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EXISTENCE AND MULTIPLICITY OF POSITIVE WEAK SOLUTIONS FOR A NEW CLASS OF (P; Q)-LAPLACIAN SYSTEMS

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Abstract. The paper is concerned with the existence of positive weak solutions for a new class of (p,q)-Laplacian elliptic systems in a bounded domain by means of the method of sub-super solutions. Particularly, we do not need any sign conditions for $\gamma(0), g(0), f(0)$ and $h(0)$. Moreover, a multiplicity result is obtained when $\gamma(0) = g(0) = f(0) = h(0) = 0$. Finally, we give some examples to verify our main results.

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1. INTRODUCTION

In this paper, we deal with the existence and multiplicity of positive weak solutions for the following (p,q)-Laplacian systems

$$\begin{cases} -\triangle_p u - |u|^{p-2} u = \lambda_1 a(x) f(v) + \mu_1 \alpha(x) h(u) \text{ in } \Omega, \\ -\triangle_q v - |v|^{q-2} v = \lambda_2 b(x) g(u) + \mu_2 \beta(x) \gamma(v) \quad \text{in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\triangle_s z = \operatorname{div}\left(|\nabla z|^{s-2} \nabla z\right), s > 1, \Omega \subset \mathbb{R}^N \ (N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega, a(x), b(x), \alpha(x), \beta(x) \in C(\overline{\Omega})$, $\lambda_1, \lambda_2, \mu_1, ,\mu_2 \geq 0$, $1 < p,q < \infty$.

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The study of (p,q)-Laplacian systems is a new and interesting topic. It arises from electrorheological fluids, nonlinear elasticity theory, etc. (see [5], [15] [1]-[6]). A lot of existence results have been obtained on this class of problems, we refer to ([4], [5], [7], [10], [13], [14], [3], [2], [11], [9], [8]). These problems originate from physical models and are widely used in many fields such as combustion, mathematical biology, chemical reactions and so on. Our method is mainly focused on the method of sub-super solutions (see [10], [11] for a more detailed discussion).

As far as we know, there are very few contributions devoted to the (p,q)-Laplacian nonlinear elliptic system. Therefore, with the help of the method of sub-super solutions method, we are inspired by the paper of [12] in which a new (p,q)-Laplacian system was discussed and extended our previous results to problem (1.1) without assuming any sign conditions for $h(0)$, $g(0)$, $f(0)$, and $\gamma(0)$. Furthermore, when $h(0)=f(0)=g(0)=\gamma(0)=0$, a multiplicity result is given.

The outline of the paper is organized as follows: Sec. 2 introduces some definitions and make appropriate assumptions, which will be used in the body of the paper. In addition, we show the proof of two important results. Sec. 3, we will illustrate our main results with some interesting examples.

2. Main results

First, in order to get our main results, we will consider the following hypothesis:
$(H1)$ There exist $a(x), \alpha(x), b(x), \beta(x) \in C(\overline{\Omega})$ such that

$$a(x) \geq a_1 > 0,\ b(x) \geq b_1 > 0,$$
$$\alpha(x) \geq \alpha_1 > 0,\ \beta(x) \geq \beta_1 > 0.$$

$(H2)$ Let $f, g, h, \gamma \in C^1([0,\infty))$ be monotone functions satisfying

$$\lim_{s\to+\infty} f(s) = \lim_{s\to+\infty} g(s) = \lim_{s\to+\infty} h(s) = \lim_{s\to+\infty} \gamma(s) = +\infty.$$

$(H3)$ $\lim_{s\to+\infty} \frac{f\left(M(g(s))^{\frac{1}{q-1}}\right)}{s^{p-1}} = 0,\ \forall M > 0.$

$(H4)$ $\lim_{s\to+\infty} \frac{h(s)}{s^{p-1}} = \lim_{s\to+\infty} \frac{\gamma(s)}{s^{q-1}} = 0.$

Next, we define weak solutions and sub-super solutions in (p,q)-Laplacian elliptic systems.

Definition 1. Let $(u,v) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$, we say that (u,v) is a weak solution of problem (1.1), if

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla\xi dx - \int_\Omega |u|^{p-2}u.\xi dx$$
$$= \lambda_1 \int_\Omega a(x) f(v)\xi dx + \mu_1 \int_\Omega \alpha(x) h(u)\xi dx \text{ in } \Omega,$$

$$\int_\Omega |\nabla v|^{q-2}\nabla v.\nabla\zeta dx - \int_\Omega |v|^{q-2}v.\zeta dx$$
$$= \lambda_2 \int_\Omega b(x) g(u)\zeta dx + \mu_2 \int_\Omega \beta(x)\gamma(v)\zeta dx \text{ in } \Omega$$

for all $(\xi,\zeta) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

Definition 2. The nonnegative functions $(\underline{u},\underline{v}), (\overline{u},\overline{v})$ in $W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ are called a weak subsolution and supersolution of problem (1.1) if they satisfy $(\underline{u},\underline{v}), (\overline{u},\overline{v}) = (0,0)$ on $\partial\Omega$

$$\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u}.\nabla\xi dx - \int_\Omega |\underline{u}|^{p-2}\underline{u}.\xi dx$$
$$\leq \lambda_1 \int_\Omega a(x) f(\underline{v})\xi dx + \mu_1 \int_\Omega \alpha(x) h(\underline{u})\xi dx \text{ in } \Omega,$$

$$\int_\Omega |\nabla \underline{v}|^{q-2}\nabla \underline{v}.\nabla\zeta dx - \int_\Omega |\underline{v}|^{q-2}\underline{v}.\zeta dx$$
$$\leq \lambda_2 \int_\Omega b(x) g(\underline{u})\zeta dx + \mu_2 \int_\Omega \beta(x)\gamma(\underline{v})\zeta dx \text{ in } \Omega$$

and

$$\int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u}.\nabla\xi dx - \int_\Omega |\overline{u}|^{p-2}\overline{u}.\xi dx$$
$$\geq \lambda_1 \int_\Omega a(x) f(\overline{v})\xi dx + \mu_1 \int_\Omega \alpha(x) h(\overline{u})\xi dx \text{ in } \Omega,$$

$$\int_\Omega |\nabla \overline{v}|^{q-2}\nabla \overline{v}.\nabla\zeta dx - \int_\Omega |\overline{v}|^{q-2}\overline{v}.\zeta dx$$
$$\geq \lambda_2 \int_\Omega b(x) g(\overline{u})\zeta dx + \mu_2 \int_\Omega \beta(x)\gamma(\overline{v})\zeta dx \text{ in } \Omega$$

for any $(\xi,\zeta) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

In what follows, we shall establish the following the existence result.

Theorem 1. *Let $(H1)-(H4)$ hold. If $\lambda_1+\mu_1$ and $\lambda_2+\mu_2$ are big enough, then problem (1.1) processes a positive weak solution.*

Proof. We will show that there exist a positive weak subsolution$(\underline{u},\underline{v}) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ and a supersolution $(\overline{u},\overline{v}) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ of (1.1) such that $\underline{u} \leq \overline{u}$, $\underline{v} \leq \overline{v}$. Moreover, $(\underline{u},\underline{v}), (\overline{u},\overline{v})$ satisfy $(\underline{u},\underline{v}) = (0,0) = (\overline{u},\overline{v})$ on $\partial\Omega$.

Let σ_r be the first eigenvalue of $-\triangle_r$, and $\phi_r > 0$ the corresponding eigenfunction with $\|\phi_r\| = 1$ for $r = p, q$. There exist $m, \eta, \delta > 0$ such that $|\nabla\phi_r|^r - \sigma_r\ \phi_r \geq m$ on $\overline{\Omega_\delta} = \{x \in \Omega, d(x, \partial\Omega) \leq \delta\}$ and $\phi_r \geq \eta$ on $\Omega\backslash\overline{\Omega}_\delta$ for $r = p, q$. Taking $k_0 > 0$ such that $a_1 f(t), \alpha_1 h(t), b_1 g(t), \beta_1\gamma(t) > -k_0$.

First, we claim that

$$(\underline{u}, \underline{v}) := \left(\left[\frac{(\lambda_1 + \mu_1) k_0}{m}\right]^{1/p-1} \left(\frac{p-1}{p}\right) \phi_p^{p/p-1}, \right. \\ \left. \left[\frac{(\lambda_2 + \mu_2) k_0}{m}\right]^{1/q-1} \left(\frac{q-1}{q}\right) \phi_q^{q/q-1} \right) \tag{2.1}$$

is a subsolution of problem (1.1) when $\lambda_1 + \mu_1$ and $\lambda_2 + \mu_2$ are big enough. Taking the test function $\xi(x) \in W_0^{1,p}(\Omega)$ with $\xi(x) \geq 0$. Thus, from $(H1)$ we get

$$\begin{aligned}
&\int_\Omega |\nabla\underline{u}|^{p-2} \nabla\underline{u}.\nabla\xi dx - \int_\Omega |\underline{u}|^{p-2} \underline{u}.\xi dx \leq \int_\Omega |\nabla\underline{u}|^{p-2} \nabla\underline{u}.\nabla\xi dx \\
&= \left(\tfrac{(\lambda_1+\mu_1)k_0}{m}\right) \int_\Omega \left\{\sigma_p\phi_p^p - |\nabla\phi_p|^p\right\} \xi dx \\
&+ \left(\tfrac{(\lambda_1+\mu_1)k_0}{m}\right) \int_{\Omega\backslash\overline{\Omega}_\delta} \left\{\sigma_p\phi_p^p - |\nabla\phi_p|^p\right\} \xi dx.
\end{aligned}$$

We have known that $|\nabla\phi_r|^r - \sigma_r\ \phi_r \geq m$ for $s = p, q$, on $\overline{\Omega_\delta}$. Also on $\Omega\backslash\overline{\Omega}_\delta\ \phi_r \geq \eta$ for $r = p, q$. If $\lambda_1 + \mu_1$ and $\lambda_2 + \mu_2$ are big enough in the definition of $\underline{u}$, $\underline{v}$, then by $(H2)$ we get

$$a_1 f(\underline{v}), \alpha_1 h(\underline{u}), b_1 g(\underline{u}), \beta_1\gamma(\underline{v}) \geq \frac{k_0}{m} \max\left\{\sigma_p, \sigma_q\right\}. \tag{2.2}$$

Therefore,

$$\begin{aligned}
&\int_\Omega |\nabla\underline{u}|^{p-2} \nabla\underline{u}.\nabla\xi dx - \int_\Omega |\underline{u}|^{p-2} \underline{u}.\xi dx \\
&\leq \left(\tfrac{(\lambda_1+\mu_1)k_0}{m}\right) \int_{\Omega_\delta} \left\{\sigma_p\phi_p^p - |\nabla\phi_p|^p\right\} \xi dx \\
&+ \left(\tfrac{(\lambda_1+\mu_1)k_0}{m}\right) \int_{\Omega\backslash\overline{\Omega}_\delta} \left\{\sigma_p\phi_p^p - |\nabla\phi_p|^p\right\} \xi dx \\
&\leq -(\lambda_1 + \mu_1) k_0 \int_{\Omega_\delta} \xi dx + \left(\tfrac{(\lambda_1+\mu_1)k_0}{m}\right) \int_{\Omega\backslash\overline{\Omega}_\delta} \sigma_p \xi dx \\
&\leq \int_{\Omega_\delta} \left[\lambda_1 a(x) f(\underline{v}) + \mu_1 \alpha(x) h(\underline{u})\ \right] \xi dx \\
&+ \int_{\Omega\backslash\overline{\Omega}_\delta} \left[\lambda_1 a(x) f(\underline{v}) + \mu_1 \alpha(x) h(\underline{u})\ \right] \xi dx \\
&= \int_\Omega \left[\lambda_1 a(x) f(\underline{v}) + \mu_1 \alpha(x) h(\underline{u})\ \right] \xi dx
\end{aligned}$$

Similarly,

$$\int_{\Omega} |\nabla v|^{q-2} \nabla v.\nabla \zeta dx - \int_{\Omega} |v|^{q-2} v.\zeta dx \leq \int_{\Omega} [\lambda_2 b(x) g(\underline{u}) + \mu_2 \beta(x) \gamma(\underline{v})] \zeta dx.$$

Thus $(\underline{u}, \underline{v})$ is a subsolution of problem (1.1).
Next, let ω_r be a unique positive solution of

$$\begin{cases} -\triangle_r \omega_r = 1 \text{ in } \Omega, \\ \omega_r = 0 \qquad \text{on } \partial\Omega. \end{cases}$$

for $r = p, q$. We denote

$$\overline{u} = \frac{C}{\nu_p} \left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1 - \nu_p^{p-1}} \right)^{\frac{1}{p-1}} \omega_p, \tag{2.3}$$

$$\overline{v} = \left[\left(\frac{\lambda_2 \|b\|_\infty + \mu_2 \|\beta\|_\infty}{1 - \nu_q^{q-1}} \right) g \left(C \left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1 - \nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right) \right]^{\frac{1}{q-1}} \omega_q, \tag{2.4}$$

where $\nu_r = \|\omega_r\|_\infty$, $r = p, q$ and $C > 0$ is big enough. We claim that $(\overline{u}, \overline{v})$ is a supersolution of (1.1) such that $(\overline{u}, \overline{v}) \geq (\underline{u}, \underline{v})$.
According to $(H3) - (H4)$, we can make C big enough so that

$$\begin{aligned} \left(\frac{C}{\nu_p}\right)^{p-1} \geq f \left(\left[\left(\frac{\lambda_2 \|b\|_\infty + \mu_2 \|\beta\|_\infty}{1 - \nu_q^{q-1}} \right) g \left(C \left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1 - \nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right) \right]^{\frac{1}{q-1}} \omega_q \right) \\ + \mu_1 h \left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1 - \nu_p^{p-1}} \right)^{\frac{1}{p-1}} \omega_p. \end{aligned} \tag{2.5}$$

Hence

$$\int_{\Omega} |\nabla \overline{u}|^{p-2} \nabla \overline{u}.\nabla \xi dx - \int_{\Omega} |\overline{u}|^{p-2} \overline{u}.\xi dx = \left(\frac{C}{\nu_p} \right)^{p-1} (\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty) \int_{\Omega} \xi dx.$$

Using (2.5)

$$\begin{aligned}
&\int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u}.\nabla \xi dx - \int_\Omega |\overline{u}|^{p-2} \overline{u}.\xi dx \\
&\geq \lambda_1 \|a\|_\infty f\left(\left[\left(\frac{\lambda_2 \|b\|_\infty + \mu_2 \|\beta\|_\infty}{1-\nu_q^{q-1}} \right) g\left(C\left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1-\nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right) \right]^{\frac{1}{q-1}} \omega_q \right) \times \\
&\int_\Omega \xi dx + \mu_1 \|\alpha\|_\infty \int_\Omega h\left(C\left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1-\nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right) \xi dx \\
&\geq \int_\Omega \left[\lambda_1 a(x) f(\overline{v}) + \mu_1 \alpha(x) h(\overline{u}) \right] \xi dx.
\end{aligned} \tag{2.6}$$

Next

$$\begin{aligned}
&\int_\Omega |\nabla \overline{v}|^{q-2} \nabla \overline{v}.\nabla \zeta dx - \int_\Omega |\overline{v}|^{q-2} \overline{v}.\zeta dx \\
&= \left\{ (\lambda_2 \|b\|_\infty + \mu_2 \|\beta\|_\infty) g\left(C\left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1-\nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right) \right\} \omega_q \int_\Omega \xi dx \\
&\geq \left[\lambda_2 \|b\|_\infty g\left(C\left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1-\nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right) \right. \\
&\left. + \mu_2 \|\beta\|_\infty g\left(C\left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1-\nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right) \right] \int_\Omega \xi dx.
\end{aligned} \tag{2.7}$$

According to $(H4)$ and choose C big enough, we obtain

$$\begin{aligned}
&g\left(C\left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1-\nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right) \\
&\geq \gamma \left(\left[\left(\frac{\lambda_2 \|b\|_\infty + \mu_2 \|\beta\|_\infty}{1-\nu_q^{q-1}} \right) g\left(C\left(\frac{\lambda_1 \|a\|_\infty + \mu_1 \|\alpha\|_\infty}{1-\nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right) \right]^{\frac{1}{q-1}} \left\| \omega_q \right\|_\infty \right).
\end{aligned}$$

Then from (2.6) we get

$$\begin{aligned}
&\int_{\Omega} |\nabla \overline{v}|^{q-2} \nabla \overline{v}.\nabla \zeta dx - \int_{\Omega} |\overline{v}|^{q-2} \overline{v}.\zeta dx \\
&\geq \lambda_2 \|b\|_{\infty} g\left(C \left(\frac{\lambda_1 \|a\|_{\infty} + \mu_1 \|\alpha\|_{\infty}}{1-\nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right) \\
&+\mu_2 \|\beta\|_{\infty} \gamma \left(\left\{ \left(\frac{\lambda_2 \|b\|_{\infty} + \mu_2 \|\beta\|_{\infty}}{1-\nu_q^{q-1}} \right) g\left(C \left(\frac{\lambda_1 \|a\|_{\infty} + \mu_1 \|\alpha\|_{\infty}}{1-\nu_p^{p-1}} \right)^{\frac{1}{p-1}} \right) \right\}^{\frac{1}{q-1}} \|\omega_q\|_{\infty} \right) \\
&\geq \int_{\Omega} \left[b(x) g(\overline{u}) + \mu_2 \beta(x) \gamma(\overline{v}) \right] \zeta dx.
\end{aligned} \tag{2.8}$$

According to (2.6) and (2.7), we can conclude that $(\overline{u}, \overline{v})$ is a supersolution of (1.1). Further $\overline{u} \geq \underline{u}$ and $\overline{v} \geq \underline{v}$ for C big enough. Thus, we get a solution $(u, v) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ of (1.1) with $\underline{u} \leq u \leq \overline{u}$, and $\underline{v} \leq v \leq \overline{v}$. The proof of theorem 1 is complete. □

Now we show that the more general system (1.1) possesses at least two distinct positive solutions.

Theorem 2. *Suppose that the conditions $(H1)-(H4)$ hold. Let f, g, h, and γ the function be smooth enough around zero with*

$$\begin{aligned}
f(0) = h(0) = g(0) = \gamma(0) = 0 \\
= f^{(k)}(0) = h^{(k)}(0) = g^{(l)}(0) = \gamma^{(l)}(0)
\end{aligned}$$

for $k = 1, 2, \ldots [p-1]$, $l = 1, 2, \ldots [q-1]$, where $[s]$ denotes the integer part of s. Then, problem (1.1) processes at least two positive solutions when $\lambda_i + \mu_i$ are big enough; $i = 1, 2$.

Proof. For problem (1.1), we will look for a strict supersolution (ζ_1, ζ_2), a subsolution (ψ_1, ψ_2), a supersolution (z_1, z_2), and a strict subsolution (ω_1, ω_2), such that

$$\begin{aligned}
(\psi_1, \psi_2) \leq (\zeta_1, \zeta_2) \leq (z_1, z_2), \\
(\psi_1, \psi_2) \leq (\omega_1, \omega_2) \leq (z_1, z_2),
\end{aligned}$$

and $(\omega_1, \omega_2) \nleq (\zeta_1, \zeta_2)$. Then, problem (1.1) processes at least three distinct solutions (u_i, v_i), $i = 1, 2, 3$, such that

$$\begin{aligned}
(u_1, v_1) \in [(\psi_1, \psi_2), (\zeta_1, \zeta_2)], \\
(u_2, v_2) \in [(\omega_1, \omega_2), (z_1, z_2)]
\end{aligned}$$

and

$$(u_3, v_3) \in [(\psi_1, \psi_2), (z_1, z_2)] \diagdown ([(\psi_1, \psi_2), (\zeta_1, \zeta_2)] \cup (\omega_1, \omega_2), (z_1, z_2)).$$

It is obvious that $(\psi_1, \psi_2) = (0, 0)$ is a (sub)solution. Moreover, we always can find a big supersolution $(z_1, z_2) = (\overline{u}, \overline{v})$. Consider

$$\begin{cases} -\triangle_p\omega_1 - |\omega_1|^{p-2}\omega_1 = \lambda_1 a(x)\widetilde{f}(\omega_2) + \mu_1\alpha(x)\widetilde{h}(\omega_1) \text{ in } \Omega, \\ -\triangle_q\omega_2 - |\omega_2|^{q-2}\omega_2 = \lambda_2 b(x)\widetilde{g}(\omega_1) + \mu_2\beta(x)\widetilde{\gamma}(\omega_2) \text{ in } \Omega, \\ \omega_1 = \omega_2 = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.9}$$

where $\widetilde{g}(s) = g(s) - 1, \widetilde{\gamma}(s) = \gamma(s) - 1, \widetilde{h}(s) = h(s) - 1, \widetilde{f}(s) = f(s) - 1$. Then by Theorem 1, when $\lambda_i + \mu_i$ are big enough, we know that the problem (2.10) processes a solution $(\omega_1, \omega_2) > 0$ $i = 1, 2$. It is clear that (ω_1, ω_2) is a strict subsolution of problem (1.1).

In the end, we will find a strict supersolution (ζ_1, ζ_2).

Let ϕ_p, ϕ_q be the corresponding eigenfunction with respect to operators $\triangle_p$ and $\triangle_q$ and there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\phi_p \leq C_1\phi_q \quad \text{and} \quad \phi_q \leq C_2\phi_p. \tag{2.10}$$

Let $(\zeta_1, \zeta_2) = (\rho\phi_p, \rho\phi_q)$, where $\rho > 0$,

$$G_p(x) := (\sigma_p - 1)x^{p-1} - \lambda_1 f(C_2 x) - \mu_1 h(x)$$

and

$$G_q(x) := (\sigma_q - 1)x^{q-1} - \lambda_2 g(C_1 x) - \mu_2\gamma(x).$$

Note that $G_p(0) = G_q(0) = 0$, $G_p^{(k)}(0) = G_q^{(l)}(0) = 0$ for $k = 1, 2, \ldots. [p-1]$ and $l = 1, 2, \ldots. [q-1]$.

$$\begin{cases} G_p^{(p-1)}(0) > 0 \text{ and } G_q^{(q-1)}(0) > 0 \text{ if } \quad p, q \in \mathbb{Z}^+, \\ \lim_{r\to+\infty} G_p^{([p])}(r) = +\infty = \lim_{r\to+\infty} G_p^{([q])}(r) \text{ if } \quad p, q \notin \mathbb{Z}^+. \end{cases}$$

Hence, there exists θ such that $G_q(x) > 0$ and $G_p(x) > 0$ for any $x \in (0, \theta]$. So, for $0 < \rho \leq \theta$ we get

$$(\sigma_p - 1)\zeta_1^{p-1} = (\sigma_p - 1)(\rho\phi_p)^{p-1} > \lambda_1 f(C_2\rho\phi_p) - \mu_1 h(\rho\phi_p).$$

By (2.10) and the monotonicity of function f, we obtain

$$\begin{aligned} &(\sigma_p - 1)\zeta_1^{p-1} = (\sigma_p - 1)(\rho\phi_p)^{p-1} \\ &> \lambda_1 f(C_2\rho\phi_p) - \mu_1 h(\rho\phi_p) \\ &\geq \lambda_1 f(\rho\phi_q) - \mu_1 h(\rho\phi_p) = \lambda_1 f(\zeta_2) - \mu_1 h(\zeta_1) \end{aligned} \tag{2.11}$$

for any $x \in \Omega$. In the same way, we also have

$$\begin{aligned} &(\sigma_q - 1)\zeta_2^{q-1} = (\sigma_q - 1)(\rho\phi_q)^{q-1} \\ &> \lambda_2 g(C_1\rho\phi_q) - \mu_2\gamma(\rho\phi_q) \\ &\geq \lambda_2 g(\rho\phi_p) - \mu_2\gamma(\rho\phi_q) = \lambda_2 g(\zeta_1) - \mu_2\gamma(\zeta_2), \end{aligned} \tag{2.12}$$

for any $x \in \Omega$. Making use of (2.11) and (2.12), we obtain

$$\begin{aligned}
&\int_\Omega |\nabla\zeta_1|^{p-2}\nabla\zeta_1.\nabla\xi dx - \int_\Omega |\zeta_1|^{p-2}\zeta_1.\xi dx \\
&= \rho^{p-1}\left\{\int_\Omega |\nabla\phi_p|^{p-2}\nabla\phi_p.\nabla\xi dx - \int_\Omega |\phi_p|^{p-2}\phi_p.\xi dx\right\} \\
&= \int_\Omega \left\{\sigma_p(\rho\phi_p)^{p-1} - (\rho\phi_p)^{p-1}\right\}\xi dx.
\end{aligned}$$

Since because $\phi_p > 0$, we have

$$\begin{aligned}
&\int_\Omega \left\{\sigma_p(\rho\phi_p)^{p-1} - (\rho\phi_p)^{p-1}\right\}\xi dx \\
&= \int_\Omega \left\{(\sigma_p - 1)(\rho\phi_p)^{p-1}\right\}\xi dx \\
&> \lambda_1 \int_\Omega f(\zeta_2)\xi dx - \mu_1 \int_\Omega h(\zeta_1).\xi dx,
\end{aligned}$$

Similarly we also have

$$\int_\Omega |\nabla\zeta_2|^{q-2}\nabla\zeta_2.\nabla\xi dx - \int_\Omega |\zeta_2|^{q-2}\zeta_2.\xi dx > \lambda_2 \int_\Omega g(\zeta_1)\xi dx - \mu_2 \int_\Omega \gamma(\zeta_2)\xi dx.$$

It follows that (ζ_1, ζ_2) is a strict supersolution. Let ρ small enough so that $(\omega_1, \omega_2) \nleq (\zeta_1, \zeta_2)$. So, we can find solutions

$$(u_1, v_1) \in [(\psi_1, \psi_2), (\zeta_1, \zeta_2)], (u_2, v_2) \in [(\omega_1, \omega_2), (z_1, z_2)]$$

and

$$(u_3, v_3) \in [(\psi_1, \psi_2), (z_1, z_2)] \diagdown ([(\psi_1, \psi_2), (\zeta_1, \zeta_2)] \cup (\omega_1, \omega_2), (z_1, z_2)).$$

This fact that $(u_1, v_1) \equiv (\psi_1, \psi_2) \equiv (0,0)$ can happen due to $(\psi_1, \psi_2) \equiv (0,0)$ is a solution. So, anyway we can find two positive solutions (u_2, v_2) and (u_3, v_3). Therefore, we conclude the proof of Theorem2. □

3. EXAMPLES

Example 1. Let

$$f(x) = \sum_{i=1}^{m} a_i x^{p_i} - c_1,\ g(x) = \sum_{j=1}^{n} b_j x^{q_j} - c_2$$

$$h(x) = \sum_{k=1}^{s} \alpha_k x^{r_k} - c_3,\ \gamma(x) = \sum_{l=1}^{\tau} \beta_l x^{d_l} - c_4,$$

where

$$d_j < (q-1), r_k < (p-1), p_i q_j < (p-1)(q-1)$$

and

$$a_i, b_j, \alpha_k, \beta_l, p_i, q_j, r_k, d_j, c_1, c_2, c_3, c_4 \geq 0.$$

So, it is clear that f, g, h and γ fulfill the assumptions of Theorem 1.

Example 2. Let

$$f(x)=\begin{cases} x^{p_1},\ x\le 1,\\ \frac{p_1}{p_2}x^{p_2}+\left(1-\frac{p_1}{p_2}\right),\ \ x>1,\end{cases},\ h(x)=\begin{cases} x^{p_3},\ x\le 1,\\ \frac{p_3}{p_4}x^{p_4}+\left(1-\frac{p_3}{p_4}\right),\ x>1,\end{cases},$$
$$g(x)=\begin{cases} x^{q_1},\ \ x\le 1,\\ \frac{q_1}{q_2}x^{q_2}+\left(1-\frac{q_1}{q_2}\right),\ \ x>1,\end{cases},\ \gamma(x)=\begin{cases} x^{q_3},\ \ x\le 1,\\ \frac{q_3}{q_4}x^{q_4}+\left(1-\frac{q_3}{q_4}\right),\ \ x>1,\end{cases}$$

where we suppose that

$$\begin{cases} p_1, p_3 > p-1 \text{ if } \ p\in\mathbb{Z}^+,\\ p_1, p_3 > [p] \ \text{ if } \ p\notin\mathbb{Z}^+,\\ q_1, q_3 > q-1 \ \text{ if } \ q\in\mathbb{Z}^+,\\ q_1, q_3 > [q] \ \text{ if } \ \ q\notin\mathbb{Z}^+,\end{cases}$$

$p_4 < p-1$, $p_2q_2 < (p-1)(q-1)$ and $q_4 < q-1$. Clearly, f, g, h and γ fulfill all the assumptions of Theorem 2

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AVAILABILITY OF DATA AND MATERIALS

Not applicable.

COMPETING INTERESTS

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AUTHORS' CONTRIBUTIONS

The authors contributed equally in this article. They have all read and approved the final manuscript.

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