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EXISTENCE AND CONVERGENCE OF FIXED POINT FOR A G-NON-LIPSCHITZIAN MAPPING

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Abstract. The existence of fixed point in uniformly convex hyperbolic metric space endowed with graph of G-nearly asymptotically nonexpansive mapping has been obtained. Further, we prove strong and $\Delta-$convergence of $M-$iteration to fixed point of G-nearly asymptotically nonexpansive mapping. We also derived some corollaries of our results in uniformly convex Banach space which are also independent new findings.

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1. INTRODUCTION

Fixed point theory is an active and vital branch of mathematics. In 1922, Banach gave the first fundamental fixed point theorem which is known as Banach contraction principle. Banach contraction principle is also notable for its simplicity. It requires only two conditions one on underlying space as complete metric space and other is on involved mapping as contraction mapping. Generalizations of Banach contraction principle in different directions is one of the important part of research in nonlinear analysis. The one way is to generalize Banach contraction principle is to change the underlying metric structure and second is to extend the involve mapping. The important generalizations came into picture due to Ran and Reurings [19] and Nieto and López [17]. They generalize the Banach contraction principle to partially ordered metric space. Ran and Reurings [19] applied their results to solve matrix equation while Nieto and López [17] applied to solve differential equation.

Nonexpansive mappings are those mappings whose Lipschitz constant are equal to one. In 1965, Browder [4,5], Göhde [7] and Kirk [11] independently proved existence of fixed point for nonexpansive mappings in Banach space. Further, it is well known fact that Picard iteration does not converege to fixed point of nonexpansive mappings even if it exists. In 1953, Mann [15] introduced a iteration to approximate the fixed point of nonexpansive mappings. After Mann iteration, many iteration came into picture. Recently in 2018, Ullah and Arshad [29] introduced a new iteration namely $M-$iteration which is more efficient than the many existing iterations.

In 2008, Jachymski [8] proved Banach contraction principle in complete metric space endowed with graph which is natural generalization of Ran and Reurings [19] and Nieto and López [17] results.

In 2016, Alfuraidan and Shukri [2] proved Browder and Göhde fixed point theorem for G-nonexpansive mappings. Recently many authors obtained convergence results for many iterations in spaces endowed with graphs see [25, 27, 28, 30].

In 2005, Sahu [22] introduced a mapping namely nearly asymptotically nonexpansive mapping, it is a non-Lipschitzian type mapping and generalize nonexpansive mappings. In 2008 Sahu et al. [21] and in 2015 Saluja et al. [23] proved some convergence results for nearly asymptotically nonexpansive mappings. In this paper, we prove the Browder and Göhde fixed point theorem for G-nearly asymptotically nonexpansive mappings. We also prove, $\Delta-$ convergence and strong convergence of M-iteration for G-nearly asymptotically nonexpansive mappings in hyperbolic metric space.

2. Preliminaries

Let us start the section by collecting some needed results.

A graph G is an ordered pair $(V(G),E(G))$ where $V(G)$ is a set called vertices and $E(G)$ is a binary relation on $V(G)$ $(i.e. E(G) \subseteq V(G)\times V(G))$ called edges of G. If the direction is imposed on each edges then we call the graph a directed graph or digraph. Here we assume that digraph have loop at every vertex (i.e. $(x,x) \in E(G)$ for each $x \in V(G)$). We also assume that G has no parallel edges. Moreover, we assume that there is a distance function d defined on the set of vertices $V(G)$ and we call it a weighted graph. G^{-1} is obtained from G by reversing the direction of edges. Thus we have

$$E(G^{-1}) = \{(y,x) : (x,y) \in E(G)\}.$$

The letter $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. It will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention, we have

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

Further, if x and y are vertices in a graph G, then a path in G form x to y of length $N \in \mathbb{N}$ is a sequence $\{x_i\}_{i=1}^{N}$ of N vertices such that $x_1 = x$, $x_N = y$ and $(x_i, x_{i+1}) \in E(G)$. A graph G is connected if there is a path between any two vertices.

In this paper we will take (X,d,G) as a weighted directed graph G defined on the set X.

Definition 1. A directed graph is said to be transitive whenever $(u,v),(v,w) \in E(G)$ then this implies $(u,w) \in E(G)$.

A point $x \in X$ is said to be a fixed point of T whenever $T(x) = x$ and the set of the fixed points of T is denoted by $F(T)$.

Fixed point theory in Banach space has been developed richly, because it have linearity and convex structure. It is a great interest of researchers to develop results of a linear space (Banach Space) to a nonlinear space (metric space). But due to unavailability of convex structure in metric space it was looking impossible to develop the results of Banach space into metric space. To keep in mind this situation Reich and Shafrir [20] introduced a nonlinear space (hyperbolic metric space) by using geodesic segment and use Menger convexity [16] in this space. In 2005, Kohlenbach [12] introduced a generalization of the classical concept of hyperbolic metric space by using Takahashi [26] convex structure as follows:

Definition 2. Let (X,d) be a metric space, then (X,d,W) will be the hyperbolic metric space if the function $W : X \times X \times [0,1] \to X$ satisfying

(i) $d(z,W(x,y,\alpha)) \leq (1-\alpha)d(z,x)+\alpha d(z,y)$,
(ii) $d(W(x,y,\alpha),W(x,y,\beta)) = |\alpha-\beta| d(x,y)$,
(iii) $W(x,y,\alpha) = W(x,y,1-\alpha)$,
(iv) $d(W(x,y,\alpha),W(z,w,\alpha)) \leq (1-\alpha)d(x,z)+\alpha d(y,w)$

for all $x,y,z,w \in X$ and $\alpha,\beta \in [0,1]$.

If only condition (i) is satisfied then it will be convex metric space in sense of Takahashi [26]. For some results of fixed point theory on convex structure see [10].

Linear example of *hyperbolic metric space* is Banach space and nonlinear examples are Hadamard manifolds, the Hilbert open unit ball equipped with the hyperbolic metric and the CAT(0) spaces.

Here we present nearly asymptotically nonexpansive mappings endowed with graphs. Originally it was introduced by Sahu [22] and is a generalization of asymptotically nonexpansive mappings.

Definition 3. A mapping $T : (X,d,G) \to (X,d,G)$ is said to be G-nearly Lipschitzian with respect to a_n if for each $n \in \mathbb{N}$, there exist a constant $k_n \geq 0$ such that

$$d(T^n x, T^n y) \leq k_n(d(x,y)+a_n),$$

where $a_n \in [0,\infty)$ with $a_n \to 0$ and for every $x,y \in X$ such that $(x,y) \in E(G)$. The infimum of constants k_n for which the last inequality hold is denoted by $\eta(T^n)$ and called the nearly Lipschitz constant. The G-nearly Lipschitz mapping T with sequence $\{(a_n, \eta(T^n))\}$ is said to be G-nearly asymptotically nonexpansive if

(1) $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and
(2) $\lim\limits_{n\to\infty} \eta(T^n) = 1$.

If we take $a_n = 0$ for all $n \in \mathbb{N}$ then it will be G-asymptotically nonexpansive mapping and if we take $k_n = 1$, $a_n = 0$ for all $n \in \mathbb{N}$ then it will be G-nonexpansive mapping.

Lemma 1 ([3, 6]). *Let C be a closed nonempty subset of uniformly convex hyperbolic metric space (X,d). Let $\tau : C \to [0,\infty)$ be a type function, i.e., there exist bounded sequence $\{x_n\} \in X$ such that*

$$\tau(x) = \limsup_{n\to\infty} d(x_n, x),$$

for any $x \in C$. Then τ is continuous. Since X is hyperbolic, τ is convex, i.e., the subset $\{x \in C; \tau(x) \leq r\}$ is convex for any $r \geq 0$ and there exist a unique minimum point $z \in C$ such that

$$\tau(z) = \inf\{\tau(x); x \in C\}.$$

Moreover, if $\{z_n\}$ is a minimizing sequence in C, i.e., $\lim_{n\to\infty} \tau(z_n) = \tau_0$, then $\{z_n\}$ converges strongly to z.

Due to unavailability of addition and scalar multiplication in general metric space it was impossible to study weak convergence. So to tackle this situation in 1976, Lim [14] introduced the concept of $\Delta-$convergence in metric space. $\Delta-$convergence is an analogue of weak convergence.

Definition 4. Let X be a complete hyperbolic metric space and $\{x_n\}$ be a bounded sequence in X. Then the type function $r(.,\{x_n\}) : X \to [0,\infty)$ is defined by

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : \text{ for } x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined as

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}.$$

Definition 5. A bounded sequence $\{x_n\}$ in X is said to $\Delta-$converge to $x \in X$ if x is the unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$. We write $x_n \rightharpoonup x$ ($\{x_n\}$ $\Delta-$converges to x).

Lemma 2 ([13]). *Let (X,d) be a complete uniformly convex hyperbolic metric space and C be a nonempty, convex and closed subset X. Then, every bounded sequence $\{x_n\} \in X$ has a unique asymptotic center with respect to C.*

Proposition 1 ([1]). *Let (X,d) be complete hyperbolic metric space then (X,d,G) is said to have property $(*)$, if for each sequence $\{x_n\}$ in X $\Delta-$ converges to $x \in X$ and $(x_n,x_{n+1}) \in E(G)$, then there is a subsequence $\{x_{n_k}\}$ with $(x_{n_k},x) \in E(G) \quad \forall n \in \mathbb{N}$. Note that if the triplet (X,d,G) has property $(*)$ and G is transitive, then we have the following property:*
$(**)$ For any $\{x_n\}$ in X, if x_n $\Delta-$converges to x and $(x_n,x_{n+1}) \in E(G)$ then $(x_n,x) \in E(G)$ for $n \geq 1$.

Lemma 3 ([9]). *Let X be a uniformly convex hyperbolic space. Let $R \in [0,\infty)$ be such that*

$$\limsup_{n\to\infty} d(x_n,a) \le R,\ \limsup_{n\to\infty} d(y_n,a) \le R \text{ and } \lim_{n\to\infty} d(a,\alpha_n x_n \oplus (1-\alpha_n)y_n) = R$$

where $\alpha_n \in [a,b]$, with $0 < a \le b < 1$. Then we have, $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Lemma 4 ([18]). *Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be three sequences of nonnegative numbers such that*

$$y_n \ge 1 \text{ and } x_{n+1} \le y_n x_n + z_n \text{ for all } n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty}(y_n - 1) < \infty$ and $\sum_{n=1}^{\infty} z_n < \infty$, then $\lim_{n\to\infty} x_n$ exists.

$M-$iteration process in hyperbolic metric space, as follows:

$$\begin{cases} x_0 \in C, \\ z_n = W(x_n, T^n x_n, \alpha_n), \\ y_n = T^n z_n, \\ x_{n+1} = T^n y_n, \forall n \in \mathbb{N}. \end{cases} \tag{2.1}$$

Through the paper we will assume that G-intervals are closed and convex. We know that a G-interval is any of the subsets

$$[a,\to) = \{x \in V(G); (a,x) \in E(G)\} \quad \text{and} \quad (\leftarrow,b] = \{x \in V(G); (x,b) \in E(G)\},$$

for every $a,b \in V(G)$.

3. Existence theorem

In this section we prove an existence theorem for a mapping involving graph.

Theorem 1. *Let (X,d,G) be a complete uniformly convex hyperbolic metric space endowed with directed graph G. Assume that G is convex and transitive and C be a nonempty, closed and convex subset of X which contain more than one point. If $T: C \to C$ is a continuous $G-$nearly asymptotically nonexpansive mapping and there exist $x_0 \in C$ such that $(x_0, Tx_0) \in E(G)$. Then, T has a fixed point.*

Proof. Since $(x_0, T(x_0)) \in E(G)$, then by edge preserving of T, $(T^n x_0, T^{n+1}x_0) \in E(G)$ for all $n \in \mathbb{N}$. Since X is a complete uniformly convex hyperbolic metric space, then by property (R),

$$C_\infty = \bigcap_{n\ge 0} [T^n x_0, \to) \cap C = \bigcap_{n\ge 0} \{x \in C; (T^n x_0, x) \in E(G)\} \ne \varnothing.$$

Choose $x \in C_\infty$, then $(T^n(x_0), x) \in E(G)$. Again by using edge preserving of T, $(T^{n+1}(x_0), Tx) \in E(G)$. Thus, by transitivity of $E(G)$, $(T^n(x_0), Tx) \in E(G)$, i.e., $T(C_\infty) \subset C_\infty$. Consider the type function $\tau : C_\infty \to [0,\infty)$ generated by $\{T^n(x_0)\}$, that is, $\tau(x) = \limsup_{n\to\infty} d(T^n x_0, x)$. By using Lemma 1, there exist a unique minimum point $z \in C_\infty$ that is $\tau(z) = \inf\{\tau(x) : x \in C_\infty\}$. Since $z \in C_\infty$, then we have $T^p(z) \in C_\infty$ for all $p \in \mathbb{N}$ which implies

$$\tau(T^p(z)) = \limsup_{n\to\infty} d(T^n x_0, T^p(z)) \leq \eta(T^p) \limsup_{n\to\infty} d(T^n x_0, z) + \eta(T^p)a_p$$
$$= \eta(T^p)\tau(z) + \eta(T^p)a_p.$$

Since, $\tau(z)$ is minimum then, $\tau(z) \leq \tau(T^p(z)) \leq \eta(T^p)\tau(z) + \eta(T^p)a_p$ for all $p \in \mathbb{N}$. By using the condition of Definition 2, $\lim_{p\to\infty} \tau(T^p(z)) = \tau(z)$. Also, $\tau(\lim_{p\to\infty} T^p(z)) = \tau(z)$. By using continuity of T, $\tau(T \lim_{p\to\infty} T^{p-1}(z)) = \tau(z)$, that is $\tau(T(z)) = \tau(z)$. By the uniqueness of the minimum point $z = T(z)$, we get that z is a fixed point of T. □

The following corollary can be directly obtained from the above result.

Corollary 1. *Let (X,d,G) be a uniformly convex Banach space endowed with directed graph G. Assume that G is convex and transitive and C be a nonempty, closed and convex subset of X which contain more than one point. If $T : C \to C$ is a continuous $G-$nearly asymptotically nonexpansive mapping and there exist $x_0 \in C$ such that $(x_0, Tx_0) \in E(G)$. Then, T has a fixed point.*

4. Δ-CONVERGENCE AND STRONG CONVERGENCE THEOREM

In this section we are going to prove convergence results. So let's start the section with the following proposition.

Proposition 2. *Let $p \in F(T)$ be such that (x_0, p) and (p, x_0) are in $E(G)$ and $\{x_n\}$ is a sequence defined by (2.1). Then $(x_n, p), (p, x_n), (y_n, p)$, $(p, y_n), (z_n, p), (z_n, p)$ and (x_n, x_{n+1}) are in $E(G)$.*

Proof. Given that $(x_0, p) \in E(G)$. Then, by using convexity of $E(G)$, $(z_0, p) \in E(G)$. Also, $(y_0, p) \in E(G)$. Since T is edge preserving, we have $(Ty_0, p) \in E(G)$ that is $(x_1, p) \in E(G)$. Again, by edge preserving of $E(G)$, $(Tx_1, p) \in E(G)$. Again by using convexity of $E(G)$ on (x_1, p) and (Tx_1, p), we get $(z_1, p) \in E(G)$. By using edge preserving of T, we get $(Tz_1, p) \in E(G)$, that is $(y_1, p) \in E(G)$. Now suppose $(x_k, p) \in E(G)$ for fix $k \in \mathbb{N}$. Then, by edge preserving of T, $(T^k x_k, p) \in E(G)$. By convexity of $E(G)$, $(z_k, p) \in E(G)$. Again, $(T^k z_k, p) \in E(G)$, that is $(y_k, p) \in E(G)$. By edge preserving of T, $(T^k y_k, p) \in E(G)$ that is, $(x_{k+1}, p) \in E(G)$. Thus, by edge preserving of $E(G)$, $(T^{k+1} x_{k+1}, p) \in E(G)$. Again by using convexity of $E(G)$ on (x_{k+1}, p) and $(T^{k+1} x_{k+1}, p)$ we get, $(z_{k+1}, p) \in E(G)$. Again, $(T^{k+1} z_{k+1}, p) \in E(G)$ we get $(y_{k+1}, p) \in E(G)$. Hence by induction, $(x_n, p), (y_n, p), (z_n, p)$ are in $E(G)$ for all $n \in \mathbb{N}$. Using the similar argument we can show that $(p, x_n), (p, y_n), (p, z_n)$ are in $E(G)$ for all $n \in \mathbb{N}$. By transitivity of G we get $(x_n, x_{n+1}) \in E(G)$. □

Theorem 2. *Let (X,d) be a complete uniformly convex hyperbolic metric space and suppose that (X,d,G) has property $(*)$. Assume that G is convex and transitive and C be a nonempty, closed and convex subset of X which contain more than* *one point. If $T : C \to C$ is a continuous $G-$nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ and $F(T) \neq \varnothing$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}(\eta(T^n)-1) < \infty$. If sequence $\{x_n\}$ is defined by (2.1) with (x_0, p), $(p, x_0) \in E(G)$ where $0 < a \leq \alpha_n, \beta_n \leq b < 1$, $p \in F(T)$ and $x_0 \in C$, then $\{x_n\}$ $\Delta-$ converges to a fixed point x^* of T.*

Proof. Let $p \in F(T)$. It follows form Proposition 1 that $(x_n, p), (y_n, p), (z_n, p)$ are in $E(G)$. Now,

$$\begin{aligned} d(x_{n+1}, p) &= d(T^n y_n, p) \\ &\leq \eta(T^n) d(y_n, p) + \eta(T^n) a_n \\ &\leq \eta(T^n) d(T^n z_n, p) + \eta(T^n) a_n \\ &\leq \eta(T^n)^2 d(z_n, p) + \eta(T^n)^2 a_n + \eta(T^n) a_n \\ &\leq \eta(T^n)^2 d(W(T^n x_n, x_n, \alpha_n), p) + \eta(T^n)^2 a_n + \eta(T^n) a_n \\ &\leq (\eta(T^n)^2 - \alpha_n \eta(T^n)^2 + \alpha_n \eta(T^n)^3) d(x_n, p) + \alpha_n \eta(T^n)^3 a_n \\ &\quad + \eta(T^n)^2 a_n + \eta(T^n) a_n \end{aligned}$$

for $n \in \mathbb{N}$. Also,

$$\sum_{n=1}^{\infty} (\eta(T^n)^2 - \alpha_n \eta(T^n)^2 + \alpha_n \eta(T^n)^3 - 1) = \sum_{n=1}^{\infty} (\eta(T^n) + 1 + \alpha_n \eta(T^n)^2)(\eta(T^n) - 1)$$

$$\leq \sup_{1 \leq n < \infty} (\eta(T^n) + 1 + \alpha_n \eta(T^n)^2) \sum_{n=1}^{\infty} ((\eta(T^n) - 1) < \infty$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} (a_n)(\alpha_n \eta(T^n)^3 + \eta(T^n)^2 + \eta(T^n)) &\leq \sup_{1 \leq n < \infty} (\alpha_n \eta(T^n)^3 + \eta(T^n)^2 + \eta(T^n)) \sum_{n=1}^{\infty} a_n \\ &< \infty. \end{aligned}$$

It follows from Lemma 3 that $\lim_{n \to \infty} d(x_n, p)$ exist.
Let $\lim_{n \to \infty} d(x_n, p) = R$. Then

$$\limsup_{n \to \infty} d(T^n x_n, p) \leq \limsup_{n \to \infty} [\eta(T^n) d(x_n, p) + \eta(T^n) a_n] = \limsup_{n \to \infty} d(x_n, p) = R.$$

Now,

$$\begin{aligned} d(z_n, p) &\leq d(W(T^n x_n, x_n, \alpha_n), p) \\ &\leq (1 - \alpha_n) d(x_n, p) + \alpha_n [\eta(T^n) d(x_n, p) + \eta(T^n) a_n] \\ &= (1 - \alpha_n + \alpha_n \eta(T^n)) d(x_n, p) + \alpha_n \eta(T^n) a_n. \end{aligned}$$

Thus, $\lim_{n \to \infty} d(z_n, p) \leq R$. Again,

$$\lim_{n \to \infty} d(x_{n+1}, p) = \lim_{n \to \infty} d(T^{2n} z_n, p)$$

$$\begin{aligned}
&\leq \lim_{n\to\infty} \eta(T^n) d(T^n z_n, p) + \eta(T^n) a_n \\
&\leq \lim_{n\to\infty} \eta(T^n)^2 d(z_n, p) + \eta(T^n)^2 a_n + \eta(T^n) a_n \\
&\leq \lim_{n\to\infty} d(z_n, p) \\
R &\leq \lim_{n\to\infty} d(z_n, p).
\end{aligned}$$

Hence

$$\lim_{n\to\infty} d(z_n, p) = R.$$

By using Lemma 3,

$$\lim_{n\to\infty} d(T^n x_n, x_n) = 0.$$

From Lemma 1, $\{x_n\}$ have unique asymptotic center. Let $A(x_n) = x^*$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ such that $A(u_n) = u$. Now, claim $x^* = u$.
On contrary suppose that $x^* \neq u$. Then,

$$\begin{aligned}
\limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x^*) \leq \limsup_{n\to\infty} d(x_n, x^*) \\
&< \limsup_{n\to\infty} d(x_n, u) = \limsup_{n\to\infty} d(u_n, u),
\end{aligned}$$

which is a contradiction and hence $\Delta - \lim_{n\to\infty} x_n = x^*$.
Now, we claim that $x^* \in F(T)$. Since by Property $(**)$, $(x^*, x_n) \in E(G)$. Then,

$$\begin{aligned}
\limsup_{n\to\infty} d(T^n x^*, x_n) &\leq \limsup_{n\to\infty} d(T^n x^*, T^n x_n) + \limsup_{n\to\infty} d(T^n x_n, x_n) \\
&\leq \limsup_{n\to\infty} [\eta(T^n) d(x^*, x_n) + \eta(T^n) a_n] + \limsup_{n\to\infty} d(T^n x_n, x_n) \\
&\leq \limsup_{n\to\infty} d(x^*, x_n).
\end{aligned}$$

Since $\Delta - \lim_{n\to\infty} x_n = x^*$, $\limsup_{n\to\infty} d(x^*, x_n) < \limsup_{n\to\infty} d(T^n x^*, x_n)$. Thus, we have $T^n x^* = x^*$, which completes the proof. □

The above nonlinear space result can be written as corollary in a linear space as follows:

Corollary 2. *Let (X,d) be a uniformly convex Banach space with Opial's property and suppose that (X,d,G) has property $(*)$. Assume that G is convex and transitive and C be a nonempty, closed and convex subset of X which contain more than one point. If $T: C \to C$ is a continuous $G-$nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ and $F(T) \neq \varnothing$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} (\eta(T^n) - 1) < \infty$. If sequence $\{x_n\}$ is defined by (2.1) with (x_0, p), $(p, x_0) \in E(G)$ where $0 < a \leq \alpha_n, \beta_n \leq b < 1$, $p \in F(T)$ and $x_0 \in C$ then, $\{x_n\}$ weakly converges to a fixed point x^* of T.*

Theorem 3. *Let (X,d,G) be a complete uniformly convex hyperbolic metric space endowed with directed graph G. Assume that G is convex and transitive and C be a nonempty, closed and convex subset of X which contain more than one point. If $T : C \to C$ is a continuous $G-$nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ and $F(T) \neq \emptyset$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}(\eta(T^n)-1) < \infty$. If sequence $\{x_n\}$ is defined by (2.1) with (x_0, p), $(p, x_0) \in E(G)$ where $0 < a \leq \alpha_n, \beta_n \leq b < 1$, $p \in F(T)$ and $x_0 \in C$, then, $\{x_n\}$ converges strongly to a fixed point x^* of T if and only if $\liminf\limits_{n\to\infty} d(x_n, F(T)) = 0$.*

Proof. It is easy to see that if $\{x_n\}$ converges to a point $x^* \in F(T)$ then

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

For converse part, suppose that $\liminf\limits_{n\to\infty} d(x_n, F(T)) = 0$. From the proof of Theorem 1, $\lim\limits_{n\to\infty} d(x_n, x^*)$ exist. But as it is given in the hypothesis that $\liminf\limits_{n\to\infty} d(x_n, F(T)) = 0$, therefore $\lim\limits_{n\to\infty} d(x_n, F(T)) = 0$.
Thus, for a given $\varepsilon > 0$ there exist a $K(\varepsilon) \in \mathbb{N}$ such that

$$d(x_n, F(T)) < \frac{\varepsilon}{2} \quad \text{whenever} \quad \mathrm{n} > \mathrm{K}(\varepsilon).$$

Particularly, $\inf\{d(x_K, x^*) : x^* \in F(T)\} < \frac{\varepsilon}{2}$. So there exist $x^* \in F(T)$ such that $d(x_K, x^*) < \frac{\varepsilon}{2}$. Now, for $n, m > K(\varepsilon)$

$$d(x_n, x_m) \leq d(x_n, x^*) + d(x^*, x_m) < \varepsilon.$$

Hence, x_n is a Cauchy sequence in C. Since C is a closed subset of X, then $\lim\limits_{n\to\infty} x_n = x^* \in C$. □

Corollary 3. *Let (X,d,G) be a uniformly convex Banach space endowed with directed graph G. Assume that G is convex and transitive and C be a nonempty, closed and convex subset of X which contain more than one point. If $T : C \to C$ is a continuous $G-$nearly asymptotically nonexpansive mapping with sequence $\{(a_n, \eta(T^n))\}$ and $F(T) \neq \emptyset$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}(\eta(T^n)-1) < \infty$. If sequence $\{x_n\}$ is defined by (2.1) with (x_0, p), $(p, x_0) \in E(G)$ where $0 < a \leq \alpha_n, \beta_n \leq b < 1$, $p \in F(T)$ and $x_0 \in C$, then, $\{x_n\}$ converges strongly to a fixed point x^* of T if and only if $\liminf\limits_{n\to\infty} ||x_n - F(T)|| = 0$.*

5. Example

In this section, we construct an example of a G-nearly asymptotically nonexpansive mapping which is not a nearly asymptotically nonexpansive mapping. Also, domain of our example is a hyperbolic metric space which is not in Banach space. The motivation is essentially taken from [24].

Example 1. Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 > 0\}$. Define $d : X \times X \to [0, \infty)$ by

$$d(x,y) = |x_1 - y_1| + |x_1x_2 - y_1y_2|$$

for all $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in X. Now for $\alpha \in [0,1]$, define a function $W : X \times X \times [0,1] \to X$ by

$$W(x,y,\alpha) = \Big(\alpha x_1 + (1-\alpha)y_1, \frac{\alpha x_1 x_2 + (1-\alpha)y_1 y_2}{\alpha x_1 + (1-\alpha)y_1}\Big).$$

Then we can easily verify that (X, d, W) is a hyperbolic metric space.
Now, define the graph G on X by

$$(x,y) \in E(G) \Leftrightarrow x_1 + x_2 = y_1 + y_2.$$

Define the mapping $T : [\frac{1}{2}, 1] \times [\frac{1}{2}, 1] \to [\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$ by

$$T(x_1, x_2) = (1 - x_1, 1 - x_2).$$

This gives

$$T^{2n}(x_1, x_2) = (x_1, x_2)$$

and

$$T^{2n+1}(x_1, x_2) = (1 - x_1, 1 - x_2)$$

for all $n \in \mathbb{N}$. This mapping is G-nearly asymptotically nonexpansive mapping but not nearly asymptotically nonexpansive mapping because

$$d(T^{2n+1}x, T^{2n+1}y) \leq d(x,y) + |x_1 + x_2 - y_1 - y_2|.$$

6. Conclusion

We are concluding our paper with the following pertinent observations:

(i) Our Theorem 1 generalize Theorem 3.3 of Alfuraidan and Shukri [2], Theorem Alfuraidan and Khamsi [3] and Theorem 3.1 of Dehaish and Khamsi [6].
(ii) Our convergence results extend the domain of convergence results of Ullah and Arshad [29].
(iii) We also furnish an example of a G-nearly asymptotically nonexpansive in hyperbolic metric space with graph.

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