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A NOTE ON COMMON FIXED POINT THEOREMS IN PARTIAL METRIC SPACES

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Abstract. In this manuscript, we consider the notion of generalized Sehgal contraction condition in a partial metric space. For the pair of two self mappings (S,T) which satisfies Sehgal contraction condition, we obtain a unique common fixed point.

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1. Introduction and preliminaries

Partial metric space (in short PMS), is one of the attempts to generalize the notion of the metric space that by replacing the condition $d(x,x) = 0$ with the condition $d(x,x) \leq d(x,y)$ for all x, y in the definition of the metric (see e.g.[14, 15]). In these initial papers, Matthews discussed not only the general topological properties of partial metric spaces but also some properties of convergence of sequences. In [14, 15], he proved a fixed point theorem for contractive mappings of partial metric spaces: Any mapping T of a complete partial metric space X into itself that satisfies, for some $0 \leq k < 1$, the inequality $d(Tx,Ty) \leq kd(x,y)$, for all $x, y \in X$, has a unique fixed point. Recently, many authors (see e.g.[1, 2, 8, 10–17, 20]) focused on this subject and generalized some fixed point theorems from the class of metric spaces to the class of partial metric spaces. In this manuscript, we discuss existence and uniqueness of a common fixed point of self-mappings S, T of partial metric spaces.

A partial metric space (See e.g.[14,15]) is a pair $(X, p : X \times X \to \mathbb{R}^+)$ (where $\mathbb{R}^+$ denotes the set of all non negative real numbers) such that

(PM1) $p(x,y) = p(y,x)$ (symmetry)
(PM2) If $0 \leq p(x,x) = p(x,y) = p(y,y)$ then $x = y$ (equality)
(PM3) $p(x,x) \leq p(x,y)$ (small self-distances)
(PM4) $p(x,z) + p(y,y) \leq p(x,y) + p(y,z)$ (triangularity)

for all $x, y, z \in X$. For a partial metric p on X, the functions $d_p, d_m : X \times X \to \mathbb{R}^+$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{1.1}$$

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and

$$d_m(x,y) = \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\} \tag{1.2}$$

are (usual) metrics on X. It is clear that d_p and d_m are equivalent. Each partial metric p on X generates a T_0 topology τ_p on X with a base consisting of the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Example 1. A basic example of partial metric is $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$. It is clear that p is not a (usual) metric. Note that in this case $d_p(x,y) = |x-y|$ and $d_m(x,y) = \frac{1}{2}|x-y|$.

Example 2. (See [8]) Let $X = \{[a,b] : a,b, \in \mathbb{R},\ a \le b\}$ and define $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then (X,p) is a partial metric spaces.

Example 3. (See [8]) Let $X := [0,1] \cup [2,3]$ and define $p : X \times X \to [0,\infty)$ by
$$p(x,y) = \begin{cases} \max\{x,y\} & \text{if } \{x,y\} \cap [2,3] \neq \varnothing, \\ |x-y| & \text{if } \{x,y\} \subset [0,1]. \end{cases}$$
Then (X, p) is a complete partial metric space.

Definition 1. (See e.g. [14, 15])

- (i) A sequence $\{x_n\}$ in a PMS (X,p) converges to $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$,
- (ii) a sequence $\{x_n\}$ in a PMS (X,p) is called a Cauchy if and only if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists (and finite),
- (iii) A PMS (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.
- (iv) A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0,\delta)) \subset B(f(x_0),\varepsilon)$.

Lemma 1. *(See e.g.[14, 15])*

- (A) *A sequence $\{x_n\}$ is Cauchy in a PMS (X,p) if and only if $\{x_n\}$ is Cauchy in a metric space (X,d_p),*
- (B) *A PMS (X,p) is complete if and only if a metric space (X,d_p) is complete. Moreover,*

$$\lim_{n\to\infty} d_p(x,x_n) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m) \tag{1.3}$$

Remark 1. Since d_p and d_m are equivalent, we can take d_p instead of d_m in this Lemma.

Let (X,p) be a PMS and denote the closure of the set $\{p(x,y) : x,y \in X\}$ by P and $P^3 = P \times P \times P$. A function $\phi : P^3 \to \mathbb{R}^+$ is right continuous if and only if

- (S1) the sequences $\{a_n\}, \{b_n\}, \{c_n\}$ decrease and converge to $a,b,c \in P$, respectively, then

$$\phi(a_n, b_n, c_n) \to \phi(a, b, c).$$

The function ϕ is called symmetric if and only if

$$\phi(a, b, c) = \phi(b, a, c), \text{ for all } (a, b, c) \in P^3.$$

In the spirit of Sehgal [19], we state the following definition for partial metric spaces.

Definition 2. Let (X, p) be a PMS and $S, T : X \to X$ be two mappings. The pair (S, T) is said to satisfy Sehgal k-condition if and only if there are maps $I_S : S \times X \to \mathbb{Z}^+$ and $I_T : T \times X \to \mathbb{Z}^+$ such that if $r(x) = I_S(S, x)$ and $q(x) = I_T(T, x)$, then

$$p(S^{r(x)}x, T^{q(y)}y) \leq k\phi(p(S^{r(x)}x, x), p(y, T^{q(y)}y), p(x, y)) \tag{1.4}$$

for all $x, y \in X$, where $k \in \mathbb{R}$ and ϕ is a symmetric right continuous. If $0 \leq k < 1$, then we say that (S, T) satisfy Sehgal contraction condition.

2. Main results

The following two lemmas are easy to prove but they will be very useful in the proof of the main theorem.

Lemma 2. *(See e.g. [1, 10]) Let (X, p) be a complete PMS. Then*

(A) *If $p(x, y) = 0$ then $x = y$,*
(B) *If $x \neq y$, then $p(x, y) > 0$.*

Lemma 3. *(See e.g. [1, 10]) Assume $x_n \to z$ as $n \to \infty$ in a PMS (X, p) such that $p(z, z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

The following theorem extends the results of [19].

Theorem 1. *Let (X, p) be a complete partial metric space. Suppose that $S, T : X \to X$ are two mappings such that the pair (S, T) satisfies Sehgal's contraction contraction.*

(A) *If $\phi(a, b, c) \leq \max\{a, b, c\}$, for $(a, b, c) \in P^3$, then S and T have a unique common fixed point in X, that is, $S^{r(z)}z = T^{q(z)}z = z$.*

Proof. Let $x_0 \in X$. Define the sequence $\{x_n\}_{n=1}^{\infty}$ in a way that $x_2 = T^{q(x_1)}x_1$ and $x_1 = S^{r(x_0)}x_0$ and inductively

$$x_{2n+2} = T^{q(x_{2n+1})}x_{2n+1} \quad \text{and} \quad x_{2n+1} = S^{r(x_{2n})}x_{2n} \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

If n is odd, due to (1.4), we have

$$p(x_{n+1}, x_{n+2}) = p(Tx_n, Sx_{n+1}) \leq k\phi(p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), p(x_n, x_{n+1})) \tag{2.1}$$

Regarding the assumption of (A),

$$p(x_{n+1},x_{n+2}) = p(Tx_n,Sx_{n+1}) \leq k\max\{p(x_n,x_{n+1}),p(x_{n+1},x_{n+2})\} \quad (2.2)$$

If $\max\{p(x_n,x_{n+1}),p(x_{n+1},x_{n+2})\} = p(x_{n+1},x_{n+2})$ then the expression (2.2) turns into

$$p(x_{n+1},x_{n+2}) \leq kp(x_{n+1},x_{n+2}).$$

Since $k < 1$, this is impossible. Thus, we have

$$p(x_{n+1},x_{n+2}) \leq kp(x_n,x_{n+1}). \quad (2.3)$$

If n is even, analogously we observe that $p(x_{n+1},x_{n+2}) \leq kp(x_n,x_{n+1})$. Observe that $\{p(x_n,x_{n+1})\}$ is a non-negative, non-increasing sequence of reals. Regarding (2.3), one can observe that

$$p(x_n,x_{n+1}) \leq k^n p(x_0,x_1), \ \forall n = 0,1,2,\cdots \quad (2.4)$$

Letting $n \to \infty$, the right hand side of (2.4) tends to zero.

Consider now

$$\begin{aligned} d_p(x_{n+1},x_{n+2}) &= 2p(x_{n+1},x_{n+2}) - p(x_{n+1},x_{n+1}) - p(x_{n+2},x_{n+2}) \\ &\leq 2p(x_{n+1},x_{n+2}) \\ &\leq 2k^{n+1}p(x_0,x_1). \end{aligned} \quad (2.5)$$

Hence, regarding (2.4), we have $\lim_{n\to\infty} d_p(x_{n+1},x_{n+2}) = 0$. Moreover,

$$\begin{aligned} d_p(x_{n+1},x_{n+s}) &\leq d_p(x_{n+s-1},x_{n+s}) + \cdots + d_p(x_{n+1},x_{n+2}) \\ &\leq 2k^{n+s}p(x_0,x_1) + \cdots + 2k^{n+1}p(x_0,x_1) \end{aligned} \quad (2.6)$$

which implies that $\{x_n\}$ is a Cauchy sequence in (X,d_p) that is, $d_p(x_n,x_m) \to 0$. Since (X,p) is complete, by Lemma 1.3, (X,d_p) is complete and the sequence $\{x_n\}$ is convergent in (X,d_p), say to $z \in X$.

By Lemma 1.3,

$$p(z,z) = \lim_{n\to\infty} p(x_n,z) = \lim_{n,m\to\infty} p(x_n,x_m) \quad (2.7)$$

Since $\{x_n\}$ is a Cauchy sequence in (X,d_p), we have $\lim_{n,m\to\infty} d_p(x_n,x_m) = 0$. Since

$$\max\{p(x_n,x_n),p(x_{n+1},x_{n+1})\} \leq p(x_n,x_{n+1}),$$

then by (2.4), it follows that

$$\max\{p(x_n,x_n),p(x_{n+1},x_{n+1})\} \leq k^{n+1}p(x_0,x_1) \quad (2.8)$$

Thus from (2.4), (2.8) and from the definition of d_p, we have $\lim_{n,m\to\infty} p(x_n,x_m) = 0$. Therefore from (2.7) we have

$$p(z,z) = \lim_{n\to\infty} p(x_n,z) = \lim_{n,m\to\infty} p(x_n,x_m) = 0. \quad (2.9)$$

We assert that $T^{q(z)}z = z$. Assume $T^{q(z)}z \neq z$, then $p(z, T^{q(z)}z) > 0$. Let $\{x_{2n(i)}\}$ be subsequence of $\{x_{2n}\}$ and hence $\{x_n\}$. Due to (PM4), we have

$$\begin{aligned} p(Sx_{2n(i)}, T^{q(z)}z) &= p(Sx_{2n(i)}, T^{q(z)}z) \\ &\leq k\phi(p(x_{2n(i)}, x_{2n(i)+1}), p(T^{q(z)}z, z), p(x_{2n(i)}, z)) \end{aligned} \tag{2.10}$$

Letting $n \to \infty$ and taking the assumption of (A) and (2.9) into account, we get that

$$p(z, T^{q(z)}z) \quad \leq k\phi(0, p(T^{q(z)}z, z), 0) \leq kp(T^{q(z)}z, z) \tag{2.11}$$

Since $k < 1$, then $p(T^{q(z)}z, z) = 0$. By Lemma 2, we get $T^{q(z)}z = z$. Analogously, if we choose a subsequence $\{x_{2n(i)+1}\}$ be subsequence of $\{x_{2n+1}\}$, we obtain $S^{r(z)}z = z$.

Assume now that there exists $w \in X$ such that $S^{r(w)} = T^{q(w)}w = w$. By (PM3)

$$p(z, z) \quad \leq p(z, w) \text{ and } p(w, w) \quad \leq p(z, w) \tag{2.12}$$

Regarding that the function ϕ satisfies the condition of (A) with (2.12), we get

$$\begin{aligned} p(z, w) = p(S^{r(z)}z, T^{q(w)}w) \quad &\leq k\phi(p(z, S^{r(z)}z), p(T^{q(w)}w, w), p(z, w)) \\ &\leq k\phi(p(z, z), p(w, w), p(z, w)) \\ &\leq kp(z, w) \end{aligned}$$

Since $k < 1$, it yields a contradiction.

Thus, $p(z, w) = 0$ and by Lemma 2 we have $z = w$. □

Corollary 1. *Let (X, p) be a complete partial metric space. Suppose I_T and I_S are defined as above. $S, T : X \to X$ are two mappings such that the pair (S, T) satisfies one of the following condition:*

(A) $p(S^{r(x)}x, T^{q(y)}y) \leq k \max\{p(S^{r(x)}x, x), p(y, T^{q(y)}y), p(x, y)\}$ *for some* $0 \leq k < 1$,

(B) $p(S^{r(x)}x, T^{q(y)}y) \leq \alpha p(S^{r(x)}x, x) + \beta p(y, T^{q(y)}y) + \gamma p(x, y)$ *for some non-negative reals* α, β, γ *with* $\alpha + \beta + \gamma < 1$.

Then S and T have a unique common fixed point in X, that is, $S^{r(z)}z = T^{q(z)}z = z$.

Proof. For (A), we choose a function $\phi(a, b, c) = \max\{a, b, c\}$ as in Theorem 1. In case of (B), set $k = \alpha + \beta + \gamma$. Then (A) implies (B). □

Notice that this corollary generalizes also some results of ([6] -[4]).

Corollary 2. *Let (X, p) be a complete partial metric space. Let $S, T : X \to X$ be two mappings such that the pair (S, T) satisfies the following condition:*

$$p(S^r x, T^q y) \leq k\phi(p(S^r x, x), p(y, T^q y), p(x, y)) \tag{2.13}$$

for all $x, y \in X$ where $0 \leq k < 1$ and ϕ is symmetric right-continuous. If $\phi(a, b, c) \leq \max\{a, b, c\}$ then S and T have a unique common fixed point theorem.

Proof. By Theorem 1,and by taking the maps I_T, I_S as a constant, we get that S^r and T^q have a unique common fixed point, say $z \in X$. Now consider

$$S^q(Sz) = S^{q+1}z = S(S^q z) = Sz$$

which says that Sz is a fixed point of S^q. Since z is the unique fixed point of S^q, then $Sz = z$. Analogously, one can get $Tz = z$. □

Corollary 3. *Let (X, p) be a complete partial metric space. Let $S, T : X \to X$ be two mappings such that the pair (S, T) satisfies the following condition:*

$$p(S^r x, T^q y) \leq k \max\{p(S^r x, x), p(y, T^q y), p(x, y)\} \tag{2.14}$$

for all $x, y \in X$ where $0 \leq k < 1$ and ϕ is symmetric right-continuous. If $\phi(a,b,c) \leq \max\{a,b,c\}$ then S and T have a unique common fixed point theorem.

Corollary 4. *Let (X, p) be a complete partial metric space. Let $S, T : X \to X$ be two mappings such that the pair (S, T) satisfies the following condition:*

$$p(S^r x, T^q y) \leq \alpha p(S^r x, x) + \beta p(y, T^q y) + \gamma p(x, y) \tag{2.15}$$

for all $x, y \in X$, where for some non-negative reals α, β, γ with $\alpha + \beta + \gamma < 1$. $0 \leq k < 1$ and ϕ is symmetric right-continuous. If $\phi(a,b,c) \leq \max\{a,b,c\}$ then S and T have a unique common fixed point theorem.

Remark 2. Consider Corollary 4 and take $S = T$.

(1) If we set $r = q$ in (2.15) then we get Reich type fixed point theorem (See e.g. [2, 18]).
(2) If we set $r = q = 1$ and $\gamma = 0$ in (2.15) we get Kannan type fixed point theorem (See e.g. [2, 9])
(3) If we set $r = q = 1$ and $\alpha = \beta = 0$ in (2.15) we get Banach type fixed point theorem (See e.g. [2, 3, 15, 17] and [4–7])

Example 4. Let $X = [0,1]$ and $p(x,y) = \max\{x,y\}$. It is clear that (X, p) is a partial metric spaces but not a metric. Suppose that $Sx = Tx = \frac{x}{2}$ and I_S, I_T are constant mappings, such as $r(x) = 2 = q(y)$. Take $\phi(a,b,c) = \frac{1}{3}[a+b+c]$. Let $p(x,y) = \max\{x,y\}$ for all $x, y \in X$. For $\frac{k}{3}$ the condition of Corollary 2 is satisfied. Clearly, 0 is the common fixed point of S, T.

Example 5. Let $X = [1,15]$ and $p(x,y) = \max\{x,y\}$. Here (X, p) is a complete metric spaces. Define the self-mappings $S, T : X \to X$ as $Tx = \frac{x^2}{1+x}$ and $Sx = \begin{cases} \frac{x}{1+x} \text{ if } 1 < x \leq 15 \\ 0 \text{ if } 0 \leq x \leq 1 \end{cases}$. Set $\phi(a,b,c) = \frac{19}{20}\max\{x,y\}$. Without loss of generality, assume $y < x$. Thus, $p(Tx, x) = x$, $p(x,y) = x$, $p(Sy, y) = y$ and $p(Tx, Sy) = \frac{x^2}{1+x}$. Clearly, $p(Tx, Sy) = \frac{x^2}{1+x} \leq \phi(x,y,x) = \frac{19}{20}x$. Hence, it satisfies the conditions of Corollary 2.14 for $r = 1$ and $q = 1$, and 0 is the unique common fixed point of S and T.

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