Miskolc Mathematical Notes
Vol. 21 (2020), No. 2, pp. 887–896

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2020.3380

EXISTENCE RESULT FOR A NEW CLASS OF KIRCHHOFF ELLIPTIC SYSTEM WITH VARIABLE PARAMETERS

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Received 08 June, 2020

Abstract. The paper studies the existence result for a new class of Kirchhoff elliptic system with variable parameters in the right hand side. Sub-super solutions method are used for proving the main result. Our study is a natural improvement result of our previous one in (Boulaaras et al. in Math. Methods Appl. Sci. 41:5203-5210, 2018).

2010 *Mathematics Subject Classification:* 35J60; 35B30; 35B40

Keywords: Kirchhoff elliptic systems, existence, positive solutions, sub-supersolution, variable parameters

1. INTRODUCTION

Consider the following system

$$\begin{cases} -A\left(\int\limits_{\Omega}|\nabla u|^2\,dx\right)\triangle u=\lambda\alpha(x)\,f(u,v)\ \text{ in }\Omega,\\ -B\left(\int\limits_{\Omega}|\nabla v|^2\,dx\right)\triangle v=\lambda\beta(x)\,g(u,v)\ \text{ in }\Omega,\\ u=v=0\text{ on }\partial\Omega, \end{cases}\tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is a bounded smooth domain with C^2 boundary $\partial\Omega$, and A, B: $\mathbb{R}^+\to\mathbb{R}^+$ are continuous functions with further conditions to be given later, λ is a positive parameter, and $\alpha,\beta\in C\left(\overline{\Omega}\right)$.

The second author would like to thank his Professors/Scientists Pr. Mohamed Haiour and Pr. Azzedine Benchettah for the important masters and PhD courses content in pure and applied mathematics which he received them during his studies. Moreover, he thanks them for the additional help they provided him during office hours in their office about the few concepts/difficulties He had encountered and appreciated their talent and dedication for their postgraduate students currently and previously.

This nonlocal problem originates from the stationary version of Kirchhoff's work [10] in 1883

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int\limits_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

where Kirchhoff extended the classical d'Alembert's wave equation by considering the effect of the changes in the length of the string during vibrations. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

Recently, the problems associated to Laplacian operator and Kirchhoff elliptic equations have been heavily studied, we refer to [1, 3–5, 8, 9, 11–13].

In [2], Alves and Correa proved the validity of Sub-super solutions method for problems of Kirchhoff class involving a single equation and a boundary condition

$$\begin{cases} -M\left(\|u\|^2\right)\Delta u=f(x,u) \text{ in } \Omega,\\ \qquad u=0 \quad \text{on } \partial\Omega,\end{cases}$$

with $f\in C\left(\overline{\Omega}\times\mathbb{R}\right)$.

By using a comparison principle that requires M to be non-negative and non-increasing in $[0,+\infty)$, with $H(t):=M\left(t^2\right)t$ increasing and $H(\mathbb{R})=\mathbb{R}$, they managed to prove the existence of positive solutions assuming f increasing in the variable u for each $x\in\Omega$ fixed.

For systems involving similar class of equations, this result can not be used directly, i.e. the existence of a subsolution and a supersolution does not guarantee the existence of the solution. Therefore, a further construction is needed. As in [6], where we studied the system

$$\begin{cases} -A\left(\int\limits_\Omega|\nabla u|^2dx\right)\triangle u=\lambda_1 f(v)+\mu_1 g(u) \text{ in } \Omega,\\ -B\left(\int\limits_\Omega|\nabla v|^2dx\right)\triangle v=\lambda_2 h(u)+\mu_2(x)\,l(v) \text{ in } \Omega,\\ u=v=0 \text{ on } \partial\Omega.\end{cases} \tag{1.3}$$

Using a weak positive supersolution as first term of a constructed iterative sequence (u_n,v_n) in $H_0^1(\Omega)\times H_0^1(\Omega)$, and a comparison principle introduced in [2], the authors established the convergence of this sequence to a positive weak solution of the considered problem.

In this paper, we generalize the previous work in [6] by considering variable parameters α,β,γ and η in the right hand side of (1.1). We also give a better subsolution providing easier computations.

2. Existence result

Definition 1. $(u,v) \in \left(H_0^1(\Omega) \times H_0^1(\Omega)\right)$, is called a weak solution of (1.1) if it satisfies

$$A\left(\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla u \nabla \phi \, dx = \lambda \int_\Omega \alpha(x) f(u,v)\phi \, dx \text{ in } \Omega,$$
$$B\left(\int_\Omega |\nabla v|^2 dx\right)\int_\Omega \nabla v \nabla \psi \, dx = \lambda \int_\Omega \beta(x) g(u,v)\psi \, dx \text{ in } \Omega,$$

for all $(\phi,\psi) \in \left(H_0^1(\Omega) \times H_0^1(\Omega)\right)$.

Definition 2. Let $(\underline{u},\underline{v}),(\overline{u},\overline{v})$ be a pair of nonnegative functions in $\left(H_0^1(\Omega) \times H_0^1(\Omega)\right)$, they are called positive weak subsolution and positive weak supersolution (respectively) of (1.1) if they satisfy the following

$$A\left(\int_\Omega |\nabla \underline{u}|^2 dx\right)\int_\Omega \nabla \underline{u} \nabla \phi \, dx \le \lambda \int_\Omega \alpha(x) f(\underline{u},\underline{v})\phi \, dx,$$
$$B\left(\int_\Omega |\nabla \underline{v}|^2 dx\right)\int_\Omega \nabla \underline{v} \nabla \psi \, dx \le \lambda \int_\Omega \beta(x) g(\underline{u},\underline{v})\psi \, dx,$$

and

$$A\left(\int_\Omega |\nabla \overline{u}|^2 dx\right)\int_\Omega \nabla \overline{u} \nabla \phi \, dx \ge \lambda \int_\Omega \alpha(x) f(\overline{u},\overline{v})\phi \, dx,$$
$$B\left(\int_\Omega |\nabla \overline{v}|^2 dx\right)\int_\Omega \nabla \overline{v} \nabla \psi \, dx \ge \lambda \int_\Omega \beta(x) g(\overline{u},\overline{v})\psi \, dx,$$

for all $(\phi,\psi) \in \left(H_0^1(\Omega) \times H_0^1(\Omega)\right)$, with $\phi \ge 0$ and $\psi \ge 0$, and $(\underline{u},\underline{v}),(\overline{u},\overline{v}) = (0,0)$ on $\partial\Omega$.

Lemma 1 (Comparison principle [2])**.** *Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous nonincreasing function such that*

$$M(s) > m_0 > 0, \text{ for all } s \ge s_0, \tag{2.1}$$

and $H(t) = tM\left(t^2\right)$ increasing on $\mathbb{R}^+$.

If u_1, u_2 are two non-negative functions verifying

$$\begin{cases} -M\left(\int_\Omega |\nabla u_1|^2 dx\right)\triangle u_1 \geq -M\left(\int_\Omega |\nabla u_2|^2 dx\right)\triangle u_2 \text{ in } \Omega, \\ u_1 = u_2 = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.2}$$

then $u_1 \geq u_2$ a.e. in Ω.

Before stating and proving our main result, here are the conditions we need.

$(H1)$ $A, B : \mathbb{R}^+ \to \mathbb{R}^+$ are two continuous and increasing functions that satisfy the monotonicity conditions of Lemma 1 so that we can use the Comparison principle, and assume further that there exists $a_i, b_i > 0$, $i = 1, 2$,

$$a_1 \leq A(t) \leq a_2, \quad b_1 \leq B(t) \leq b_2 \text{ for all } t \in \mathbb{R}^+.$$

$(H2)$ $\alpha, \beta \in C\left(\overline{\Omega}\right)$ and

$$\alpha(x) \geq \alpha_0 > 0,\ \beta(x) \geq \beta_0 > 0$$

for all $x \in \Omega$.

$(H3)$ f, g are continuous on $[0, +\infty[$, C^1 on $(0, +\infty)$, and increasing functions of infinite growth

$$\lim_{s,t\to+\infty} f(s,t) = +\infty, \lim_{s,t\to+\infty} g(s,t) = +\infty.$$

$(H4)$ For all $K > 0$

$$\lim_{t\to+\infty} \frac{f(t, K(g(t,t)))}{t} = 0.$$

$(H5)$

$$\lim_{t\to+\infty} \frac{g(t,t)}{t} = 0.$$

Theorem 1. *For large values of $\lambda\alpha_0$ and $\lambda\beta_0$, system (1.1) admits a large positive weak solution if conditions $(H1)-(H5)$ are satisfied.*

Proof of Theorem 1. Consider σ the first eigenvalue of $-\triangle$ with Dirichlet boundary conditions and ϕ_1 the corresponding positive eigenfunction with $\|\phi_1\| = 1$ and $\phi_1 \in C^\infty\left(\overline{\Omega}\right)$ (see [7]).

Let $S = \sup_{x\in\Omega}\{\sigma\phi_1^2 - |\nabla\phi_1|^2\}$, then from growth conditions (H3)

$$f(t,t) \geq S,\ g(t,t) \geq S, \quad \text{for } t \text{ large enough.}$$

For each α_0 large, let us define

$$\underline{u} = \left(\frac{\lambda\alpha_0}{2a_2}\right)\phi_1^2$$

and

$$\underline{v} = \left(\frac{\lambda\beta_0}{2b_2}\right)\phi_1^2,$$

where a_2, b_2 are given by condition $(H1)$. Let us show that $(\underline{u}, \underline{v})$ is a subsolution of problem (1.1) for $\lambda\alpha_0$ large enough. Indeed, let $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$ in Ω. By $(H1) - (H3)$, we get

$$\begin{aligned}
&A\left(\int_\Omega |\nabla\underline{u}|^2\,dx\right)\int_\Omega \nabla\underline{u}.\nabla\phi\,dx = A\left(\int_\Omega |\nabla\underline{u}|^2\,dx\right)\left(\frac{\lambda\alpha_0}{a_2}\right)\int_\Omega \phi_1\nabla\phi_1.\nabla\phi\,dx\\
&= \left(\frac{\lambda\alpha_0}{a_2}\right)A\left(\int_\Omega |\nabla\underline{u}|^2\,dx\right)\times\left\{\int_\Omega \nabla\phi_1\nabla(\phi_1.\phi)\,dx - \int_\Omega |\nabla\phi_1|^2\phi\,dx\right\}\\
&= \left(\frac{\lambda\alpha_0}{a_2}\right)A\left(\int_\Omega |\nabla\underline{u}|^2\,dx\right)\int_\Omega\left(\sigma\phi_1^2 - |\nabla\phi_1|^2\right)\phi\,dx\\
&\leq \lambda\alpha_0\int_\Omega S\phi\,dx \leq \lambda\int_\Omega \alpha(x)f(\underline{u},\underline{v})\phi\,dx
\end{aligned}$$

for $\lambda\alpha_0 > 0$ large enough, and all $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$ in Ω.

Similarly,

$$B\left(\int_\Omega |\nabla\underline{v}|^2\,dx\right)\int_\Omega \nabla\underline{v}\nabla\psi\,dx \leq \lambda\int_\Omega \beta(x)g(\underline{u},\underline{v})\psi\,dx \text{ in } \Omega$$

for $\lambda\beta_0 > 0$ large enough and all $\psi \in H_0^1(\Omega)$ with $\psi \geq 0$ in Ω.
Also notice that $\underline{u} > 0$ and $\underline{v} > 0$ in Ω, $\underline{u} \to +\infty$ and $\underline{v} \to +\infty$ as $\lambda\alpha_0 \to +\infty$, $\lambda\beta_0 \to +\infty$.

For the supersolution part, consider e the solution of the following problem

$$\begin{cases} -\triangle e = 1 \text{ in } \Omega, \\ e = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.3}$$

We give the supersolution of problem (1.1) by

$$\overline{u} = \frac{C}{\mu}(\lambda\|\alpha\|_\infty)e, \quad \overline{v} = \left(\frac{\lambda\|\beta\|_\infty}{b_2}\right)g(C\lambda, C\lambda)e,$$

where $\mu = \|e\|_\infty$, $C > 0$ is a large positive real number to be given later.
Indeed, for all $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$ in Ω, we get from (2.3) and the condition $(H1)$

$$A\left(\int_\Omega |\nabla\overline{u}|^2\,dx\right)\int_\Omega \nabla\overline{u}.\nabla\phi\,dx = A\left(\int_\Omega |\nabla\overline{u}|^2\,dx\right)\frac{C}{\mu}(\lambda\|\alpha\|_\infty)\int_\Omega \nabla e.\nabla\phi\,dx$$

$$= A\left(\int_\Omega |\nabla \overline{u}|^2 dx\right)\frac{C\lambda}{\mu}\left(\|\alpha\|_\infty\right)\int_\Omega \phi\, dx \geq \frac{a_1 C\lambda}{\mu}\left(\|\alpha\|_\infty\right)\int_\Omega \phi\, dx.$$

By $(H4)$ and $(H5)$, we can choose C large enough so that

$$\frac{a_1 C\lambda}{\mu}\int_\Omega \phi\, dx \geq \lambda \int_\Omega f\left(C\lambda, \left(\frac{\lambda\|\beta\|_\infty}{b_2}\right) g\left(C\lambda, C\lambda\right)\mu\right)\phi\, dx.$$

Therefore,

$$\begin{aligned} A\left(\int_\Omega |\nabla \overline{u}|^2 dx\right)\int_\Omega \nabla\overline{u}.\nabla\phi\, dx &\geq \lambda\|\alpha\|_\infty \int_\Omega f\left(C\lambda, \left(\frac{\lambda\|\beta\|_\infty}{b_2}\right) g\left(C\lambda, C\lambda\right)\mu\right)\phi\, dx \\ &\geq \lambda\|\alpha\|_\infty \int_\Omega f\left(\frac{C}{\mu}\lambda e, \left(\frac{\lambda\|\beta\|_\infty}{b_2}\right) g\left(C\lambda, C\lambda\right)e\right)\phi\, dx \\ &\geq \lambda \int_\Omega \alpha(x) f\left(\overline{u}, \overline{v}\right)\phi\, dx. \end{aligned} \tag{2.4}$$

Also,

$$\begin{aligned} B\left(\int_\Omega |\nabla \overline{v}|^2 dx\right)\int_\Omega \nabla\overline{v}\nabla\psi dx &= \frac{\lambda\|\beta\|_\infty}{b_2} g\left(C\lambda, C\lambda\right) B\left(\int_\Omega |\nabla \overline{v}|^2 dx\right)\int_\Omega \nabla e\nabla\psi dx \\ &\geq \lambda\|\beta\|_\infty g\left(C\lambda, C\lambda\right)\int_\Omega \psi dx. \end{aligned} \tag{2.5}$$

Using $(H4)$ and $(H5)$ again for C large enough we get

$$\frac{1}{\frac{\lambda\|\beta\|_\infty}{b_2}\mu} \geq \frac{g\left(C\lambda, C\lambda\right)}{C\lambda}$$

Hence

$$\begin{aligned} g\left(C\lambda, C\lambda\right)\int_\Omega \psi dx &\geq \int_\Omega g\left(C\lambda, g\left(C\lambda, C\lambda\right)\frac{\left(\lambda\|\beta\|_\infty\right)}{b_2}\mu\right)\psi dx \\ &\geq \int_\Omega g\left(\lambda C\left(\frac{e}{\mu}\right), g\left(C\lambda, C\lambda\right)\frac{\left(\lambda\|\beta\|_\infty\right)}{b_2}e\right)\psi dx \\ &= \int_\Omega g\left(\overline{u}, \overline{v}\right)\psi dx. \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we obtain

$$B\left(\int_{\Omega}|\nabla\overline{v}|^{2}\,dx\right)\int_{\Omega}\nabla\overline{v}\nabla\psi dx\geq\lambda\int_{\Omega}\beta(x)\,g(\overline{u},\overline{v})\,\psi dx. \tag{2.7}$$

By (2.4) and (2.7) we conclude that $(\overline{u},\overline{v})$ is a supersolution of problem (1.1). Furthermore, $\underline{u}\leq\overline{u}$ and $\underline{v}\leq\overline{v}$ for C choosen large enough.

Now, we use a similar argument to [6] in order to obtain a weak solution of our problem. Consider the following sequence $\{(u_n,v_n)\}\subset\left(H_0^1(\Omega)\times H_0^1(\Omega)\right)$ where: $u_0:=\overline{u}, v_0=\overline{v}$ and (u_n,v_n) is the unique solution of

$$\begin{cases} -A\left(\int_{\Omega}|\nabla u_n|^2\,dx\right)\triangle u_n=\lambda\alpha(x)\,f(u_{n-1},v_{n-1})\ \text{ in }\Omega,\\[2mm] -B\left(\int_{\Omega}|\nabla v_n|^2\,dx\right)\triangle v_n=\lambda\beta(x)\,g(u_{n-1},v_{n-1})\,\text{in }\Omega,\\[2mm] u_n=v_n=0\text{ on }\partial\Omega. \end{cases} \tag{2.8}$$

Since A and B satisfy $(H1)$ and $\alpha(x)\,f(u_{n-1},v_{n-1})$, $\beta(x)\,g(u_{n-1},v_{n-1})\in L^2(\Omega)$ (in x), we deduce from a result in [2] that system (2.8) has a unique solution $(u_n,v_n)\in\left(H_0^1(\Omega)\times H_0^1(\Omega)\right)$.

Using (2.8) and the fact that (u_0,v_0) is a supersolution of (1.1), we get

$$\begin{cases} -A\left(\int_{\Omega}|\nabla u_0|^2\,dx\right)\triangle u_0\geq\lambda\alpha(x)\,f(u_0,v_0)=-A\left(\int_{\Omega}|\nabla u_1|^2\,dx\right)\triangle u_1,\\[2mm] -B\left(\int_{\Omega}|\nabla v_0|^2\,dx\right)\triangle v_0\geq\lambda\beta(x)\,g(u_0,v_0)\ =-B\left(\int_{\Omega}|\nabla v_1|\,dx\right)\triangle v_1 \end{cases}$$

then by Lemma 1, $u_0\geq u_1$ and $v_0\geq v_1$. Also, since $u_0\geq\underline{u}$, $v_0\geq\underline{v}$ and the monotonicity of f, g, h, and l one has

$$-A\left(\int_{\Omega}|\nabla u_1|^2\,dx\right)\triangle u_1=\lambda\alpha(x)\,f(u_0,v_0)\geq\lambda\alpha(x)\,f(\underline{u},\underline{v})\geq-A\left(\int_{\Omega}|\nabla\underline{u}|^2\,dx\right)\triangle\underline{u},$$

$$-B\left(\int_{\Omega}|\nabla v_1|^2\,dx\right)\triangle v_1=\lambda\beta(x)\,g(u_0,v_0)\geq\lambda\beta(x)\,g(\underline{u},\underline{v})\geq-B\left(\int_{\Omega}|\nabla\underline{v}|^2\,dx\right)\triangle\underline{v}$$

according to Lemma 1 again, we obtain $u_1\geq\underline{u}$, $v_1\geq\underline{v}$. Repeating the same argument for u_2,v_2, observe that

$$-A\left(\int_{\Omega}|\nabla u_1|^2\,dx\right)\triangle u_1=\lambda\alpha(x)\,f(u_0,v_0)\geq\lambda\alpha(x)\,f(u_1,v_1)=-A\left(\int_{\Omega}|\nabla u_2|^2\,dx\right)\triangle u_2,$$

$$-B\left(\int_\Omega |\nabla v_1|\,dx\right)\triangle v_1=\lambda\beta(x)\,g(u_0,v_0)\geq\lambda\alpha(x)\,g(u_1,v_1)=-B\left(\int_\Omega |\nabla v_2|^2\,dx\right)\triangle v_2,$$

then $u_1\geq u_2$, $v_1\geq v_2$. Similarly, we get $u_2\geq \underline{u}$ and $v_2\geq \underline{v}$ from

$$-A\left(\int_\Omega |\nabla u_2|^2\,dx\right)\triangle u_2=\lambda\alpha(x)\,f(u_1,v_1)\geq\lambda\alpha(x)\,f(\underline{u},\underline{v})\geq -A\left(\int_\Omega |\nabla \underline{u}|^2\,dx\right)\triangle \underline{u},$$

$$-B\left(\int_\Omega |\nabla v_2|^2\,dx\right)\triangle v_2=\lambda\beta(x)\,g(u_1,v_1)\geq\lambda\beta(x)\,g(\underline{u},\underline{v})\geq -B\left(\int_\Omega |\nabla \underline{v}|^2\,dx\right)\triangle \underline{v}.$$

By repeating these implementations we construct a bounded decreasing sequence $\{(u_n,v_n)\}\subset\left(H_0^1(\Omega)\times H_0^1(\Omega)\right)$ verifying

$$\overline{u}=u_0\geq u_1\geq u_2\geq\ldots\geq u_n\geq\ldots\geq\underline{u}>0, \tag{2.9}$$

$$\overline{v}=v_0\geq v_1\geq v_2\geq\ldots\geq v_n\geq\ldots\geq\underline{v}>0. \tag{2.10}$$

By continuity of functions f, g, h, and l and the definition of the sequences (u_n) and (v_n), there exist positive constants $C_i>0$, $i=1,\ldots,4$ such that

$$|f(u_{n-1},v_{n-1})|\leq C_1,\quad |g(u_{n-1},v_{n-1})|\ \leq C_2 \text{ for all } n. \tag{2.11}$$

From (2.11), multiplying the first equation of (2.8) by u_n, integrating, using Holder inequality and Sobolev embedding we check that

$$a_1\int_\Omega |\nabla u_n|^2\,dx\leq A\left(\int_\Omega |\nabla u_n|^2\,dx\right)\int_\Omega |\nabla u_n|^2\,dx=\lambda\int_\Omega \alpha(x)\,f(u_{n-1},v_{n-1})\,u_n dx$$
$$\leq\lambda\|\alpha\|_\infty\int_\Omega |f(u_{n-1},v_{n-1})|\ \ |u_n|\,dx\leq C_1\int_\Omega\ |u_n|\,dx\leq C_3\,\|u_n\|_{H_0^1(\Omega)}$$

or

$$\|u_n\|_{H_0^1(\Omega)}\leq C_3,\ \forall n, \tag{2.12}$$

where $C_3>0$ is a constant independent of n. Similarly, there exist $C_4>0$ independent of n such that

$$\|v_n\|_{H_0^1(\Omega)}\leq C_4,\ \ \forall n. \tag{2.13}$$

From (2.12) and (2.13), we deduce that $\{(u_n,v_n)\}$ admits a weakly converging subsequence in $H_0^1\left(\Omega,\mathbb{R}^2\right)$ to a limit (u,v) satisfying $u\geq\underline{u}>0$ and $v\geq\underline{v}>0$. Being monotone and also using a standard regularity argument, $\{(u_n,v_n)\}$ converges itself to (u,v). Now, letting $n\to+\infty$ in (2.13), we conclude that (u,v) is a positive weak solution of system (1.1). □

ACKNOWLEDGMENT

The authors are grateful to the anonymous referees for the careful reading and their important observations/suggestions for the sake of improving this paper. In memory of Mr. Mahmoud ben Mouha Boulaaras (1910–1999).

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