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INTEGRATIONS ON LATTICES

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Abstract. In this paper, we introduce the notion of integration with respect to a given derivation on a lattice. More precisely, we give the definitions of integrable elements of a lattice and their integral sets. We investigate several characterizations and properties of integrations on a lattice. Also, we give a lattice structure to the family of integral sets with respect to a given integration. Further, we provide a representation theorem for the lattice of fixed points of an isotone derivation based on the family of integral sets. As an application of this notion of integration, we use the integrable elements of a Boolean lattice to determine the necessary and sufficient conditions under which a linear differential equation on this Boolean lattice has a solution.

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1. INTRODUCTION

In 1957, the author E.C. Posner introduced the notion of derivations in prime rings [9]. Later on, this notion had many applications (see, e.g. [2]). Szász [10] has extended this notion of derivation to the setting of lattice structures. He has defined a derivation on a given lattice L as a function d satisfying the following two conditions:

$$d(x\wedge y)=(d(x)\wedge y)\vee(x\wedge d(y)) \quad \text{and} \quad d(x\vee y)=d(x)\vee d(y) \quad \text{for any } x,y\in L.$$

Ferrari [4] has investigated some properties of this notion and provided some interesting examples in particular classes of lattices. Xin et al. [14] have ameliorated this notion of derivation on a lattice by considering only the first condition, and they have shown that the second condition obviously holds for the isotone derivations on a distributive lattice. In the same paper, they have characterized distributive and modular lattices in terms of their isotone derivations. Later on, Xin [13] has focused his attention on the structure of the set of the fixed points of a derivation on a lattice, and has shown some relationships between this set and the notion of an ideal of a lattice.

This notion of derivation on a lattice is witnessing increased attention. It was applied in partially ordered sets [1, 17], in distributive lattices [16], in semilattices

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[15], in bounded hyperlattices [11], in quantales and residuated lattices [5, 12] and in several kinds of algebra [6–8].

Inspired by the notion of integrations on ring structures introduced by Banič [3], we extend this notion of integration to any lattice structure. More precisely, we introduce the notions of an integrable element and its integral set with respect to a given derivation on a lattice. We investigate several characterizations and properties of integrations on lattices. Moreover, we pay particular attention to the lattice structure of the family of integral sets with respect to a given integration on a lattice. We provide a representation theorem for the lattice of fixed points of an isotone derivation based on the family of integral sets. As an application, we use integrations on a lattice to determine the necessary and sufficient conditions under which a linear differential equation on an arbitrary Boolean lattice has a solution.

The rest of the paper is organized as follows. In Section 2, we recall some necessary concepts and properties of derivations on lattices. In Section 3, we introduce the notion of integration on a lattice and investigate several characterizations and properties for this concept. In Section 4, we give a lattice structure to the family of integral sets with respect to a given integration on a lattice. Moreover, we provide a representation theorem for the lattice of fixed points of an isotone derivation based on the family of integral sets. In Section 5, we give an application of integrations on lattices. Finally, we present some conclusions and discuss a future research in Section 6.

2. DERIVATIONS ON LATTICES

In this section, we recall some basic concepts and properties of derivations on lattices that will be needed throughout this paper.

Definition 1 ([14, Definition 3.1])**.** Let $(L,\wedge,\vee)$ be a lattice and $\leqslant$ be its order relation. A function $d\colon L\to L$ is called a *derivation* on L if it satisfies the following condition:

$$d(x\wedge y)=(d(x)\wedge y)\vee(x\wedge d(y))\quad \text{for any } x,y\in L.$$

In the rest of the paper, we shortly write dx instead of $d(x)$.

Definition 2 ([14, Definition 3.7])**.** Let $(L,\wedge,\vee)$ be a lattice and d be a derivation on L. d is called *isotone* if it satisfies the following condition:

$$x\leqslant y \text{ implies } dx\leqslant dy \qquad \text{for any } x,y\in L.$$

Example 1 ([14, Example 3.8]). Let $(L,\wedge,\vee)$ be a lattice, α be an element of L and $d_\alpha\colon L\to L$ be a function defined as $d_\alpha(x)=\alpha\wedge x$ for any $x\in L$. The function d_α is called a *principal derivation* on L.

As a remark, any principal derivation d_α on L is isotone.

Remark 1 ([14, Remark 2.]). Let $(L,\wedge,\vee)$ be a lattice and d be a derivation on L. If $Fix_d(L)=\{x\in L\mid dx=x\}$ the set of fixed points of d is non-empty, then it is a down-set. Moreover, if d is isotone, then $Fix_d(L)$ is an ideal of L.

The following proposition gives some properties of derivations on a lattice.

Proposition 1 ([14, Proposition 3.6])**.** *Let $(L, \wedge, \vee)$ be a lattice and d be a derivation on L. Then it holds that*

(i) $dx \leqslant x$ *for any* $x \in L$*;*
(ii) $d(dx) = dx$ *for any* $x \in L$*;*
(iii) *if* L *has a least element* $0 \in L$*, then* $d0 = 0$*;*
(iv) d *is isotone if and only if* $d(x \wedge y) = dx \wedge dy$ *for any* $x, y \in L$*;*
(v) *if* L *is distributive and* d *is isotone, then* $d(x \vee y) = dx \vee dy$ *for any* $x, y \in L$*.*

For more details concerning derivations on lattices, we refer to [13, 14].

3. INTEGRATIONS ON A LATTICE

The notion of integrations on a ring was introduced by Banič [3]. Inspired by this notion, we introduce the notion of integration with respect to a given derivation on a lattice. Also, we investigate several characterizations and properties of integrations on a lattice.

3.1. *Definitions and examples*

Definition 3. Let $(L, \wedge, \vee)$ be a lattice and d be a derivation on L. Let $i_d : L \to \mathcal{P}(L)$ be a function defined as $i_d(x) = d^{-1}(x) = \{z \in L \mid dz = x\}$ for any $x \in L$. The function i_d is called the integration with respect to d (the *d-integration*, for short) on L. The set $i_d(x)$ is called the integral set of x with respect to d (the *d-integral set* of x, for short).

Definition 4. Let $(L, \wedge, \vee)$ be a lattice and i_d be a d-integration on L. An element x of L is called an integrable element with respect to i_d (a d-integrable element, for short) if $i_d(x) \neq \varnothing$.

Remark 2. In any lattice $(L, \wedge, \vee, 0)$ with 0, the least element 0 is an integrable element with respect to any d-integration on L. Indeed, let i_d be a d-integration on L. Since $d0 = 0$ (see Proposition 1 (*iii*)) we conclude that $i_d(0) \neq \varnothing$. Hence 0 is a d-integrable element of L.

In the following, we give two illustrative examples of integrations on lattices.

Example 2. Let $(D(12), gcd, lcm)$ be the lattice of positive divisors of 12 ordered by the divisibility order relation $\mid$ and given by the Hasse diagram in Figure 1. Let d be a principal derivation on $D(12)$ defined as $dx = gcd(6, x)$ for any $x \in D(12)$. The d-integration function i_d on $D(12)$ is defined in the following table:

x	1	2	3	4	6	12
dx	1	2	3	2	6	6
$i_d(x)$	$\{1\}$	$\{2,4\}$	$\{3\}$	$\varnothing$	$\{6,12\}$	$\varnothing$

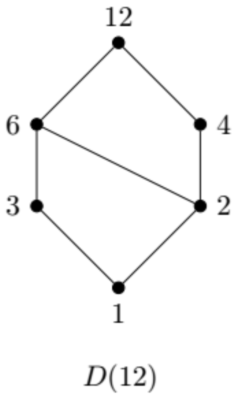


FIGURE 1. The Hasse diagram of the lattice $(D(12), gcd, lcm)$.

Example 3. Let $(\mathbb{N}^*, gcd, lcm)$ be the lattice of positive integers ordered by the divisibility order relation $|$ and d be a derivation on $\mathbb{N}^*$ defined as $dx = gcd(2,x)$ for any $x \in \mathbb{N}^*$. We denote by $\mathbb{P}$ the set of all positive even numbers and by $\mathbb{I}$ the set of all positive odd numbers. One can easily verify that

$$dx = \begin{cases} 1 & \text{if } x \text{ is odd;} \\ 2 & \text{if } x \text{ is even}\,. \end{cases}$$

The d-integration mapping i_d on $\mathbb{N}^*$ is defined as:

$$i_d(x) = \begin{cases} \mathbb{I} & \text{if } x = 1; \\ \mathbb{P} & \text{if } x = 2; \\ \varnothing & \text{otherwise}\,. \end{cases}$$

3.2. *Properties of the integrable elements of a lattice with respect to an integration*

In this subsection, we investigate several properties concerning the integrable elements of a lattice and their integral sets. First, we show the following two characterizations of an integrable element of a lattice.

Lemma 1. *Let $(L, \wedge, \vee)$ be a lattice and i_d be a d-integration on L. Then an element x of L is a d-integrable element if and only if $x \in Fix_d(L)$.*

Proof. To prove the direct implication, suppose that x is a d-integrable element of L. Then there exists $z \in L$ such that $dz = x$. Applying Proposition 1 (ii), we obtain $dx = d(dz) = dz = x$. Thus, x is a fixed point of d, i.e., $x \in Fix_d(L)$. The converse implication is immediate. ☐

The following propositions show several properties of integrable elements of a lattice.

Proposition 2. *Let $(L, \wedge, \vee)$ be a lattice, i_d be a d-integration on L and x be a d-integrable element of L. Then it holds that*

(i) x is the least element of $i_d(x)$;
(ii) if $y \leqslant x$, then y is also a d-integrable element of L for any $y \in L$.

Proof. Suppose that x is a d-integrable element of L.

(i) On the one hand, Lemma 1 guarantees that $x \in i_d(x)$. On the other hand, let $z \in i_d(x)$, then $dz = x$. Since d is a derivation on L, it holds from Proposition 1 (i) that $dz \leqslant z$. Hence, $x \leqslant z$. Thus, x is the least element of $i_d(x)$.
(ii) Using Lemma 1 guarantees that $x \in Fix_d(L)$. Let $y \in L$ such that $y \leqslant x$. The fact that $Fix_d(L)$ is a down-set (see Remark 1) implies $y \in Fix_d(L)$. Therefore, y is also a d-integrable element of L.

□

Here, we mention that if x is a d-integrable element of L, then it holds from Proposition 2 (ii) that $m \wedge x$ is also a d-integrable element of L for any $m \in L$.

Proposition 3. *Let $(L, \wedge, \vee)$ be a lattice and i_{d_1}, i_{d_2} be two integrations on L such that $d_1 \leq d_2$ (i.e., $d_1(x) \leqslant d_2(x)$ for any $x \in L$). If x is a d_1-integrable element of L, then x is also a d_2-integrable element.*

Proof. Let x be a d_1-integrable element of L. Then from Lemma 1, we have x is a fixed point of d_1. On the one hand, the fact that $d_1 \leq d_2$ implies $x = d_1(x) \leqslant d_2(x)$. Hence, $x \leqslant d_2(x)$. On the other hand, since d_2 is a derivation on L, we obtain from Proposition 1 (i) that $d_2(x) \leqslant x$. Thus, $d_2(x) = x$, i.e., x is also a fixed point of d_2. Therefore, Lemma 1 guarantees that x is also a d_2-integrable element. □

Proposition 4. *Let $(L, \wedge, \vee)$ be a lattice and i_d be a d-integration on L such that d is isotone. The following implications hold:*

(i) if $y_1 \in i_d(x_1)$ and $y_2 \in i_d(x_2)$, then $y_1 \wedge y_2 \in i_d(x_1 \wedge x_2)$ for any $x_1, x_2, y_1, y_2 \in L$;
(ii) if $y_1 \in i_d(x_1)$, $y_2 \in i_d(x_2)$ and L is distributive, then $y_1 \vee y_2 \in i_d(x_1 \vee x_2)$ for any $x_1, x_2, y_1, y_2 \in L$.

Proof. Let $x_1, x_2, y_1, y_2 \in L$ such that $y_1 \in i_d(x_1)$ and $y_2 \in i_d(x_2)$, then $dy_1 = x_1$ and $dy_2 = x_2$.

(i) The fact that d is an isotone derivation on L implies from Proposition 1 (iv) that $d(y_1 \wedge y_2) = dy_1 \wedge dy_2 = x_1 \wedge x_2$. Thus, $y_1 \wedge y_2 \in i_d(x_1 \wedge x_2)$.
(ii) Since L is distributive and d is an isotone derivation on L, we conclude from Proposition 1 (v) that $d(y_1 \vee y_2) = dy_1 \vee dy_2 = x_1 \vee x_2$. Therefore, $y_1 \vee y_2 \in i_d(x_1 \vee x_2)$.

□

Next, we give relationships between the integral sets on a lattice with zero.

Proposition 5. *Let $(L,\wedge,\vee,0)$ be a lattice with a least element 0 and i_d be a d-integration on L. Let x be a d-integrable element of L. Then the following statements hold:*

(i) $m\wedge i_d(x)\subseteq i_d(m\wedge x)\subseteq i_d(m\wedge dx)$ *for any* $m\in i_d(0)$*, where* $m\wedge i_d(x)=\{m\wedge z\mid z\in i_d(x)\}$*;*

(ii) if L is distributive and d is isotone, then $m\vee i_d(x)\subseteq i_d(x)$ *for any* $m\in i_d(0)$*, where* $m\vee i_d(x)=\{m\vee z\mid z\in i_d(x)\}$*.*

Proof. Let x be a d-integrable element of L and $m\in i_d(0)$. We know that $m\wedge x$ is also a d-integrable elements of L. Hence, $i_d(m\wedge x)\neq\varnothing$.

(i) Let $y\in m\wedge i_d(x)$, then $y=m\wedge z$ with $dz=x$. Since d is a derivation on L and $dm=0$, it follows that

$$dy=d(m\wedge z)=(dm\wedge z)\vee(m\wedge dz)=m\wedge dz=m\wedge x.$$

Hence, $y\in i_d(m\wedge x)$. Thus, $m\wedge i_d(x)\subseteq i_d(m\wedge x)$. Next, let $t\in i_d(m\wedge x)$, then $dt=m\wedge x$. Using the fact that d is a derivation on L and Proposition 1 (ii) we obtain

$$dt=d(dt)=d(m\wedge x)=(dm\wedge x)\vee(m\wedge dx)=m\wedge dx.$$

Hence, $t\in i_d(m\wedge dx)$. Therefore, $i_d(m\wedge x)\subseteq i_d(m\wedge dx)$. Consequently,

$$m\wedge i_d(x)\subseteq i_d(m\wedge x)\subseteq i_d(m\wedge dx).$$

(ii) Let $y\in m\vee i_d(x)$, then $y=m\vee z$ with $dz=x$. Since L is distributive and d is isotone derivation, it follows from Proposition 1 (v) that $dy=d(m\vee z)=dm\vee dz=dz=x$. Hence, $y\in i_d(x)$. Therefore, $m\vee i_d(x)\subseteq i_d(x)$

□

4. A LATTICE STRUCTURE OF INTEGRAL SETS ON A LATTICE

In this section, we give a lattice structure to the family of integral sets with respect to a given integration on a lattice. Also, we provide a representation theorem for the lattice of fixed points of an isotone derivation based on the family of integral sets.

Notation 1. Let $(L,\wedge,\vee)$ be a lattice and i_d be an integration on L. We denote by

(i) $\mathbb{I}_d(L):=\{i_d(x)\mid x\in L\}$ the family of d-integral sets of L;

(ii) $I_d(L):=\{i_d(x)\mid x\in Fix_d(L)\}$ the family of d-integral sets of the d-integrable elements of L.

In the following theorem, we give a poset structure to the family of integral sets of a lattice with respect to a given integration.

Theorem 1. *Let $(L,\wedge,\vee)$ be a lattice and i_d be an integration on L with d is isotone. Let $\sqsubseteq$ be a binary relation on $\mathbb{I}_d(L)$ defined for any $i_d(x),i_d(y)\in\mathbb{I}_d(L)$ as:*

$$i_d(x)\sqsubseteq i_d(y)\quad \text{if and only if}\quad i_d(x)=\varnothing \text{ or } \exists t\in i_d(y) \text{ such that } x\leqslant t.$$

Then the structure $(\mathbb{I}_d(L),\sqsubseteq)$ is a partially ordered set.

Proof. Let $i_d(x) \in \mathbb{I}_d(L)$, if $i_d(x) = \varnothing$, then $i_d(x) \sqsubseteq i_d(x)$. If $i_d(x) \neq \varnothing$, this means x is a d-integrable element of L. Lemma 1 guarantees $x \in i_d(x)$. Since $x \in i_d(x)$ and $x \leqslant x$, we conclude $i_d(x) \sqsubseteq i_d(x)$. Thus, $\sqsubseteq$ is reflexive. Next, let $i_d(x), i_d(y) \in \mathbb{I}_d(L)$ with $i_d(x) \sqsubseteq i_d(y)$ and $i_d(y) \sqsubseteq i_d(x)$. Then, $[i_d(x) = \varnothing$ or $(\exists t_1 \in i_d(y)$ such that $x \leqslant t_1)]$ and $[i_d(y) = \varnothing$ or $(\exists t_2 \in i_d(x)$ such that $y \leqslant t_2)]$. Here, we discuss the following four possible cases:

(i) if $i_d(x) = \varnothing$ and $i_d(y) = \varnothing$, then $i_d(x) = i_d(y)$;
(ii) if $i_d(x) = \varnothing$ and $\exists t_2 \in i_d(x)$ such that $y \leqslant t_2$, this is an impossible case;
(iii) if $i_d(y) = \varnothing$ and $\exists t_1 \in i_d(y)$ such that $x \leqslant t_1$, this is also an impossible case;
(iv) if $\exists t_1 \in i_d(y)$ such that $x \leqslant t_1$ and $\exists t_2 \in i_d(x)$ such that $y \leqslant t_2$, then x and y are d-integrable elements of L. Lemma 1 assures $x, y \in Fix_d(L)$, i.e., $dx = x$ and $dy = y$. Since $t_1 \in i_d(y)$ and $t_2 \in i_d(x)$, we get $dt_1 = y$ and $dt_2 = x$. Using the hypotheses of d is an isotone derivation on L, we obtain $x = dx \leqslant dt_1 = y$ and $y = dy \leqslant dt_2 = x$. Thus, $x = y$, i.e., $i_d(x) = i_d(y)$.

Therefore, $\sqsubseteq$ is antisymmetric. Now, let $i_d(x), i_d(y), i_d(z) \in \mathbb{I}_d(L)$ with $i_d(x) \sqsubseteq i_d(y)$ and $i_d(y) \sqsubseteq i_d(z)$. Then $[i_d(x) = \varnothing$ or $(\exists t_1 \in i_d(y)$ such that $x \leqslant t_1)]$ and $[i_d(y) = \varnothing$ or $(\exists t_2 \in i_d(z)$ such that $y \leqslant t_2)]$. Here, we have the following three possible cases:

(i) if $i_d(x) = \varnothing$, then $i_d(x) \sqsubseteq i_d(z)$;
(ii) if $\exists t_1 \in i_d(y)$ such that $x \leqslant t_1$ and $i_d(y) = \varnothing$, this is an impossible case;
(iii) if $\exists t_1 \in i_d(y)$ such that $x \leqslant t_1$ and $\exists t_2 \in i_d(z)$ such that $y \leqslant t_2$, then $dt_1 = y$ and $dt_2 = z$. Since, $i_d(x) \neq \varnothing$, $i_d(y) \neq \varnothing$ and $i_d(z) \neq \varnothing$, it holds from Lemma 1 that $dx = x$, $dy = y$ and $dz = z$. The fact that d is an isotone derivation on L gives $x = dx \leqslant dt_1 = y$ and $y = dy \leqslant dt_2 = z$. Hence, $x \leqslant z$ and $z \in i_d(z)$. Thus, $i_d(x) \sqsubseteq i_d(z)$.

Therefore, $\sqsubseteq$ is transitive. Consequently, $\sqsubseteq$ is an order relation on $\mathbb{I}_d(L)$, i.e., the structure $(\mathbb{I}_d(L), \sqsubseteq)$ is a partially ordered set. □

In the following proposition, we present the least element of $(\mathbb{I}_d(L), \sqsubseteq)$

Proposition 6. *Let $(L, \wedge, \vee)$ be a lattice and d be an isotone derivation on L diffirent from the identity function of L. Then the empty-set $\varnothing$ is the least element of $\mathbb{I}_d(L)$.*

Proof. The fact that d is not the identity function implies that there exists $x \in L$ where x is not a fixed point of d. Lemma 1 guarantees that x is not a d-integrable element of L. Hence, $i_d(x) = \varnothing$, i.e., $\varnothing \in \mathbb{I}_d(L)$. Since $\varnothing \sqsubseteq i_d(y)$ for any $i_d(y) \in \mathbb{I}_d(L)$, it holds that $\varnothing$ is the least element of $\mathbb{I}_d(L)$. □

The following lemma is the key to prove the next main results of this section.

Lemma 2. *Let $(L, \wedge, \vee)$ be a lattice and d be an isotone derivation on L. Then $i_d(x) \sqsubseteq i_d(y)$ if and only if $x \leqslant y$ for any $x, y \in Fix_d(L)$.*

Proof. Let $x, y \in Fix_d(L)$, then from Lemma 1 we have x and y are two d-integrable elements of L, i.e., $i_d(x) \neq \varnothing$ and $i_d(y) \neq \varnothing$. Firstly, suppose that $i_d(x) \sqsubseteq i_d(y)$, then there exists $t \in i_d(y)$ such that $x \leqslant t$. Since $x \in Fix_d(L)$ and d is an isotone derivation on L, it holds that $x = dx \leqslant dt = y$. Thus, $x \leqslant y$. Conversely, suppose that $x \leqslant y$. Using Lemma 1 gives that $y \in i_d(y)$. Thus, there exists $t = y$ such that $t \in i_d(y)$ and $x \leqslant t$. Therefore, $i_d(x) \sqsubseteq i_d(y)$. □

In the following theorem, we give a lattice structure to the family of integral sets of a lattice.

Theorem 2. *Let $(L, \wedge, \vee)$ be a lattice and d be an isotone derivation on L. Define on $\mathbb{I}_d(L)$ two binary operations for any $i_d(x), i_d(y) \in \mathbb{I}_d(L)$ as:*

$$i_d(x) \sqcap i_d(y) = \begin{cases} i_d(x \wedge y) & \text{if } x, y \in Fix_d(L), \\ \varnothing & \text{otherwise}. \end{cases}$$

and

$$i_d(x) \sqcup i_d(y) = \begin{cases} i_d(x \vee y) & \text{if } x, y \in Fix_d(L), \\ i_d(y) & \text{if } x \notin Fix_d(L), \\ i_d(x) & \text{if } y \notin Fix_d(L). \end{cases}$$

Then the structure $(\mathbb{I}_d(L), \sqcap, \sqcup)$ is a lattice with respect to the order relation $\sqsubseteq$.

Proof. Let $i_d(x), i_d(y) \in \mathbb{I}_d(L)$. Firstly, we are aiming to prove that $i_d(x) \sqcap i_d(y)$ is the greatest lower bound of $i_d(x)$ and $i_d(y)$. On the one hand, if $x \notin Fix_d(L)$ or $y \notin Fix_d(L)$, then $i_d(x) \sqcap i_d(y) = \varnothing$. Hence, $i_d(x) \sqcap i_d(y) \sqsubseteq i_d(x)$ and $i_d(x) \sqcap i_d(y) \sqsubseteq i_d(y)$. If $x, y \in Fix_d(L)$, then $i_d(x) \sqcap i_d(y) = i_d(x \wedge y)$. Remark 1 assures that $Fix_d(L)$ is a down-set, then $x \wedge y \in Fix_d(L)$. Since $x \wedge y \leqslant x$ and $x \wedge y \leqslant y$, it holds from Lemma 2 that $i_d(x) \sqcap i_d(y) = i_d(x \wedge y) \sqsubseteq i_d(x)$ and $i_d(x) \sqcap i_d(y) = i_d(x \wedge y) \sqsubseteq i_d(y)$. Thus, $i_d(x) \sqcap i_d(y)$ is a lower bound of $i_d(x)$ and $i_d(y)$. On the other hand, let $i_d(z) \in \mathbb{I}_d(L)$ be a lower bound of $i_d(x)$ and $i_d(y)$. If $i_d(z) = \varnothing$, then $i_d(z) \sqsubseteq i_d(x) \sqcap i_d(y)$. Otherwise, $i_d(z) \sqsubseteq i_d(x)$ and $i_d(z) \sqsubseteq i_d(y)$ such that $x, y, z \in Fix_d(L)$. So, Lemma 2 guarantees that $z \leqslant x$ and $z \leqslant y$. Hence, $z \leqslant x \wedge y$. Thus, $i_d(z) \sqsubseteq i_d(x \wedge y) = i_d(x) \sqcap i_d(y)$. Therefore, $i_d(x) \sqcap i_d(y)$ is the greatest lower bound of $i_d(x)$ and $i_d(y)$, i.e., the poset $(\mathbb{I}_d(L), \sqcap)$ is a *meet*-semilattice.

Secondly, we show that $i_d(x) \sqcup i_d(y)$ is the least upper bound of $i_d(x)$ and $i_d(y)$. On the one hand,

(i) if $x \notin Fix_d(L)$, then $i_d(x) = \varnothing$ and $i_d(x) \sqcup i_d(y) = i_d(y)$. Hence, $i_d(x) \sqsubseteq i_d(x) \sqcup i_d(y)$ and $i_d(y) \sqsubseteq i_d(x) \sqcup i_d(y)$;

(ii) if $y \notin Fix_d(L)$, then $i_d(y) = \varnothing$ and $i_d(x) \sqcup i_d(y) = i_d(x)$. Hence, $i_d(x) \sqsubseteq i_d(x) \sqcup i_d(y)$ and $i_d(y) \sqsubseteq i_d(x) \sqcup i_d(y)$;

(iii) if $x, y \in Fix_d(L)$, then $i_d(x) \sqcup i_d(y) = i_d(x \vee y)$. Since d is an isotone derivation on L, it holds from Remark 1 that $Fix_d(L)$ is an ideal of L, then $x \vee y \in Fix_d(L)$. The fact that $x \leqslant x \vee y$ and $y \leqslant x \vee y$ with $x, y, x \vee y \in Fix_d(L)$ imply from Lemma 2 that $i_d(x) \sqsubseteq i_d(x \vee y) = i_d(x) \sqcup i_d(y)$ and $i_d(y) \sqsubseteq i_d(x \vee y) = i_d(x) \sqcup i_d(y)$.

Therefore, $i_d(x) \sqcup i_d(y)$ is an upper bound of $i_d(x)$ and $i_d(y)$. On the other hand, let $i_d(t) \in \mathbb{I}_d(L)$ be an upper bound of $i_d(x)$ and $i_d(y)$. Two possible cases to consider.

(i) If $i_d(t) = \varnothing$, since $i_d(x) \sqsubseteq i_d(t)$ and $i_d(y) \sqsubseteq i_d(t)$ we obtain $i_d(x) = \varnothing$ and $i_d(y) = \varnothing$. Hence, $i_d(x) \sqcup i_d(y) = \varnothing \sqsubseteq i_d(t)$.

(ii) If $i_d(t) \neq \varnothing$, then $t \in Fix_d(L)$. So, if $x \notin Fix_d(L)$ or $y \notin Fix_d(L)$, then $i_d(x) \sqcup i_d(y) = i_d(y) \sqsubseteq i_d(t)$ or $i_d(x) \sqcup i_d(y) = i_d(x) \sqsubseteq i_d(t)$. If $x, y, t \in Fix_d(L)$ with $i_d(x) \sqsubseteq i_d(t)$ and $i_d(y) \sqsubseteq i_d(t)$, then Lemma 2 gives that $x \leqslant t$ and $y \leqslant t$. Hence, $x \vee y \leqslant t$ such that $t, x \vee y \in Fix_d(L)$. Thus, Lemma 2 guarantees that $i_d(x) \sqcup i_d(y) = i_d(x \vee y) \sqsubseteq i_d(t)$.

Therefore, $i_d(x) \sqcup i_d(y)$ is the least upper bound of $i_d(x)$ and $i_d(y)$, i.e., the poset $(\mathbb{I}_d(L), \sqcup)$ is a *join*-semilattice. Consequently, the structure $(\mathbb{I}_d(L), \sqcap, \sqcup)$ is a lattice. ☐

The following proposition shows that $I_d(L)$ is a sublattice of $\mathbb{I}_d(L)$.

Proposition 7. *Let $(L, \wedge, \vee)$ be a lattice and d be an isotone derivation on L. Then $(I_d(L), \sqcap, \sqcup)$ is a sublattice of $(\mathbb{I}_d(L), \sqcap, \sqcup)$, where $i_d(x) \sqcap i_d(y) = i_d(x \wedge y)$ and $i_d(x) \sqcup i_d(y) = i_d(x \vee y)$ for any $i_d(x), i_d(y) \in I_d(L)$.*

Proof. Let $i_d(x), i_d(y) \in I_d(L)$, then $x, y \in Fix_d(L)$. Since $Fix_d(L)$ is an ideal of L (see Remark 1), we have $x \wedge y \in Fix_d(L)$ and $x \vee y \in Fix_d(L)$. Then we conclude that $i_d(x) \sqcap i_d(y) = i_d(x \wedge y) \in I_d(L)$ and $i_d(x) \sqcup i_d(y) = i_d(x \vee y) \in I_d(L)$. Thus, $I_d(L)$ is closed under $(\sqcap)$ and $(\sqcup)$ operations. Therefore, the structure $(I_d(L), \sqcap, \sqcup)$ is a sublattice of $(\mathbb{I}_d(L), \sqcap, \sqcup)$. ☐

In the following theorem, we provide a representation theorem for the lattice of fixed points of an isotone derivation based on the family of integral sets of the integrable elements.

Theorem 3. *Let $(L, \wedge, \vee)$ be a lattice and d be an isotone derivation on L. Then $(I_d(L), \sqcap, \sqcup)$ is isomorphic to $(Fix_d(L), \wedge, \vee)$, where $(Fix_d(L), \wedge, \vee)$ is the lattice of fixed points of d.*

Proof. Since $Fix_d(L)$ is an ideal of L (see Remark 1), we conclude that $(Fix_d(L), \wedge, \vee)$ is a lattice. Also, Proposition 7 guarantees that $(I_d(L), \sqcap, \sqcup)$ is a lattice. Next, let $\psi\colon Fix_d(L) \to I_d(L)$ be a mapping defined as $\psi(x) = i_d(x)$ for any $x \in Fix_d(L)$. One can easily verify that ψ is surjective. Further, Lemma 2 guarantees

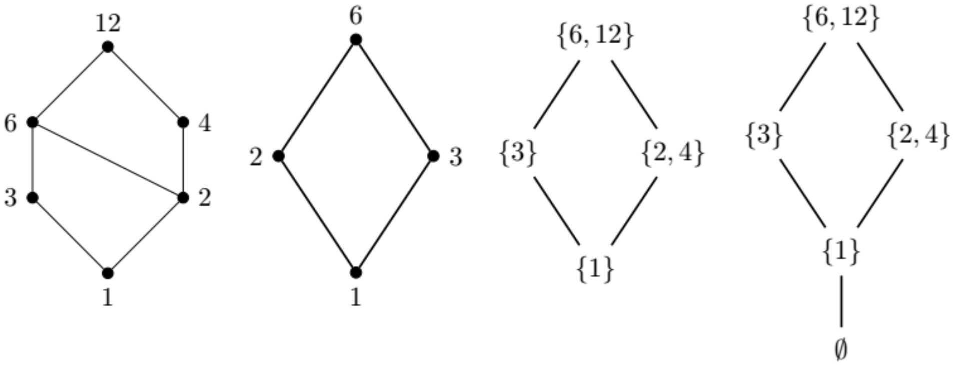
$$x \leqslant y \quad \text{if and only if} \quad \psi(x) \sqsubseteq \psi(y) \text{ for any } x, y \in Fix_d(L).$$

Now, we show that ψ is injective. Let $x_1, x_2 \in Fix_d(L)$ such that $\psi(x_1) = \psi(x_2)$, so $\psi(x_1) \sqsubseteq \psi(x_2)$ and $\psi(x_2) \sqsubseteq \psi(x_1)$. Then $x_1 \leqslant x_2$ and $x_2 \leqslant x_1$. Therefore, $x_1 = x_2$. Hence, ψ is injective.

Thus, ψ is an order isomorphism between the lattices $Fix_d(L)$ and $I_d(L)$. Therefore, ψ is a lattice isomorphism from $Fix_d(L)$ into $I_d(L)$. Consequently, $(I_d(L), \sqcap, \sqcup)$ and $(Fix_d(L), \wedge, \vee)$ are isomorphic. $\square$

Next, we give an illustrative example to present this isomorphism.

Example 4. Let $(D(12), gcd, lcm)$ be the lattice of positive divisors of 12 ordered by the divisibility order relation $\mid$ and i_d be the d-integration function defined in the table of Example 2.



$D(12)$ $Fix_d(D(12))$ $\mathcal{I}_d(D(12))$ $\mathbb{I}_d(D(12))$

FIGURE 2. The Hasse diagrams of the lattices $(D(12), gcd, lcm)$, $(Fix_d(D(12)), gcd, lcm)$, $(I_d(D(12)), \sqcap, \sqcup)$ and $(\mathbb{I}_d(D(12)), \sqcap, \sqcup)$.

The above Theorem 3 leads to the following corollary.

Corollary 1. *Let $(L, \wedge, \vee)$ be a lattice and d be an isotone derivation on L. If $(L, \wedge, \vee)$ is modular (resp. distributive), then $(I_d(L), \sqcap, \sqcup)$ is also modular (resp. distributive).*

Remark 3. Let $(L, \wedge, \vee)$ be a lattice and d be an isotone derivation on L different from the identity function. Then $\mathbb{I}_d(L) = I_d(L) \cup \{\varnothing\}$. Indeed, $\mathbb{I}_d(L) = \{i_d(x) \mid x \in L\} = \{i_d(x) \mid x \in Fix_d(L)\} \cup \{i_d(x) \mid x \notin Fix_d(L)\} = I_d(L) \cup \{\varnothing\}$.

Combining Proposition 6, Remark 3 and Corollary 1 leads to the following corollary.

Corollary 2. *Let $(L, \wedge, \vee)$ be a lattice and d be an isotone derivation on L different from the identity function. If $(L, \wedge, \vee)$ is modular (resp. distributive), then $(\mathbb{I}_d(L), \sqcap, \sqcup)$ is also modular (resp. distributive).*

5. AN APPLICATION OF INTEGRATIONS ON LATTICES

In this section, we use this notion of integrations to determine the necessary and sufficient conditions under which a linear differential equation on an arbitrary Boolean lattice has a solution.

Definition 5. Let $(B,\wedge,\vee,0,1,')$ be a Boolean lattice and $\leqslant$ be its order relation. Let i_d be an integration on B and a,b be two elements of B. A *linear differential equation* with respect to d is any equation with the form

$$a\cdot d(X)+b=0\,, \tag{5.1}$$

where $a\cdot d(X)=a\wedge d(X)$ and $x+y=(x\wedge y')\vee(x'\wedge y)$ for any $x,y\in B$.

Theorem 4. *Let $(B,\wedge,\vee,0,1,')$ be a Boolean lattice and i_d be an integration on B. Then the equation (5.1) has a solution if and only if $b\leqslant a$ and b is a d-integrable element of B.*

Proof. Let x_0 be a solution of the equation (5.1), then $a\cdot d(x_0)+b=0$, i.e.,

$$a\cdot d(x_0)=b.$$

Thus, $b\leqslant a$. On an other hand, the fact that d is a derivation on B implies that

$$\begin{aligned}d(b)&=d(a\cdot d(x_0))\\&=(d(a)\cdot d(x_0))\vee(a\cdot d(d(x_0)))\\&=(d(a)\cdot d(x_0))\vee(a\cdot d(x_0))\\&=(d(a)\cdot d(x_0))\vee b\,.\end{aligned}$$

Thus, $b\leqslant d(b)$. Also, from Proposition 1 (i) we have $d(b)\leqslant b$. Then $d(b)=b$, i.e., b is a fixed point of d. Therefore, Lemma 1 guarantees that b is a d-integrable element of B. Conversely, suppose that $b\leqslant a$ and b a d-integrable element of B. Then $a\cdot b=b$ and $d(b)=b$. Hence, $a\cdot d(b)+b=a\cdot b+b=b+b=0$. Thus, b is a solution of the equation (5.1). ☐

Example 5. Let $(D(30),gcd,lcm,1,30,')$ be the Boolean lattice of positive divisors of 30 ordered by the divisibility order relation $\mid$ and given by the Hasse diagram in Figure 3. Let d be a principal derivation on $D(30)$ defined as $dx=gcd(6,x)$ for any $x\in D(30)$. The d-integration function i_d on $D(30)$ is defined in the following table:

x	1	2	3	5	6	10	15	30
$d(x)$	1	2	3	1	6	2	3	6
$i_d(x)$	$\{1,5\}$	$\{2,10\}$	$\{3,15\}$	$\varnothing$	$\{6,30\}$	$\varnothing$	$\varnothing$	$\varnothing$

Let $10\cdot d(X)+2=0$ and $10\cdot d(X)+5=0$ be two linear differential equations on $D(30)$. Since $2\mid 10$ and 2 is an integrable element of $D(30)$, then Theorem 4 guarantees that the $10\cdot d(X)+2=0$ has a solution. The solutions are given by this set:

$$S=\{2,6,10,30\}\,.$$

Moreover, the fact that $5 \mid 10$ but 5 is not an integrable element of $D(30)$, then Theorem 4 guarantees that the $10 \cdot d(X) + 5 = 0$ has not a solution on the Boolean lattice $D(30)$.

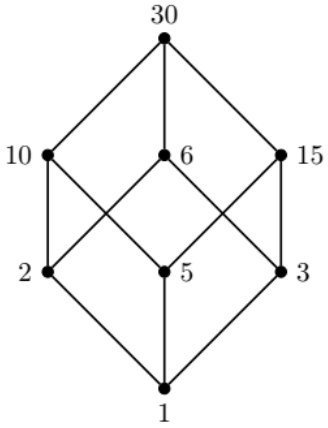


FIGURE 3. The Hasse diagram of the Boolean lattice $(D(30), gcd, lcm, 1, 30, ')$.

Theorem 5. *Let* $(B, \wedge, \vee, 0, 1, ')$ *be a Boolean lattice and* i_d *be an integration on* B. *If* x_0 *is a solution of the equation* (5.1)*, then* $b \leqslant d(x_0) \leqslant a + b + 1$.

Proof. Let $x_0 \in B$ is a solution of the equation (5.1), i.e., $a \cdot d(x_0) + b = 0$. Then $a \cdot d(x_0) = b$, so on the one hand $b \leqslant d(x_0)$. On the other hand, $(a + b + 1) \cdot d(x_0) = a \cdot d(x_0) + b \cdot d(x_0) + 1 \cdot d(x_0) = b + b + d(x_0) = d(x_0)$. Thus, $d(x_0) \leqslant (a + b + 1)$. Therefore, $b \leqslant d(x_0) \leqslant a + b + 1$. ☐

6. CONCLUSION AND FUTURE RESEARCH

In this paper, we have introduced the notion of integration with respect to a given derivation on a lattice. More precisely, we have given the definitions of integrable elements of a lattice and their integral sets. We have investigated several characterizations and properties concerning the integrable elements and their integral sets. Moreover, we have given a lattice structure to the family of integral sets with respect to a given integration on a lattice. We have provided a representation theorem for the lattice of fixed points of an isotone derivation based on the family of integral sets. As an application, we have used this notion of integration to determine the necessary and sufficient conditions under which a linear differential equation on an arbitrary Boolean lattice has a solution.

Finally, we anticipate that there exist relationships between isotone derivations and their integrations on a lattice. Also, we intend to extend this notion of integration to several interesting algebraic structures.

REFERENCES

[1] A. Y. Abdelwanis and A. Boua, "On generalized derivations of partially ordered sets," *Communications in Mathematics*, vol. 27, no. 1, pp. 69–78, 2019, doi: 10.2478/cm-2019-0006.

[2] M. Ashraf, S. Ali, and C. Haetinger, "On derivations in rings and their applications," *The Aligarh Bull of Math*, vol. 25, no. 2, pp. 79–107, 2006.

[3] I. Banič, "Integrations on rings," *Open Mathematics*, vol. 15, no. 1, pp. 365–373, 2017, doi: 10.1515/math-2017-0034.

[4] L. Ferrari *et al.*, "On derivations of lattices," *Pure Math. Appl*, vol. 12, no. 4, pp. 365–382, 2001.

[5] P. He, X. Xin, and J. Zhan, "On derivations and their fixed point sets in residuated lattices," *Fuzzy Sets and Systems*, vol. 303, pp. 97–113, 2016, doi: 10.1016/j.fss.2016.01.006.

[6] K. H. Kim and B. Davvaz, "On f-derivations of be-algebras," *Journal of the Chungcheong Mathematical Society*, vol. 28, no. 1, pp. 127–138, 2015, doi: 10.14403/jcms.2015.28.1.127.

[7] S. M. Lee and K. H. Kim, "A note on f-derivations of bcc-algebras," *Pure Math. Sci*, vol. 1, no. 2, pp. 87–93, 2012.

[8] S. A. Ozbal and A. Firat, "On f- derivations of incline algebras," *International Electronic Journal of Pure and Applied Mathematics*, vol. 3, no. 1, pp. 83–90, 2011.

[9] E. C. Posner, "Derivations in prime rings," *Proceedings of the American Mathematical Society*, vol. 8, no. 6, pp. 1093–1100, 1957, doi: 10.1090/S0002-9939-1957-0095863-0.

[10] G. Szász, "Derivations of lattices," *Acta Scientiarum Mathematicarum*, vol. 37, no. 1-2, pp. 149–154, 1975.

[11] J. Wang, Y. Jun, X. Xin, and Y. Zou, "On derivations of bounded hyperlattices," *Journal of Mathematical Research with Applications*, vol. 36, no. 2, pp. 151–161, 2016, doi: 10.3770/j.issn:2095-2651.2016.02.003.

[12] Q. Xiao and W. Liu, "On derivations of quantales," *Open Mathematics*, vol. 14, no. 1, pp. 338–346, 2016, doi: 10.1515/math-2016-0030.

[13] X. L. Xin, "The fixed set of a derivation in lattices," *Fixed Point Theory and Applications*, vol. 2012, no. 1, pp. 1–12, 2012, doi: 10.1186/1687-1812-2012-218.

[14] X. L. Xin, T. Y. Li, and J. H. Lu, "On derivations of lattices," *Information sciences*, vol. 178, no. 2, pp. 307–316, 2008, doi: 10.1016/j.ins.2007.08.018.

[15] Y. H. Yon and K. H. Kim, "On f-derivations from semilattices to lattices," *Communications of the Korean Mathematical Society*, vol. 29, no. 1, pp. 27–36, 2014, doi: 10.4134/CKMS.2014.29.1.027.

[16] L. Zedam, M. Yettou, and A. Amroune, "f-fixed points of isotone f-derivations on a lattice," *Discussiones Mathematicae-General Algebra and Applications*, vol. 39, pp. 69–89, 2019, doi: 10.7151/dmgaa.1308.

[17] H. Zhang and Q. Li, "On derivations of partially ordered sets," *Mathematica Slovaca*, vol. 67, no. 1, pp. 17–22, 2017, doi: 10.1515/ms-2016-0243.

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