large body of results concerning the regularity like in [8] and [5] where some results of local L^2 boundedness have been obtained for Fourier integral operators associated to smooth amplitudes $a(x,\xi)\in S^m_{\rho,\delta}$ and phase functions $\phi(x,\xi)\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\})$, those are non degenerate and positively homogeneous of degree 1 in the frequency variable ξ. These results of local boundedness have been generalized in L^p for $1\leq p\leq\infty$ by Seeger-Sogge-Stein [15], Gala-Liu-Ragusa [6], Khelouki-Ikassoulene [10] and Ragusa [12].

Since 1970s, motivated by applications in microlocal analysis and hyperbolic partial differential equations, many authors extended local L^2 boundedness results to

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global $L^2(\mathbb{R}^n)$ regularity. The first result has been given in Asada-Fujiwara [2] considering Fourier integral operators associated to amplitudes in $S^0_{0,0}$ and inhomogeneous phase functions. This result was extended to a general class of amplitudes in $S^m_{\rho,\delta}$, satisfying some conditions, see [11]. For the global L^p boundedness, Cordiro, Nicola and Rodino have considered Fourier integral operators with amplitudes in $S^m_{1,0}$, see [3]. Another result of global L^p boundedness of Fourier integral operators, due to Coriasco and Ruzhanky [4], was to consider their amplitudes in some subspaces of $S^0_{1,0}$.

In the semiclassical case an h-Fourier integral operator has the following form

$$(I_h(a,\phi)f)(x) = (2\pi h)^{-n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\phi(x,\xi)}a(x,\xi)\widehat{f}(\xi)\,d\xi,\ f\in S(\mathbb{R}^n),$$

where $h\in\,]0,h_0]$ is a semiclassical parameter and S denotes the Schwartz space. These operators appear naturally in the expression of the solution of the semiclassical hyperbolic partial differential equations, and when expressing the C^∞ -solution of the associated Cauchy's problem. Comparatively with the study of FIOs, there has been a smaller amount of activity for h-FIOs concerning the investigation of the corresponding L^p boundedness properties. Some results of L^2 boundedness and L^2 compactness are obtained for h -Fourier integral operators with amplitudes introduced by Hörmander and non degenerate and homogeneous phase functions, see [7] and [16] . Recently we have establish in our paper [1] the L^p boundedness for h-Fourier integral operators with rough amplitudes in the class $L^pS^m_\rho$ introduced by [14] and non degenerate and homogeneous phase functions in Φ^2. We have obtained results of boundedness of these operators from L^q into L^r, where the numbers q,r satisfy some relations with the others parameters p, n, m, ρ.

Motivated by the lack of L^p boundedness results for h-FIOs in the literature of semiclassical analysis, the aim of this wok is to extend the aforementioned results for a general class of h-FIOs. Our investigation is to study rough h-Fourier integral operators associated to amplitudes in the class $L^\infty S^m_\rho$ and phase functions in the class $L^\infty\Phi^k$ as introduced in [9].

To summarize the paper is organized as follows. In the second section we introduce some definitions and notations of a general class of amplitudes and phase functions associated to the class of h-FIOs treated in this paper. Tools and preliminary lemmas are mentioned in the third section. The last section is devoted to establish the results of the boundedness of this class of h-FIOs.

2. Rough amplitudes and phase functions

In this section we define the classes of linear amplitudes and the class of phase functions with rough spatial behaviour that appear in the definition of operators treated here. But at first we recall some definitions and notations of classical amplitudes and phase functions introduced by Hörmander [8].

For the sake of simplicity we use the notation $\langle\xi\rangle$ for $\left(1+|\xi|^2\right)^{\frac{1}{2}}$.

Definition 1. Let $m\in\mathbb{R}$, $0\le\rho,\delta\le 1$. We denote $S^m_{\rho,\delta}$ the class of all functions $a(x,\xi)\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ such that

$$\forall\alpha,\beta\in\mathbb{N}^n:\sup_{\xi\in\mathbb{R}^n}\langle\xi\rangle^{-m+\rho|\alpha|-\delta|\beta|}\left|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)\right|<+\infty.$$

Notation 1. We denote by Φ^k, $k\in\mathbb{N}$, the space of real valued functions $\varphi(x,\xi)\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n\backslash\{0\})$, such that $\varphi(x,\xi)$ is positively homogeneous of degree 1 in the frequency variable ξ, and satisfies the following condition:
$\forall\alpha,\beta\in\mathbb{N}^n$, with $|\alpha|+|\beta|\ge k$, $\exists C_{\alpha,\beta}>0$:

$$\sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\backslash\{0\}}|\xi|^{-1+|\alpha|}\left|\partial_\xi^\alpha\partial_x^\beta\varphi(x,\xi)\right|\le C_{\alpha,\beta}.$$

Definition 2. A real valued phase $\varphi(x,\xi)\in C^2(\mathbb{R}^n\times\mathbb{R}^n\backslash\{0\})$ satisfies the *Strong Non-Degeneracy* condition or the SND condition for short, if there exists a constant $c>0$ such that

$$\left|\det\frac{\partial^2\varphi(x,\xi)}{\partial x_j\partial\xi_k}\right|\ge c \text{ for all } (x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\backslash\{0\}.$$

Example 1. The function $\phi(x,\xi)=\langle x,\xi\rangle$ is in Φ^2 and satisfies the strong non-degenerate condition. A Fourier integral operator with such phase function is called pseudodifferential operator.

Now we introduce the definitions and conditions of rough amplitudes and phase functions which we will use for our study of the L^p boundedness for h-Fourier integral operators associated to these amplitudes and phase functions.

Definition 3. Let $m\in\mathbb{R}$ and $0\le\rho\le 1$. We denote by $L^\infty S^m_\rho$, the space of functions $a(x,\xi)$, $x,\xi\in\mathbb{R}^n$ such that $a(x,\xi)$ is measurable in $x\in\mathbb{R}^n$, $a(x,\xi)\in C^\infty\left(\mathbb{R}^n_\xi\right)$ a.e. $x\in\mathbb{R}^n$, and for each multi-index α, there exists a constant C_α such that

$$\left\|\partial_\xi^\alpha a(.,\xi)\right\|_{L^\infty(\mathbb{R}^n)}\le C_\alpha\langle\xi\rangle^{m-\rho|\alpha|},\ \forall\xi\in\mathbb{R}^n.$$

Remark 1. It is clear that $S^m_{\rho,\delta}\subset L^\infty S^m_\rho$ for all $m\in\mathbb{R}$ and $0\le\rho,\delta\le 1$.

Notation 2. Let $k\in\mathbb{N}$, we denote by $L^\infty\Phi^k$ the space of real valued functions $\phi(x,\xi)$ such that $\phi(x,\xi)$ is smooth and homogeneous of degree 1 in the frequency variable ξ, $\phi(x,\xi)$ is measurable in $x\in\mathbb{R}^n$ and satisfies the following condition:
$\forall\alpha\in\mathbb{N}^n$ with $|\alpha|\ge k$, $\exists C_\alpha>0$:

$$\sup_{\xi\in\mathbb{R}^n\backslash\{0\}}|\xi|^{-1+|\alpha|}\left\|\partial_\xi^\alpha\phi(.,\xi)\right\|_{L^\infty(\mathbb{R}^n)}\le C_\alpha.$$

Example 2. For each $f \in L^\infty(\mathbb{R}^n)$ the phase function $\phi(x,\xi) = f(x)|\xi| + \langle x,\xi\rangle$ belongs to the class $L^\infty\Phi^2$.

Instead of the non-degeneracy condition in case of smooth phase function, we need in our case to an analogous condition which we will call rough non-degeneracy condition.

Definition 4 (The rough non-degeneracy condition)**.** We say that a real valued phase ϕ satisfies the *Rough Non-Degeneracy condition*, if it is C^1 on $\mathbb{R}^n\setminus\{0\}$ in the frequency variable ξ, bounded measurable in the spatial variable x, and there exists a constant $c > 0$ such that

$$\left|\partial_\xi\phi(x,\xi) - \partial_\xi\phi(y,\xi)\right| \geq c\,|x-y|\,, \text{ for all } x,y \in \mathbb{R}^n \text{ and } \xi \in \mathbb{R}^n\setminus\{0\}\,.$$

The aim of this work is to obtain some results of L^p boundedness for h -Fourier integral operators of the form

$$(I_h(a,\phi)f)(x) = (2\pi h)^{-n}\int_{\mathbb{R}^n} e^{\frac{i}{h}\phi(x,\xi)}a(x,\xi)\,\widehat{f}(\xi)\,d\xi, \tag{2.1}$$

where $a \in L^\infty S^m_\rho$, $\phi \in L^\infty\Phi^k$ and $h \in]0,1]$ is a parameter. Here

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-\frac{i}{h}x\xi} f(x)\,dx.$$

3. Preliminary tools

For our study we collect here the main tools in proving our boundedness results of boundedness of h-FIOs. First we need Seeger-Sogge-Stein partition of unity.

For each $s \in]0,1]$ let $\{\xi^\nu\}_{1\leq\nu\leq N}$ a finite collection of unit vectors of $\mathbb{S}^{n-1}$ satisfying

1) $\left|\xi^\nu - \xi^{\nu'}\right| \geq h^{-\frac{j}{2}}$, if $\nu \neq \nu'$,

2) $\forall\xi \in \mathbb{S}^{n-1}$, $\exists\xi^\nu$ so that $|\xi - \xi^\nu| \leq s^{\frac{1}{2}}$.

For any $\nu \in \{1,\dots,N\}$ let Γ^ν the cone defined by

$$\Gamma^\nu = \left\{\xi \in \mathbb{R}^n : \left|\frac{\xi}{|\xi|} - \xi^\nu\right| \leq s^{\frac{1}{2}}\right\}.$$

In order to cover the annulus $\left\{\xi; \frac{1}{2} \leq |\xi| \leq 2\right\}$, we need $s^{-\frac{n-1}{2}}$ cones such Γ^ν.

Next we construct an associated partition of unity subordinated to $\{\Gamma^\nu\}_{1\leq\nu\leq N}$ given by functions χ^ν, homogenous of degree 0 in ξ with support in Γ^ν, i.e.

$$\sum_{\nu=1}^{N}\chi^\nu(\xi) = 1 \text{ for all } \xi \neq 0,$$

and such that

$$\forall\alpha \in \mathbb{N}^n, \exists C_\alpha > 0 : |\partial^\alpha\chi^\nu(\xi)| \leq C_\alpha s^{-\frac{|\alpha|}{2}}.$$

According to this partition of unity we have get the following lemma (see also [1]).

Lemma 1. *Any h-Fourier integral operator I_h of the type (2.1) with amplitude $a(x,\xi) \in L^\infty S^m_\rho$ and phase function $\phi(x,\xi) \in L^\infty \Phi^2$, can be written as a finite sum of operators of the form*

$$(2\pi h)^{-n} \int \sigma(x,\xi)\, e^{\frac{i}{h}\psi(x,\xi)+\frac{i}{h}\langle \nabla_\xi \phi(x,\zeta),\xi\rangle} \widehat{u}(\xi)\, d\xi,$$

where ζ is a point on the unit sphere S^{n-1}, $\psi(x,\xi) \in L^\infty \Phi^1$ and $\sigma(x,\xi) \in L^\infty S^m_\rho$ is localized in the ξ variable around the point ζ.

But in order to study the L^p boundedness for rough h-Fourier integral operators with amplitudes in $L^\infty S^m_\rho$ and phase functions in $L^\infty \Phi^2$ we need the Littlewood-Paley partition of unity (see [15]):

$$\Psi_0(\xi) + \sum_{j=1}^{\infty} \Psi_j(\xi) = 1,$$

where $\operatorname{supp}\Psi_0 \subset \{\xi; |\xi| \le 2\}$, and $\Psi_j(\xi) = \Psi\left(2^{-j}\xi\right)$, with $\operatorname{supp}\Psi \subset \left\{\xi; \frac{1}{2} \le |\xi| \le 2\right\}$.

So we can write

$$\begin{aligned} I_h(a,\phi) &= \Psi_0(\xi)\, I_h(a,\phi) + \sum_{j=1}^{\infty} \Psi_j(\xi)\, I_h(a,\phi) \\ &= I_{h,0}(a,\phi) + \sum_{j=1}^{\infty} I_{h,j}(a,\phi). \end{aligned}$$

Applying the Seeger-Sogge-Stein decomposition in the annulus $\left\{\xi; \frac{1}{2} \le |\xi| \le 2\right\}$ to the operator $I_{h,j}(a,\phi)$ we obtain

$$I_{h,j}(a,\phi) = \sum_{\nu=1}^{N(j)} \chi^\nu(\xi)\, I_{h,j}(a,\phi) = \sum_{\nu=1}^{N(j)} I^\nu_{h,j}.$$

The operator $I^\nu_{h,j}(a,\phi)$ has kernel

$$\begin{aligned} I^\nu_{h,j}(x,y) &= \frac{(2\pi h)^{-n}}{2^{jn}} \int e^{\frac{i}{h}2^j(\phi(x,\xi)-\langle x,\xi\rangle)} \Psi(\xi)\, \chi^\nu(\xi)\, a\left(x, 2^j\xi\right) d\xi \\ &= \frac{(2\pi h)^{-n}}{2^{jn}} \int e^{\frac{i}{h}2^j\langle \nabla_\xi \phi(x,\xi^\nu)-y,\xi\rangle} b^\nu_j(x,\xi)\, d\xi, \end{aligned} \tag{3.1}$$

where $b^\nu_j(x,\xi) = e^{\frac{i}{h}2^j\langle \nabla_\xi\phi(x,\xi)-\nabla_\xi\phi(x,\xi^\nu),\xi\rangle} \Psi(\xi)\, \chi^\nu(\xi)\, a\left(x, 2^j\xi\right)$.

In the sequel we will also need the following lemmas for the proof (see [9]).

Lemma 2. *Let $a \in L^\infty S^m_\rho$ and $\phi(x,\xi) \in L^\infty \Phi^2$, then the symbol $b^\nu_j(x,\xi)$ above satisfies the estimate*

$$\forall \alpha \in \mathbb{N}^n, \exists C_\alpha > 0 : \sup_\xi \left\| \partial^\alpha_\xi b^\nu_j(.,\xi) \right\|_{L^\infty} \le C_\alpha \left(2^j\right)^{m+|\alpha|(1-\rho)+\frac{|\alpha'|}{2}}.$$

Lemma 3. *Let $b(x,\xi)$ be a bounded function which is $C^{n+1}\left(\mathbb{R}^n_\xi \backslash \{0\}\right)$ and compactly supported in the frequency variable ξ and $L^\infty(\mathbb{R}^n_x)$ in the space variable x, satisfying*

$$\sup_{\xi\in\mathbb{R}^n\backslash\{0\}} |\xi|^{-1+|\alpha|} \left\|\partial^\alpha_\xi b(.,\xi)\right\|_{L^\infty(\mathbb{R}^n)} < +\infty,\ \forall |\alpha| \leq n+1.$$

Then for all $\mu \in [0,1[$ we have

$$\sup_{x,y\in\mathbb{R}^n} \langle y\rangle^{n+\mu} \left|\int e^{-i\langle y,\xi\rangle} b(x,\xi)\, d\xi\right| < +\infty.$$

Finally we give the semiclassical version of Hausdorff-Young and Minkowsky's inequality.

Hausdorff-Young inequality: For all $p,q\in\mathbb{R}$ such that $1\leq p\leq 2$ and $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\exists C>0 : \left\|\widehat{f_h}\right\|_{L^q(\mathbb{R}^n)} \leq Ch^{\frac{n}{q}} \|f_h\|_{L^q(\mathbb{R}^n)},\ f\in L^p(\mathbb{R}^n).$$

Minkowsky's inequality for integrals: For all $p,q\in\mathbb{R}$ such that $1\leq p\leq\infty$ and all measurable function f on $\mathbb{R}^n_x\times\mathbb{R}^n_y$, we have

$$\left(\int_{\mathbb{R}^n_y}\left|\int_{\mathbb{R}^n_x} f(x,y)\,dx\right|^p dy\right)^{\frac{1}{p}} \leq \int_{\mathbb{R}^n_x}\left(\int_{\mathbb{R}^n_y}|f(x,y)|^p\,dy\right)^{\frac{1}{p}} dx.$$

4. Main results

In this section we establish the results of L^p boundedness for rough h-Fourier integral operators, which permits us to obtain those of smooth h-Fourier integral operators as consequences.

Let us begin with rough h-Fourier integral operators associated to amplitudes with compact support in the frequency variable ξ.

Theorem 1. *Let $m\in\mathbb{R}$ and $\rho\in[0,1]$, let $a(x,\xi)\in L^\infty S^m_\rho$ such that $\operatorname{supp}_\xi a(x,\xi)$ is compact, and let $\phi(x,\xi)\in L^\infty\Phi^2$ a phase function satisfying the rough non-degeneracy condition. Then the h-Fourier integral operator given by (2.1) is bounded on L^p for every $p\in[1,\infty]$.*

Proof. According to Lemma 1, it suffices to prove the L^p boundedness of each Fourier integral operator of the form

$$Au(x) = \int \sigma(x,\xi)\, e^{\frac{i}{h}\psi(x,\xi)+\frac{i}{h}\langle\nabla_\xi\phi(x,\zeta),\xi\rangle}\,\widehat{u}(\xi)\,d\xi,$$

where $\zeta\in S^{n-1}$, $\sigma\in L^\infty S^m_\rho$ and $\psi\in L^\infty\Phi^1$.

Let us rewrite the operator A with its kernel, i.e.

$$Au(x) = \int A(x,y)\,u(y)\,dy,$$

where the kernel of A is given by

$$A(x,y)=\int \sigma(x,\xi)\, e^{\frac{i}{h}\psi(x,\xi)+\frac{i}{h}\langle\nabla_\xi\phi(x,\zeta)-y,\xi\rangle}d\xi.$$

To get an estimate of $\|Au\|_{L^p}$ we will estimate $A(x,y)$ in L^1. Set $b(x,\xi)=\sigma(x,\xi)\,e^{\frac{i}{h}\psi(x,\xi)}$, this function is bounded because σ and ψ are bounded uniformly in x and $\mathrm{supp}_\xi\sigma$ is compact, further since $\psi\in L^\infty\Phi^1$ so

$$\forall\,|\alpha|\geq 1:\sup_{x\in\mathbb{R}^n}\sup_{|\xi|\neq 0}|\xi|^{-1+|\alpha|}\left|\partial_\xi^\alpha b(x,\xi)\right|<\infty,$$

in particular for $|\alpha|\leq n+1$, we obtain from Lemma 3

$$\sup_{x\in\mathbb{R}^n}|A(x,y)|\leq C\left\langle\nabla_\xi\phi(x,\zeta)-y\right\rangle^{-n-\mu}\text{ for all }\mu\in[0,1[,$$

which implies (by choosing some $\mu\in]0,1[$)

$$\sup_x\int|A(x,y)\,dy|<\infty.$$

On the other hand from the rough non-degeneracy assumption we get

$$\int v\circ\partial_\xi\phi(x,\zeta)\,dx=\int v(z)\,J(z)\,dz,\ \forall v\in L^1,$$

where J is the Jacobian of $\partial_\xi\phi$. Thus we obtain

$$\int|A(x,y)|\,dx\leq C\int\left\langle\nabla_\xi\phi(x,\zeta)-y\right\rangle^{-n-\mu}dx\leq C'\int\langle z\rangle^{-n-\mu}dz<\infty,$$

uniformly in y.

Finally from this estimate and using Young's inequality we can easily obtain an estimate of $\|Au\|_{L^p}$ in term of $\|u\|_{L^p}$, and therefore we get the L^p boundedness of the operator A. □

Next let us consider the general case of rough h-Fourier integral operators, without assumption of compactness on the amplitude.

Theorem 2. *Let $0\leq\rho\leq 1$ and $m\in\mathbb{R}$ such that $m<-\frac{n-1}{2}+n(\rho-1)$. If $a\in L^\infty S^m_\rho$ and $\phi\in L^\infty\phi^2$ satisfying the rough non- degeneracy condition, then the h-Fourier integral operator given by (2.1) is bounded on $\mathcal{S}(\mathbb{R}^n)$ provided with the norm L^1, i.e.*

$$\exists C>0:\|I_h(a,\phi)\,u\|_{L^1}\leq C\|u\|_{L^1},\ \forall u\in\mathcal{S}(\mathbb{R}^n).$$

Proof. By the Littlewood-Paley partition of unity, we can decompose the operator $I_h(a,\phi)$ as a sum of operators of the form

$$\left(I_{h,j}f\right)(x)=\int_{\mathbb{R}^n}e^{\frac{i}{h}\phi(x,\xi)}\Psi_j(\xi)\,a(x,\xi)\,\widehat{f}(\xi)\,d\xi,\ j=0,1,\ldots$$

For $j=0$, the operator $I_{h,0}$ has amplitude $\Psi_0(\xi)\,a(x,\xi)$ which is supported around the origin, so $I_{h,0}$ is bounded on L^1 from Theorem 1.

For $j = 1, 2, ...$ using the Seeger-Sogge-Stein decomposition for $s = 2^{-j}$, we have $I_{h,j} = \sum_{\nu=1}^{N(j)} I_{h,j}^{\nu}$, where the operator $I_{h,j}^{\nu}$ has kernel defined by (3.1). In order to estimate this kernel we consider the differential operator

$$L = 1 - \partial_{\xi_1}^2 - 2^{-j}\partial_{\xi'}^2.$$

from Lemma 2, we have for any $k \in \mathbb{N}$

$$\sup_{\xi} \left\| L^k b_j^{\nu}(.,\xi) \right\|_{L^\infty} \leq C_k \left(2^{-j}\right)^{-m-2k(1-\rho)}.$$

Using an integrations by parts in ξ we get

$$\begin{aligned} \left| I_{h,j}^{\nu}(x,y) \right| &\leq \frac{(2\pi h)^{-n}}{2^{jn}} \left(1 + g\left(y - \nabla_\xi \phi(x,\xi^\nu)\right)\right)^{-k} \int \left| L^k b_j^{\nu}(x,\xi) \right| d\xi \\ &\leq C_k \left(2^{-j}\right)^{-m-\frac{n+1}{2}-2k(1-\rho)} \left(1 + g\left(y - \nabla_\xi \phi(x,\xi^\nu)\right)\right)^{-k}, \end{aligned}$$

where for simplify $g(y) = 2^{2j} y_1^2 + 2^j |y'|^2$. This estimate remains valid if the integer k is replaced by any positive number M. We can check that by some interpolation inequality.

So for any real number $M > n$ we obtain

$$\sup_x \int \left| I_{h,j}^{\nu}(x,y) \right| dy \leq C_M \left(2^{-j}\right)^{-m-M(1-\rho)}.$$

On the other hand from the rough non-degeneracy assumption on $\phi(x,\xi)$ we get

$$\sup_y \int \left| I_{h,j}^{\nu}(x,y) \right| dx \leq C'_M \left(2^{-j}\right)^{-m-M(1-\rho)}.$$

Hence summing over ν and using Young's inequality we obtain

$$\left\| I_{h,j} u \right\|_{L^1} \leq \sum_{\nu=1}^{N(j)} \left\| I_{h,j}^{\nu} u \right\|_{L^1} \leq C''_M \left(2^{-j}\right)^{-m-\frac{n-1}{2}-M(1-\rho)} \|u\|_{L^1}.$$

(Note that $N(j) = s^{-\frac{n-1}{2}} = \left(2^{-j}\right)^{-\frac{n-1}{2}}$). Since $m < -\frac{n-1}{2} - n(1-\rho) < -\frac{n-1}{2} - M(1-\rho)$ so $\sum \left(2^{-j}\right)^{-m-\frac{n-1}{2}-M(1-\rho)} < \infty$ and therefore by summing over $j = 1, ...$ we find

$$\|I_h(a,\phi)u\|_{L^1} \leq \|I_{h,0}u\|_{L^1} + \sum_{j=1}^{\infty} \left\| I_{h,j} u \right\|_{L^1} \leq C \|u\|_{L^1}.$$

The proof of the theorem is complete. □

Remark 2. In case of $\rho = 0$, i.e. $a \in L^\infty S_0^m$, the result can be extended for all $m < -n$.

Now we will establish the result of L^2 boundedness of rough h-Fourier integral operators, but we must prove an intermediary result when the phase function $\phi(x,\xi)$ is in Φ^2 instead of $L^\infty \Phi^2$.

Theorem 3. *Let $a \in L^\infty S^m_\rho$ with $m < \frac{n}{2}(\rho - 1)$ and $\phi \in \Phi^2$ satisfying the SND condition. Then the h-Fourier integral operator $I_h(a,\phi)$ given by (2.1) is bounded from L^2 to itself.*

Proof. As in Theorem 2, we write the operator $I_h(a,\phi)$ as a sum of operators $I_{h,j}$ and the boundedness of $I_{h,0}$ follows from 1. Next for $I_{h,j}$ we deal with the operator $T_{h,j} = I_{h,j}I^*_{h,j}$ where its kernel is given by

$$T_{h,j}(x,y) = \frac{(2\pi h)^{-n}}{2^{jn}} \int e^{\frac{i}{h}2^j(\phi(x,\xi)-\phi(y,\xi))}\chi^2(\xi)\, a(x,\xi)\, \overline{a}(y,\xi)\, d\xi.$$

From the SND condition on the function ϕ there exists a constant $C > 0$ such that

$$\left|\nabla_\xi \phi(x,\xi) - \nabla_\xi \phi(y,\xi)\right| \geq c\,|x-y|\,,\ \forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}\,,$$

so by the non-stationary phase estimate in [8] we have for all integer k

$$\forall k \in \mathbb{N}, \exists C_k > 0 : \left|T_{h,j}(x,y)\right| \leq C_k 2^{j[2m+n+(1-\rho)k]} \left\langle 2^j(x-y)\right\rangle^{-k}.$$

This estimate remains valid for $k \in \mathbb{R}^*_+$ by using the integer part, thus

$$\forall M > n, \exists C_M > 0 : \sup_x \int \left|T_{h,j}(x,y)\right| dy \leq C_M 2^{j[2m+(1-\rho)M]},$$

hence by Cauchy-Schwarz and Young inequalities we get for all $M > n$,

$$\left\|I^*_{h,j}u\right\|^2_{L^2} \leq \left\|T_{h,j}u\right\|_{L^2} \|u\|_{L^2} \leq C_M 2^{j[2m+(1-\rho)M]} \|u\|^2_{L^2}.$$

Since $m < (\rho - 1)\frac{n}{2}$ we can choose $M > n$ such that $m < (\rho-1)\frac{M}{2}$ in order that the series $\sum 2^{j[2m+(1-\rho)M]}$ converges and therefore by summing over $j = 0, 1, \ldots$, we obtain

$$\|I_h(a,\phi)\,u\|_{L^2} \leq C\,\|u\|_{L^2}.$$

Thus the proof of Theorem 3 is complete. □

Remark 3. Note that Theorem 3 is not valid for $m = (\rho-1)\frac{n}{2}$, there are pseudodifferential with symbols $a \in S^m_{\rho,1}$ which are not bounded in L^2 (see [13]). Recall that $S^m_{\rho,1} \subset L^\infty S^m_\rho$ and $\phi(x,\xi) = \langle x,\xi\rangle$ is in $L^\infty\Phi^2$ and satisfies the SND condition.

Finally we prove the L^2 boundedness of rough h-Fourier integral operator, i.e. with rough amplitude and rough phase function.

Theorem 4. *Let $a \in L^\infty S^m_\rho$ with $0 \leq \rho \leq 1$ and $m < n\frac{(\rho-1)}{2} - \frac{(n-1)}{4}$, and let $\phi \in L^\infty\Phi^2$ satisfying the rough non-degeneracy condition. Then the h-Fourier integral operator given by (2.1) is bounded from L^2 to itself.*

Proof. We will follows the steps of the proof of Theorem 3. First we write the operator $I_h(a,\phi)$ as a sum of operators $I_{h,j}$ and the boundedness of $I_{h,0}$ follows from

Theorem 1. Next for $I_{h,j}$ we deal with the operator $T_{h,j} = I_{h,j} I_{h,j}^*$ where its kernel is given by

$$T_{h,j}(x,y) = \frac{(2\pi h)^{-n}}{2^{jn}} \int e^{\frac{i}{h} 2^j (\phi(x,\xi) - \phi(y,\xi))} \chi^2(\xi)\, a(x,\xi)\, \overline{a}(y,\xi)\, d\xi.$$

Here we need the Seeger-Sogge-Stein decomposition, so $T_{h,j} = \sum_{\nu=1}^{N(j)} T_{h,j}^{\nu}$ where $T_{h,j}^{\nu}$ has the kernel

$$T_{h,j}^{\nu}(x,y) = \frac{(2\pi h)^{-n}}{2^{jn}} \int e^{\frac{i}{h} 2^j \langle \nabla_\xi \phi(x,\xi^\nu) - \nabla_\xi \phi(y,\xi^\nu), \xi \rangle} b_j^{\nu}(x,\xi)\, \overline{b_j^{\nu}}(y,\xi)\, d\xi.$$

After as in the the proof of Theorem 2, we introduce the differential operator

$$L = 1 - \partial_{\xi_1}^2 - 2^{-j} \partial_{\xi'}^2,$$

from Lemma 2, we have for any $k \in \mathbb{N}$

$$\sup_{\xi} \left\| L^k b_j^{\nu}(.,\xi) \right\|_{L^\infty} \leq C_k \left(2^{-j}\right)^{-m-2k(1-\rho)}.$$

Taking account the rough non-degeneracy assumption on ϕ and using integration by parts we obtain in the same manner

$$\forall M > \frac{n}{2}, \exists C > 0 : \left\| T_{h,j}^{\nu} u \right\|_{L^2} \leq C \left(2^{-j}\right)^{-2m-2M(1-\rho)} \|u\|_{L^2},$$

which gives for the operator $I_{h,j}^*$ the estimate

$$\left\| I_{h,j}^* u \right\|_{L^2}^2 \leq \sum_{\nu=1}^{N(j)} \left\| T_{h,j}^{\nu} u \right\|_{L^2} \|u\|_{L^2} \leq C \left(2^{-j}\right)^{-2m-\frac{n+1}{2}-2M(1-\rho)} \|u\|_{L^2}^2.$$

Since $m < (\rho - 1)\frac{n}{2} - \frac{n-1}{4}$ and $M > \frac{n}{2}$ then $-2m - \frac{n+1}{2} - 2M(1-\rho) > 0$, therefore $\sum_{j=1}^{\infty} \left(2^{-j}\right)^{-2m-\frac{n+1}{2}-2M(1-\rho)}$ converges. Thus by summing over $j = 0, 1, \ldots$ we get

$$\|I_h(a,\phi)\, u\|_{L^2} \leq \widetilde{C} \|u\|_{L^2},$$

which means the boundedness of the operator $I_h(a,\phi)$ in L^2. □

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