



SHARP ESTIMATION OF HERMITIAN-TOEPLITZ DETERMINANTS FOR JANOWSKI TYPE STARLIKE AND CONVEX FUNCTIONS

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Abstract. The sharp upper and lower bounds for the third-order Hermitian-Toeplitz determinant are investigated for the classes of Janowski type starlike and convex functions. The results presented in this paper generalize several recent works in this direction.

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1. INTRODUCTION

Investigating the sharp bound for the coefficient functionals of normalised analytic functions defined over the unit disk has gained much focus after the Bieberbach conjecture (1916) came into the picture. Even after the historical proof of the Bieberbach conjecture, in 1984, by L. de Branges, investigation of the bounds for coefficients of various classes of analytic functions did not stop as the bounds on coefficient also unfold many geometric properties of analytic functions. For example, the growth and distortion of an analytic function can be estimated using the bound on the second coefficient of an analytic univalent function [8]. The Hankel determinants have been among the most studied topics in Geometric Function Theory (GFT) in recent years; such studies can be back to the 1960s (see [10, 18]). In many of the recently-published works dealing extensively with the Hankel and Toeplitz determinants, use is made also of the basic (or q -) calculus (see, for example, [9, 17, 20, 22, 23]; see also a survey-cum-expository review article by Srivastava [21]).

In this line, using Hankel determinant of coefficients of analytic functions, Cantor [4] gave a criterion for rationality of such functions. The results of this kind

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makes the estimation of coefficient functionals much interesting. The following few paragraphs give a short review of basic notations that we will be using in this paper.

From now on, let \mathcal{A} be the class of normalised holomorphic functions defined on the open unit disk

$$\mathbb{D} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also a subclass of \mathcal{A} consisting of functions which are also univalent in \mathbb{D} shall be denoted by \mathcal{S} . Because of the normalisation $f(0) = 0 = f'(0) - 1$, the functions in the class \mathcal{A} have the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

and the same for functions in the class \mathcal{S} too. For α ($0 \leq \alpha < 1$), let

$$p_\alpha(z) := \frac{1 + (1 - 2\alpha)z}{1 - z}$$

be an analytic function in the unit disk \mathbb{D} which maps the unit disk \mathbb{D} onto the right of the line $x = \alpha$ of the complex plane. Robertson [19], in 1936, considered the classes of starlike and convex functions of order α ($0 \leq \alpha < 1$) defined as follows:

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{S} \text{ and } \frac{zf'(z)}{f(z)} \prec p_\alpha(z) \right\}$$

and

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{S} \text{ and } 1 + \frac{zf''(z)}{f'(z)} \prec p_\alpha(z) \right\}.$$

Let $\mathcal{S}^*(0) =: \mathcal{S}^*$ and $\mathcal{K}(0) =: \mathcal{K}$ be the classes of starlike and convex functions, respectively. Janowski [11] gave a generalisation to the classes of starlike and convex functions of order α by replacing the function p_α with

$$p_{A,B}(z) := \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1).$$

The classes of Janowski type starlike and Janowski type convex functions are defined, respectively, by

$$\mathcal{S}^*[A, B] := \left\{ f : f \in \mathcal{S} \text{ and } \frac{zf'(z)}{f(z)} \prec p_{A,B}(z) \right\}$$

and

$$\mathcal{K}[A, B] := \left\{ f : f \in \mathcal{S} \text{ and } 1 + \frac{zf''(z)}{f'(z)} \prec p_{A,B}(z) \right\}.$$

Janowski investigated growth and distortion rates for functions in the classes $\mathcal{S}^*[A, B]$ and $\mathcal{K}[A, B]$. Several recent investigations involving Janowski type analytic and univalent functions include, for example, those published in [2, 24–26] (see also the aforementioned review article by Srivastava [21]), in each of which the basic (or q -) calculus is also used.

Janteng *et al.* [12] computed the sharp bound on the second Hankel determinant for the classes \mathcal{S}^* and \mathcal{K} . Later, Babalola [3] investigated the non-sharp bounds for third Hankel determinant for these classes. In 2018, Kowalczyk *et al.* [13] obtained the sharp bound of third Hankel determinant to be $4/135$ for the class of convex functions. However, the best known estimate for starlike functions till date is $8/9$ (see [14]). For more details, we refer the reader to [6, 15]. In line of the investigation on Hankel determinants in recent years, Ali *et al.* [1] investigated the sharp bound for the second and third order symmetric Toeplitz determinants. Cudna *et al.* [7] considered the q^{th} -order Hermitian-Toeplitz determinants with its entries as coefficients of the function $f(z)$ given by (1.1) as follows:

$$T_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ \bar{a}_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{n+q-1} & \bar{a}_{n+q-2} & \cdots & a_n \end{vmatrix}.$$

There it is reported that $T_{q,n}(f)$ is rotationally invariant and if a_n 's are real, then $T_{q,n}(f)$ is Hermitian and, therefore, the determinant $T_{q,n}(f)$ is a real number. They obtained the sharp lower and upper bound for the third-order Hermitian-Toeplitz determinants for the classes of starlike and convex functions of order α ($0 \leq \alpha < 1$). Inspired by their work, Cho and Kumar [5] investigated the sharp lower and upper bounds of the third-order Hermitian-Toeplitz determinants for the class of analytic functions with bounded turning.

In this paper, the authors aim to generalise the work of Cudna *et al.* [7] for the class of Janowski type starlike and convex functions.

2. JANOWSKI TYPE STARLIKE FUNCTIONS

This section gives the estimation for lower and upper bounds of the third-order Hermitian-Toeplitz determinant $T_{3,1}(f)$ for Janowski type starlike functions. From the definition, it is easy to verify that $T_{2,1}(f) = 1 - |a_2|^2$ and

$$T_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ \bar{a}_2 & 1 & a_2 \\ \bar{a}_3 & \bar{a}_2 & 1 \end{vmatrix} = 2\Re(a_2^2 \bar{a}_3) - 2|a_2|^2 - |a_3|^2 + 1.$$

Estimating the coefficient bounds in many cases, in our case too, depends on the comparison coefficients of functions under consideration with that of the functions with positive real part. For this reason, let \mathcal{P} denotes the class of analytic functions $p : \mathbb{D} \rightarrow \mathbb{C}$ with $p(0) = 1$ and $\Re p(z) > 0$.

To keep the proof brief and avoid repetitions, let us note that for any function $f \in \mathcal{S}^*[A, B]$, there is a function with positive real part $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$ such

that

$$\frac{zf'(z)}{f(z)} = \frac{(1+A)p(z) + 1 - A}{(1+B)p(z) + 1 - B}. \tag{2.1}$$

On expanding in the Taylors series and comparing the coefficients, we get

$$a_2 = \frac{A - B}{2} p_1, \text{ and } a_3 = \frac{A - B}{8} [(A - 2B - 1)p_1^2 + 2p_2]. \tag{2.2}$$

Theorem 1. *Let $f \in \mathcal{S}^*[A, B]$. Then, the following sharp estimation holds*

$$1 - (A - B)^2 \leq T_{2,1}(f) \leq 1.$$

Proof. Since $f \in \mathcal{S}^*[A, B]$, in view of the known fact $|p_n| \leq 2$, we get $|a_2| \leq A - B$ and thus $T_{2,1}(f) = 1 - |a_2|^2 \geq 1 - (A - B)^2$ and $T_{2,1}(f) = 1 - |a_2|^2 \leq 1$. The extremal functions for lower and upper bounds are, respectively

$$\tilde{h}_0(z) = \begin{cases} z(1 + Bz)^{\frac{A-B}{B}}, & B \neq 0; \\ ze^{Az}, & B = 0 \end{cases} \tag{2.3}$$

and

$$\tilde{h}_1(z) = \begin{cases} z(1 + Bz^2)^{\frac{A-B}{2B}}, & B \neq 0; \\ ze^{Az^2/2}, & B = 0. \end{cases} \tag{2.4}$$

This ends the proof. □

Before we proceed further in this section, let Ω_1 be the set of points (A, B) such that

$$\left(-1 \leq B \leq -\frac{1}{2} \text{ and } 0 < A \leq \frac{4B + 2\sqrt{B^2 + 6}}{3} \right) \text{ or } \left(-\frac{1}{2} < B \leq 0 \text{ and } 0 < A \leq 1 \right).$$

Further, let Ω_2, Ω_3 and τ be the set of points (A, B) satisfying

$$\Omega_2 := \left\{ -1 \leq B \leq -\frac{1}{2} \text{ and } \frac{4B + 2\sqrt{B^2 + 6}}{3} < A \leq 1 \right\},$$

$$\Omega_3 := \left\{ -1 \leq B < 0 \text{ and } \delta_1 := \frac{\sqrt{16B^2 - 16B + 49} + 8B - 1}{6} < A \leq 1 \right\}$$

and

$$\tau := \frac{(3A^2 - 5AB + 2B^2 - 2)(A^2 - 3AB + 2B^2 - 2)}{4}.$$

Theorem 2. *Let $f \in \mathcal{S}^*[A, B]$. Then, the following best possible estimations hold:*

$$T_{3,1}(f) \leq \begin{cases} \tau, & (A, B) \in \Omega_1; \\ 1, & (A, B) \in \Omega_2 \text{ or } -1 \leq B < A \leq 0 \end{cases}$$

and

$$T_{3,1}(f) \geq \begin{cases} -\frac{(A^2-2AB+A+B^2-1)^2}{(A-2B+1)(3A-2B-1)}, & (A, B) \in \Omega_3; \\ \tau, & -1 \leq B \leq 0 < A \leq \delta_1 \text{ or } -1 \leq B < A \leq 0. \end{cases}$$

We owe to Libera and Zlotkiewicz for a result related to the class \mathcal{P} :

Lemma 1 ([16, Lemma 3, p. 254]). *Let $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \dots \in \mathcal{P}$. Then $2p_2 = p_1^2 + (4 - p_1^2)\zeta$ for some $\zeta \in \mathbb{D}$.*

Proof of Theorem 2. The well-known fact about the class of functions with positive real part is that it is rotationally invariant and for this reason, we shall limit ourselves to a consideration of non-negative value of p_1 and since $|p_1| \leq 2$, we can let $0 \leq p_1 \leq 2$ and hereafter we use $x := p_1^2 \in [0, 4]$ and $y := |\zeta| \in [0, 1]$. Keeping these in mind from (2.2), a computation gives

$$T_{3,1}(f) = 1 + \frac{1}{64}(A-2B)(3A-2B)(A-B)^2p_1^4 - \frac{(A-B)^2}{2}p_1^2 - \frac{(A-B)^2}{64}(4-p_1^2)^2|\zeta|^2 + \frac{1}{32}A(A-B)^2(4-p_1^2)p_1^2\Re\zeta \quad (2.5)$$

$$=: \Psi(p_1^2, |\zeta|, \Re\zeta). \quad (2.6)$$

We proceed in the proof in various cases.

Case I. If $-1 \leq B < A = 0$, then from (2.5), we have

$$\begin{aligned} T_{3,1}(f) &= \frac{1}{16}(4-B^2p_1^2)^2 - \frac{B^2}{64}(4-p_1^2)^2|\zeta|^2 \\ &\leq \frac{1}{16}(4-B^2p_1^2)^2 \\ &\leq 1 \end{aligned}$$

and the minimum is given by

$$\begin{aligned} T_{3,1}(f) &\geq \frac{1}{16}(4-B^2p_1^2)^2 - \frac{B^2}{64}(4-p_1^2)^2 \\ &\geq \frac{4-B^2}{4}. \end{aligned}$$

Case II. If $-1 \leq B \leq 0 < A \leq 1$, then with the settings $p_1^2 =: x \in [0, 4]$ and $|\zeta| =: y \in [0, 1]$, we have

$$T_{3,1}(f) = \Psi(p_1^2, |\zeta|, \Re\zeta) \leq \Psi(p_1^2, |\zeta|, |\zeta|) = \Psi(p_1^2, |\zeta|) = G(x, y).$$

Note that G is a continuously differentiable function of two variables x and y defined over the rectangular region $[0, 4] \times [0, 1]$ and is given by

$$G(x, y) := 1 + \frac{1}{64}(A-2B)(3A-2B)(A-B)^2x^2 - \frac{(A-B)^2}{2}x$$

$$-\frac{(A-B)^2}{64}(4-p_1^2)^2y^2 + \frac{1}{32}A(A-B)^2(4-x)xy. \quad (2.7)$$

On the boundary lines of the rectangular region $[0, 4] \times [0, 1]$, we see that

$$G(0, y) = 1 - \frac{1}{4}y^2(A-B)^2 \leq 1,$$

$$G(4, y) = \frac{1}{4}(3A^2 - 5AB + 2B^2 - 2)(A^2 - 3AB + 2B^2 - 2) =: \tau$$

and

$$G(x, 0) = 1 - \frac{1}{2}x(A-B)^2 + \frac{1}{64}x^2(A-2B)(3A-2B)(A-B)^2 =: f_1(x). \quad (2.8)$$

Now the positivity of the second derivative $f_1''(x) > 0$ assures that there is no maxima of f_1 for $0 < x < 4$. Therefore, the possibilities left are the end points and we have

$$f_1(x) \leq \max\{f_1(0), f_1(4)\} = f_1(4) = \tau.$$

Thus, for $-1 \leq B \leq 0 < A \leq 1$, we have

$$G(x, y) = \max\{1, \tau\} = \begin{cases} 1, & (A, B) \in \Omega_1; \\ \tau, & (A, B) \in \Omega_2. \end{cases}$$

To find the lower bound, we write

$$T_{3,1}(f) = \Psi(p_1^2, |\zeta|, \Re \zeta) \geq F(p_1^2, |\zeta|, -|\zeta|) \geq F(p_1^2, 1, -1) = f_2(x),$$

where the function $f_2 : [0, 4] \rightarrow \mathbb{R}$ is defined by

$$f_2(x) := 1 - \frac{(A-B)^2}{4} - \frac{(A+3)(A-B)^2}{8}x + \frac{(A-2B+1)(3A-2B-1)(A-B)^2}{64}x^2. \quad (2.9)$$

At the end points, we see that

$$f_2(0) = 1 - \frac{1}{4}(A-B)^2$$

and

$$f_2(4) = \frac{1}{4}(3A^2 - 5AB + 2B^2 - 2)(A^2 - 3AB + 2B^2 - 2) = \tau.$$

Now $f_2''(x) > 0$ holds for

$$\left(-1 \leq B \leq -\frac{1}{2} \text{ and } 0 < A \leq 1\right) \text{ or } \left(-\frac{1}{2} < B \leq 0 \text{ and } \frac{1}{3}(2B+1) < A \leq 1\right)$$

and the only critical point of f_2 is

$$x = x^* = \frac{4(A+3)}{(A-2B+1)(3A-2B-1)}.$$

A computation shows that $x^* \in (0, 4)$ if and only if $(A, B) \in \Omega_3$. But, since

$$2(2B + 1) \leq \sqrt{16B^2 - 16B + 49} + 8B - 1$$

holds for all $-1 \leq B \leq 0 < A \leq 1$, it follows that the minimum of f_2 attains at x^* for $(A, B) \in \Omega_3$ and

$$f_2(x^*) = -\frac{(A^2 - 2AB + A + B^2 - 1)^2}{(A - 2B + 1)(3A - 2B - 1)}.$$

From the above analysis, we conclude that

$$T_{3,1}(f) \geq \begin{cases} -\frac{(A^2 - 2AB + A + B^2 - 1)^2}{(A - 2B + 1)(3A - 2B - 1)}, & (A, B) \in \Omega_3; \\ \tau, & -1 \leq B \leq 0 \text{ and } 0 < A \leq \delta_1. \end{cases}$$

Case III. Let $-1 \leq B < A < 0$. Then, using (2.5) and the notations $p_1^2 =: x \in [0, 4]$ and $|\zeta| =: y \in [0, 1]$, we can write

$$T_{3,1}(f) = F(p_1^2, |\zeta|, \Re \zeta) \leq F(p_1^2, |\zeta|, -|\zeta|) = H(x, y),$$

where $H : [0, 4] \times [0, 1] \rightarrow \mathbb{R}$ is defined by

$$\begin{aligned} H(x, y) := & 1 + \frac{1}{64}(A - 2B)(3A - 2B)(A - B)^2 x^2 - \frac{(A - B)^2}{2} x \\ & - \frac{(A - B)^2}{64}(4 - p_1^2)^2 y^2 - \frac{1}{32} A(A - B)^2 (4 - x)xy. \end{aligned} \quad (2.10)$$

Now, at the boundary lines, we see that

$$H(0, y) = 1 - \frac{1}{4}(A - B)^2 y^2 \leq 1, \quad H(4, y) = \tau,$$

$$H(x, 0) = 1 - \frac{1}{2}(A - B)^2 x + \frac{1}{64}(A - 2B)(3A - 2B)(A - B)^2 x^2 = f_3(x)$$

and $H(x, 1) = f_2(x)$, where f_2 is as defined in (2.9). The first derivative of f_3 vanishes for $x = x_2 = 16/(A - 2B)(3A - 2B)$ and since $(A - 2B)(3A - 2B) < 4$ for $-1 \leq B < A < 0$, it follows that $x_2 \notin (0, 4)$ and therefore it is sufficient to consider the values of f_3 at the end points of the interval $(0, 4)$. Further, as previous, we see that $f_2'(x) = 0$ holds for $x = x_1$ and further, for $-1 \leq B < A < 0$, it can be verified that $x_1 \notin (0, 4)$. Thus the above discussion make us to write

$$T_{3,1}(f) \leq \max\{1, \tau\} = 1.$$

Now it remains to look for the claimed lower bound in the case $-1 \leq B < A < 0$. For this purpose, we write

$$T_{3,1}(f) = F(p_1^2, |\zeta|, \Re \zeta) \geq F(p_1^2, |\zeta|, |\zeta|) \geq F(p_1^2, 1, 1) = f_1(x),$$

where f_1 is defined by (2.8). As before, we see at the end points of $[0, 4]$ that

$$f_1(0) = 1 - \frac{1}{4}(A - B)^2 \text{ and } f_1(4) = \tau.$$

Also f_1 has no critical points inside the interval $(0, 4)$ and therefore,

$$T_{3,1}(f) \geq \min \left\{ 1 - \frac{1}{4}(A - B)^2, \tau \right\} = \tau.$$

Case IV. In this case, we consider the situation when $-1 \leq B < A = 0$. Upon setting $A = 0$ in (2.5), we have

$$\begin{aligned} T_{3,1}(f) &= \frac{(4 - Bp_1^2)^2}{16} - \frac{B^2(4 - p_1^2)^2}{64} |\zeta|^2 \\ &\leq \frac{(4 - Bp_1^2)^2}{16} \\ &\leq 1 \end{aligned}$$

and

$$\begin{aligned} T_{3,1}(f) &= \frac{(4 - Bp_1^2)^2}{16} - \frac{B^2(4 - p_1^2)^2}{64} |\zeta|^2 \\ &\geq \frac{(4 - Bp_1^2)^2}{16} - \frac{B^2(4 - p_1^2)^2}{64} \\ &\geq \frac{(4 - B)^2}{4}. \end{aligned}$$

The functional $T_{3,1}(f)$ equals τ in case of the function \tilde{h}_0 defined in (2.3). Further $T_{3,1}(f)$ equals τ for the function

$$\tilde{h}_2(z) = \begin{cases} z(1 + Bz^3)^{\frac{A-B}{3B}}, & B \neq 0; \\ ze^{Az^3/3}, & B = 0. \end{cases} \tag{2.11}$$

For $(A, B) \in \Omega_3$, the extremal function \tilde{h}_3 satisfying (2.1) with the function p is replaced by \tilde{p} defined as

$$\tilde{p}(z) = \frac{1 - z^2}{1 - 2\sqrt{t}z + z^2}, \quad t := x^*/4.$$

The Taylor series of \tilde{h}_3 is given by

$$\tilde{h}_3(z) = z + \sqrt{t}(A - B)z^2 + \frac{(A - B)((A - 2B + 1)t - 1)}{2}z^3 + \dots$$

and thus we have

$$T_{3,1}(f) = 1 + 2a_2^2a_3 - |a_2|^2 - |a_3|^2 = -\frac{(A^2 - 2AB + A + B^2 - 1)^2}{(A - 2B + 1)(3A - 2B - 1)}.$$

Cases I to IV, together, bring the proof to an end. □

Remark 1. For $A = 1 - 2\alpha$ and $B = -1$, Theorems 1 and 2 reduce to the results [7, Theorem 2] and [7, Theorem 3], respectively.

3. JANOWSKI TYPE CONVEX FUNCTIONS

In this section, the sharp bounds for the Hermitian-Toeplitz determinant of the third order for the classes of Janowski type convex functions are investigated.

For any function $f \in \mathcal{K}[A, B]$, there is a function with positive real part $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$ such that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{(1+A)p(z) + 1 - A}{(1+B)p(z) + 1 - B}. \quad (3.1)$$

Expanding both sides of the above in Taylor series and comparing the coefficients of similar terms, we get

$$a_2 = \frac{A-B}{4} p_1 \quad \text{and} \quad a_3 = \frac{A-B}{4} [(A-2B-1)p_1^2 + 2p_2]. \quad (3.2)$$

Clearly, since $|a_2| \leq (A-B)/2$, it follows that $(4 - (A-B)^2)/4 \leq T_{3,1}(f) \leq 1$. These lower bound is sharp in case of the function k_0 satisfying (3.1) with $p(z) = (1+z)/(1-z)$ and that the upper bound is sharp in case of the function k_1 satisfying (3.1) with $p(z) = (1+z^2)/(1-z^2)$. Thus, we have the following:

Theorem 3. *Let $f \in \mathcal{K}[A, B]$. Then, the following best possible estimations hold:*

$$\frac{4 - (A-B)^2}{4} \leq T_{2,1}(f) \leq 1.$$

Theorem 4. *Let $f \in \mathcal{K}[A, B]$. Then, the following best possible estimations hold:*

$$\sigma \leq |T_{3,1}(f)| \leq 1,$$

where

$$\sigma := \frac{1}{36} (2A^2 - 3AB + B^2 - 6) (A^2 - 3AB + 2B^2 - 6).$$

Proof. Using (3.2) computations give

$$T_{3,1}(f) = 1 + \frac{(A-B)^2(A-2B)(2A-B)}{576} p_1^4 - \frac{(A-B)^2}{8} p_1^2 - \frac{(A-B)^2}{576} (4-p_1^2)^2 |\zeta|^2 + \frac{(A-B)^2(A+B)}{576} (4-p_1^2) p_1^2 \Re \zeta =: \Upsilon(p_1^2, |\zeta|, \Re \zeta). \quad (3.3)$$

Now the proof will be completed in a few cases.

Case I. Let $A+B > 0$. Then, since the coefficient of $\Re \zeta$ is positive or equal to zero, it follows that

$$\Upsilon(p_1^2, |\zeta|, \Re \zeta) \leq \Upsilon(p_1^2, |\zeta|, |\zeta|) = G(x, y),$$

where $x = p_1^2, y = |\zeta|$ and the function $G : [0, 4] \times [0, 1] \rightarrow \mathbb{R}$ is defined by

$$G(x, y) := 1 - \frac{(A-B)^2}{8} x + \frac{(2A-B)(A-2B)(A-B)^2}{576} x^2$$

$$+ \frac{(A+B)(A-B)^2}{576}(4-x)xy - \frac{(A-B)^2}{576}(4-x)^2y^2.$$

Now on the boundary lines of the rectangular region $[0, 4] \times [0, 1]$, we see that

$$\begin{aligned} G(0, y) &= 1 - \frac{1}{36}y^2(A-B)^2, \\ G(4, y) &= \frac{1}{36}(2A^2 - 3AB + B^2 - 6)(A^2 - 3AB + 2B^2 - 6) =: \sigma, \\ G(x, 0) &= 1 - \frac{(A-B)^2}{8}x + \frac{(2A-B)(A-2B)(A-B)^2}{576}x^2 =: g_1(x) \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} G(x, 1) &= 1 - \frac{1}{36}(A-B)^2 + \frac{1}{144}(A+B-16)(A-B)^2x \\ &\quad + \frac{1}{576}(2A-B+1)(A-2B-1)(A-B)^2x^2 =: g_2(x). \end{aligned} \quad (3.5)$$

Consider the functions g_1 and g_2 . The first derivative of these functions, namely $g_1'(x)$ and $g_2'(x)$ vanish for $x = x_3 = 36/(A-2B)(2A-B)$ and $x = x_4 = -2(A+B-16)/(A-2B-1)(2A-B+1)$, respectively. Since $(A-2B)(2A-B) < 9$ and $-(A-2B-1)(2A-B+1) < 9$, it follows that none of the x_3 and x_4 lie inside the interval $(0, 4)$. At the end points, we have $g_1(0) = 1$, $g_1(4) = \sigma = g_2(4)$ and $g_2(0) = 1 - (A-B)^2/36$. Therefore, we conclude that

$$T_{3,1}(f) \leq \max \left\{ 1, \sigma, 1 - \frac{(A-B)^2}{36} \right\} = 1.$$

Next we find the minimum for $T_{3,1}(f)$. For $A+B > 0$, with the setting $p_1^2 = x$, we can write

$$\Upsilon(p_1^2, |\zeta|, \Re \zeta) \geq \Upsilon(p_1^2, |\zeta|, -|\zeta|) \geq \Upsilon(p_1^2, 1, -1) = g_3(x),$$

where the function $g_3 : [0, 4] \rightarrow \mathbb{R}$ is defined by

$$\begin{aligned} g_3(x) := & 1 - \frac{(A-B)^2}{36} - \frac{(A+B+16)(A-B)^2}{144}x \\ & + \frac{(2A-B-1)(A-2B+1)(A-B)^2}{576}x^2. \end{aligned} \quad (3.6)$$

The first derivative of g_3 vanishes for $x = x_5 = 2(A+B+16)/(A-2B+1)(2A-B-1)$. But since $2(A+B+16) > 32$ and $(A-2B+1)(2A-B-1) < 8$, we conclude that $x_5 \notin (0, 4)$ and therefore, it suffices to consider the values of g_3 at the end points of the interval $[0, 4]$, and we get

$$g_3(0) = 1 - \frac{(A-B)^2}{36} \quad \text{and} \quad g_3(4) = \sigma.$$

From the above discussion, we conclude that

$$T_{3,1}(f) \geq \min \left\{ \sigma, 1 - \frac{(A-B)^2}{36} \right\} = \sigma.$$

Case II. Let $A+B < 0$. Then, since the coefficient of $\Re \zeta$ is negative or equal to zero, it follows that

$$\Upsilon(p_1^2, |\zeta|, \Re \zeta) \leq \Upsilon(p_1^2, |\zeta|, -|\zeta|) = H(x, y),$$

where $x = p_1^2, y = |\zeta|$ and the function $H : [0, 4] \times [0, 1] \rightarrow \mathbb{R}$ is defined by

$$H(x, y) = 1 - \frac{(A-B)^2}{8}x + \frac{(2A-B)(A-2B)(A-B)^2}{576}x^2 - \frac{(A+B)(A-B)^2}{576}(4-x)xy - \frac{(A-B)^2}{576}(4-x)^2y^2.$$

Now on the boundary of the rectangular region $[0, 4] \times [0, 1]$, we have

$$H(0, y) = 1 - \frac{(A-B)^2}{36}y^2 \leq 1, H(4, y) = \sigma, H(x, 0) = g_1(x) \text{ and } H(x, 1) = g_3(x),$$

where the functions g_1 and g_3 are defined by (3.4) and (3.6), respectively. Note that at the end point of $[0, 4]$, $g_1(0) = 1 = g_3(0)$, $g_1(4) = \sigma = g_3(4)$ and $g_1'(x) = 0$ has no critical point inside the interval $(0, 4)$ as $(A-2B)(2A-B) < 9$ for all $-1 \leq B < A \leq 1$. Similarly, it can be proved that the function $g_3'(x) = 0$ has no root inside the interval $(0, 4)$. From the above discussion, for $A+B < 0$, we conclude that

$$T_{3,1}(f) \leq \max \left\{ 1, \sigma, 1 - \frac{(A-B)^2}{36} \right\} = 1.$$

We now proceed to find the minimum of $T_{3,1}(f)$ for the case $-1 \leq B < A < 0$. For this, with the setting $x = p_1^2$, we can write

$$\Upsilon(p^2, |\zeta|, \zeta) \geq \Upsilon(p^2, |\zeta|, |\zeta|) \geq \Upsilon(p^2, 1, 1) = g_2(x),$$

where the function g_2 is defined by (3.6). As in the Case II, it can be established that g_2 has no root in the interval $(0, 4)$ and $g_2(0) = 1 - (A-B)^2/36$ and $g_2(4) = \sigma$. Thus, for $-1 \leq B < A < 0$, we have

$$|T_{3,1}(f)| \geq \min \left\{ 1 - \frac{(A-B)^2}{36}, \sigma \right\} = \sigma.$$

Case III. Let $A+B = 0$. Then, from (3.3), we get

$$\begin{aligned} T_{3,1}(f) &= 1 - \frac{A^2}{2}x + \frac{A^4}{16}x^2 - \frac{A^2}{144}(4-x)^2y^2 \\ &\leq \frac{1}{16}(A^2x-4)^2 \\ &\leq 1. \end{aligned}$$

We also have

$$\begin{aligned} T_{3,1}(f) &= 1 - \frac{A^2}{2}x + \frac{A^4}{16}x^2 - \frac{A^2}{144}(4-x)^2y^2 \\ &\geq 1 - \frac{A^2}{9} - \frac{4A^2}{9}x + \frac{(9A^2-1)A^2}{144}x^2. \\ &=: h(x). \end{aligned}$$

The first derivative of h vanishes only at $x = x_5 = 32/(9A^2 - 1)$ and $x_5 \notin (0, 4)$. At the end points of the interval $(0, 4)$, we have $h(0) = 1 - A^2/9$ and $h(4) = (A^2 - 1)^2$. Thus for $A + B = 0$, we conclude that

$$(A^2 - 1)^2 \leq |T_{3,1}(f)| \leq 1, \text{ or equivalently } (B^2 - 1)^2 \leq |T_{3,1}(f)| \leq 1.$$

We put together the results discussed in the above three cases to conclude the assertion of the theorem. These lower bound is sharp in case of the function k_0 satisfying (3.1) with $p(z) = (1+z)/(1-z)$ and that the upper bound is sharp in case of the function k_1 satisfying (3.1) with $p(z) = (1+z^2)/(1-z^2)$. This ends the proof. \square

Remark 2. For $A = 1 - 2\alpha$ and $B = -1$, Theorems 3 and 4 reduce to the results [7, Theorem 4] and [7, Theorem 5], respectively.

REFERENCES

- [1] M. F. Ali, D. K. Thomas, and A. Vasudevarao, "Toeplitz determinants whose elements are the coefficients of analytic and univalent functions," *Bull. Aust. Math. Soc.*, vol. 97, no. 2, pp. 253–264, 2018, doi: [10.1017/S0004972717001174](https://doi.org/10.1017/S0004972717001174). [Online]. Available: <https://doi.org/10.1017/S0004972717001174>
- [2] M. Arif, O. Barkub, H. M. Srivastava, S. Abdullah, and S. A. Khan, "Some Janowski type harmonic q -starlike functions associated with symmetrical points," *Mathematics*, vol. 8, no. 629, pp. 1–16, 2020, doi: [10.3390/math8040629](https://doi.org/10.3390/math8040629). [Online]. Available: <https://doi.org/10.3390/math8040629>
- [3] K. O. Babalola, "On $H_{3,1}$ Hankel determinants for some classes of univalent functions. In: Cho, Y. J. and Kim, J. K. and Dragomir, S. S. (eds.), *Inequality theory and applications*, vol. 6, pp. 1–7, 2010," *Nova Science Publishers, New York*, 2009.
- [4] D. G. Cantor, "Power series with integral coefficients," *Bull. Amer. Math. Soc.*, vol. 69, pp. 362–366, 1963, doi: [10.1090/S0002-9904-1963-10923-4](https://doi.org/10.1090/S0002-9904-1963-10923-4). [Online]. Available: <https://doi.org/10.1090/S0002-9904-1963-10923-4>
- [5] N. E. Cho and V. Kumar, "Third order Hermitian-Toeplitz determinant for functions with bounded turning," *preprint*, 2020.
- [6] N. E. Cho, V. Kumar, O. S. Kwon, and Y. J. Sim, "Coefficient bounds for certain subclasses of starlike functions," *J. Inequal. Appl.*, pp. Paper No. 276, 13, 2019, doi: [10.1186/s13660-019-2231-3](https://doi.org/10.1186/s13660-019-2231-3). [Online]. Available: <https://doi.org/10.1186/s13660-019-2231-3>
- [7] K. Cudna, K. O. S., A. Lecko, Y. J. Sim, and B. Śmiarowska, "The second and third-order Hermitian-Toeplitz determinants for starlike and convex functions of order α ," *Bol. Soc. Mat. Mex. (3)*, 2020, doi: [10.1007/s40590-019-00271-1](https://doi.org/10.1007/s40590-019-00271-1). [Online]. Available: <https://doi.org/10.1007/s40590-019-00271-1>
- [8] A. W. Goodman, *Univalent functions. Vol. II*. Mariner Publishing Co., Inc., Tampa, FL, 1983.

- [9] H. O. Güney, G. Murugusundaramoorthy, and H. M. Srivastava, "The second Hankel determinant for a certain class of bi-close-to-convex functions," *Results Math.*, vol. 74, no. 3, pp. Paper No. 93, 13, 2019, doi: [10.1007/s00025-019-1020-0](https://doi.org/10.1007/s00025-019-1020-0). [Online]. Available: <https://doi.org/10.1007/s00025-019-1020-0>
- [10] W. K. Hayman, "On the second Hankel determinant of mean univalent functions," *Proc. London Math. Soc. (3)*, vol. 18, pp. 77–94, 1968, doi: [10.1112/plms/s3-18.1.77](https://doi.org/10.1112/plms/s3-18.1.77). [Online]. Available: <https://doi.org/10.1112/plms/s3-18.1.77>
- [11] W. Janowski, "Some extremal problems for certain families of analytic functions. I," *Ann. Polon. Math.*, vol. 28, pp. 297–326, 1973, doi: [10.4064/ap-28-3-297-326](https://doi.org/10.4064/ap-28-3-297-326). [Online]. Available: <https://doi.org/10.4064/ap-28-3-297-326>
- [12] A. Janteng, S. A. Halim, and M. Darus, "Hankel determinant for starlike and convex functions," *Int. J. Math. Anal. (Ruse)*, vol. 1, no. 13-16, pp. 619–625, 2007.
- [13] B. a. Kowalczyk, A. Lecko, and Y. J. Sim, "The sharp bound for the Hankel determinant of the third kind for convex functions," *Bull. Aust. Math. Soc.*, vol. 97, no. 3, pp. 435–445, 2018, doi: [10.1017/S0004972717001125](https://doi.org/10.1017/S0004972717001125). [Online]. Available: <https://doi.org/10.1017/S0004972717001125>
- [14] O. S. Kwon, A. Lecko, and Y. J. Sim, "The bound of the Hankel determinant of the third kind for starlike functions," *Bull. Malays. Math. Sci. Soc.*, vol. 42, no. 2, pp. 767–780, 2019, doi: [10.1007/s40840-018-0683-0](https://doi.org/10.1007/s40840-018-0683-0). [Online]. Available: <https://doi.org/10.1007/s40840-018-0683-0>
- [15] S. K. Lee, V. Ravichandran, and S. Supramaniam, "Bounds for the second Hankel determinant of certain univalent functions," *J. Inequal. Appl.*, pp. 2013:281, 17, 2013, doi: [10.1186/1029-242X-2013-281](https://doi.org/10.1186/1029-242X-2013-281). [Online]. Available: <https://doi.org/10.1186/1029-242X-2013-281>
- [16] R. J. Libera and E. J. Zlotkiewicz, "Coefficient bounds for the inverse of a function with derivative in \mathcal{P} ," *Proc. Amer. Math. Soc.*, vol. 87, no. 2, pp. 251–257, 1983, doi: [10.2307/2043698](https://doi.org/10.2307/2043698). [Online]. Available: <https://doi.org/10.2307/2043698>
- [17] S. Mahmood, H. M. Srivastava, N. Khan, Q. Z. Ahmad, B. Khan, and I. Ali, "Upper bound of the third Hankel determinant for a subclass of q -starlike functions," *Symmetry*, vol. 11, no. 3(347), pp. 1–13, 2019, doi: [10.3390/sym11030347](https://doi.org/10.3390/sym11030347). [Online]. Available: <https://doi.org/10.3390/sym11030347>
- [18] C. Pommerenke, "On the coefficients and Hankel determinants of univalent functions," *J. London Math. Soc.*, vol. 41, pp. 111–122, 1966, doi: [10.1112/jlms/s1-41.1.111](https://doi.org/10.1112/jlms/s1-41.1.111). [Online]. Available: <https://doi.org/10.1112/jlms/s1-41.1.111>
- [19] M. I. S. Robertson, "On the theory of univalent functions," *Ann. of Math. (2)*, vol. 37, no. 2, pp. 374–408, 1936, doi: [10.2307/1968451](https://doi.org/10.2307/1968451). [Online]. Available: <https://doi.org/10.2307/1968451>
- [20] L. Shi, H. M. Srivastava, M. Arif, S. Hussain, and H. Khan, "An investigation of the third Hankel determinant problem for certain subfamilies of univalent functions involving the exponential function," *Symmetry*, vol. 11, no. 5, pp. 1–14, 2019, doi: [10.3390/sym11050598](https://doi.org/10.3390/sym11050598). [Online]. Available: <https://doi.org/10.3390/sym11050598>
- [21] H. M. Srivastava, "Operators of Basic (or q -) Calculus and Fractional q -Calculus and Their Applications in Geometric Function Theory of Complex Analysis," *Iran. J. Sci. Technol. Trans. A Sci.*, vol. 44, no. 1, pp. 327–344, 2020, doi: [10.1007/s40995-019-00815-0](https://doi.org/10.1007/s40995-019-00815-0). [Online]. Available: <https://doi.org/10.1007/s40995-019-00815-0>
- [22] H. M. Srivastava, Q. Z. Ahmad, M. Darus, N. Khan, B. Khan, N. Zaman, and H. H. Shah, "Upper bound of the third Hankel determinant for a subclass of close-to-convex functions associated with the lemniscate of Bernoulli," *Mathematics*, vol. 7, no. 848, pp. 1–10, 2019, doi: [10.3390/math7090848](https://doi.org/10.3390/math7090848). [Online]. Available: <https://doi.org/10.3390/math7090848>
- [23] H. M. Srivastava, Q. Z. Ahmad, N. Khan, and B. Khan, "Hankel and Toeplitz determinants for a subclass of q -starlike functions associated with a general conic domain," *Mathematics*, vol. 7, no. 2, pp. 1–15, 2019, doi: [10.3390/math7020181](https://doi.org/10.3390/math7020181). [Online]. Available: <https://doi.org/10.3390/math7020181>

- [24] H. M. Srivastava, B. Khan, N. Khan, and Q. Z. Ahmad, “Coefficient inequalities for q -starlike functions associated with the Janowski functions,” *Hokkaido Math. J.*, vol. 48, no. 2, pp. 407–425, 2019, doi: [10.14492/hokmj/1562810517](https://doi.org/10.14492/hokmj/1562810517). [Online]. Available: <https://doi.org/10.14492/hokmj/1562810517>
- [25] H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad, and N. Khan, “Some general classes of q -starlike functions associated with the Janowski functions,” *Symmetry*, vol. 11, no. 2, pp. 1–14, 2019, doi: [10.3390/sym11020292](https://doi.org/10.3390/sym11020292). [Online]. Available: <https://doi.org/10.3390/sym11020292>
- [26] H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad, and N. Khan, “Some general families of q -starlike functions associated with the Janowski functions,” *Filomat*, vol. 33, no. 9, pp. 2613–2626, 2019, doi: [10.2298/FIL1909613S](https://doi.org/10.2298/FIL1909613S). [Online]. Available: <https://doi.org/10.2298/FIL1909613S>

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