Miskolc Mathematical Notes
Vol. 22 (2021), No. 2, pp. 625–638

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2021.3325

AN EFFICIENT METHOD FOR SOLVING SECOND-ORDER DELAY DIFFERENTIAL EQUATION

ERKAN CIMEN AND SEVKET UNCU

Received 29 April, 2020

Abstract. In this paper, the initial-value problem for a linear second order delay differential equation is considered. To solve this problem numerically, an appropriate difference scheme is constructed by using the method of integral identities which contains basis functions and interpolating quadrature rules with weight and remainder term in integral form. Besides, the method is proved to be first-order convergent in discrete maximum norm. The numerical illustration provided support the theoretical results. Finally, the proposed method is compared with the implicit Euler method.

2010 *Mathematics Subject Classification:* 34K06; 34K28; 65L05; 65L12; 65L20

Keywords: delay differential equation, initial value problem, finite difference method, error estimate, convergence

1. INTRODUCTION

Delay differential equations (DDEs) are very important and useful in many areas of science and engineering, for instance, physics, biomathematics, medicine, economics, chemistry, etc [7, 8, 13, 16]. Firstly, DDEs occur in modeling effects and interactions between cancer cells, namely tumor population [17]. In engineering, time delays often arise in many life systems like sensors and dynamical processes [14]. Additionally, DDEs can be used in modelling in bridge constructions, traffic control, robot technology, physiological processes and diseases, climate systems, signal transmission [2, 12, 13].

Besides, metal cutting processes [9], bistable device [22], the controlled flexible structures [23] are modelled by second order DDEs. For example, the damped linear oscillator with delayed velocity feedback described by the second order delay differential equation

$$x''(t)+cx'(t)+kx(t)=vx(t-\tau)+wx'(t-\tau)+f\cos(\lambda t),$$

where $c>0$, $k>0$, v and w are constants, representing the damping coefficient, the stiffness coefficient, the feedback gains of displacement path and velocity path, respectively and τ is constant time delay, f is the external force (for details, see [10]).

Motivated by the above works, we consider the following initial value problem with delay in the interval $\bar{I} = [0, T]$:

$$Lu := u''(t) + a(t)u'(t) + b(t)u'(t-r) + c(t)u(t-r) = f(t), \qquad t \in I, \tag{1.1}$$

$$u(t) = \varphi(t), \qquad t \in I_0; \qquad u'(0) = \gamma \tag{1.2}$$

where $I = (0, T] = \cup_{p=1}^{m} I_p$, $I_p = \{t : r_{p-1} < t \le r_p\}$, $1 \le p \le m$ and $r_s = sr$, $0 \le s \le m$, and $I_0 = [-r, 0]$ (for simplicity we suppose that T/r is integer; i.e., $T = mr$). $a(t) \ge \alpha > 0$, $b(t)$, $c(t)$, $f(t)$ and $\varphi(t)$ are sufficiently smooth functions on $\bar{I}$ and I_0, γ is a real number, r is a constant large delay. The existence and uniqueness of a solution to second order DDEs is discussed in [3,5,6,11,13,24].

In this study, the delay term r is chosen as large. Because if r is taken as small, it can be easily reduced to ordinary differential equation with the help of Taylor series, which such equations take place a lot in the literature. However, the interest in large delay equations has been remarkable in recent years.

Even if this problem is linear, it may not always be possible to find the exact solution. Therefore, it is important to develop effective numerical methods to solve DDEs. In the past two decades, several numerical methods have been proposed for the second order DDEs problem including finite difference method [4,20], two-point block method [18], Adams–Moulton method [19], variational iteration method [15], Legendre-Gauss spectral collocation method [25], initial value method [21].

The aim of this study is to present a more effective numerical method than classical methods such as Euler and Runge–Kutta. This method to be used for the numerical solution of (1.1)–(1.2) consists of a finite difference scheme given on a uniform mesh. The scheme is constructed by the method based on using appropriate quadrature rules with the weight and remainder terms in integral form. These results in a local truncation error containing only first derivative of the exact solution and hence facilitates examination of the convergence. This method of approximation has the advantage that the schemes can also be more effective than classical schemes such as Euler in the case when the continuous problem is considered under certain restrictions.

The remainder of this paper is as follows. We put forward some important properties of the exact solution in Section 2 and introduce the finite difference discretization in Section 3. In Section 4, we analyze the error estimates for the approximate solution and prove the convergence in the discrete maximum norm. In Section 5, we present numerical results which confirm the theoretical analysis. The paper ends with a summary of the main conclusions.

Throughout the paper, C denotes a generic positive constant. Some specific, fixed constants of this kind are indicated by subscripting C and $\overline{C}$.

2. Properties of Exact Solution

Here, we give some properties of the exact solution of (1.1)–(1.2), which are needed in the analysis of appropriate numerical solution. For any continuous function

$g(t)$, we use

$$\|g\|_\infty \equiv \|g\|_{\infty,\bar{I}} = \max_{0\le t\le T} |g(t)|, \qquad \|g\|_1 \equiv \|g\|_{1,I} = \int_0^T |g(t)|\,dt,$$
$$\|g\|_{\infty,p} \equiv \|g\|_{\infty,I_p}, \qquad \|g\|_{1,p} \equiv \|g\|_{1,I_p}, \qquad 0\le p\le m.$$

Lemma 1. *If a, b, c, $f \in C(\bar{I})$ and $\varphi \in C^1(I_0)$, then for the solution u of the problem (1.1)–(1.2) we have*

$$\|u\|_{\infty,p} \le C_p, \qquad 1\le p\le m, \tag{2.1}$$
$$\|u'\|_{\infty,p} \le \overline{C}_p, \qquad 1\le p\le m, \tag{2.2}$$

where

$$C_1 = \alpha^{-1}(|\gamma| + [\|f\|_{1,1} + \|b\|_{1,1}\|\varphi'\|_{\infty,0} + \|c\|_{1,1}\|\varphi\|_{\infty,0}]),$$
$$\overline{C}_1 = |\gamma| + \alpha^{-1}[\|f\|_{\infty,1} + \|b\|_{\infty,1}\|\varphi'\|_{\infty,0} + \|c\|_{\infty,1}\|\varphi\|_{\infty,0}],$$
$$C_p = \alpha^{-1}(\overline{C}_{p-1} + \|f\|_{1,p} + \|b\|_{1,p}\overline{C}_{p-1} + \|c\|_{1,p}C_{p-1}), \qquad 2\le p\le m,$$
$$\overline{C}_p = \overline{C}_{p-1} + \alpha^{-1}[\|f\|_{\infty,p} + \|b\|_{\infty,p}\overline{C}_{p-1} + \|c\|_{\infty,p}C_{k-1}], \qquad 2\le p\le m.$$

Proof. The proof is by induction in p. First, for $t\in I_p$, from (1.1) we have

$$u'(t) = u'(r_{p-1})e^{-\int_{r_{p-1}}^{t} a(s)ds} + \int_{r_{p-1}}^{t} F(\xi)e^{-\int_{\xi}^{t} a(s)ds}\,d\xi, \tag{2.3}$$

$$u(t) = u'(r_{p-1})\int_{r_{p-1}}^{t} e^{-\int_{r_{p-1}}^{\xi} a(s)ds}\,d\xi + \int_{r_{p-1}}^{t} ds\int_{r_{p-1}}^{s} F(\xi)e^{-\int_{\xi}^{t} a(\eta)d\eta}\,d\xi$$
$$= u'(r_{p-1})\int_{r_{p-1}}^{t} e^{-\int_{r_{p-1}}^{\xi} a(s)ds}\,d\xi + \int_{r_{p-1}}^{t} d\xi F(\xi)\int_{\xi}^{t} e^{-\int_{\xi}^{s} a(\eta)d\eta}\,ds \tag{2.4}$$

with

$$F(t) = f(t) - b(t)u'(t-r) - c(t)u(t-r).$$

Now, for $p=1$ ($t\in I_1$), from (2.4) we have

$$|u(t)| \le u'(0)\int_0^t e^{-\int_0^{\xi}\alpha ds}\,d\xi + \int_0^t d\xi\,|F(\xi)|\int_{\xi}^{t} e^{-\int_{\xi}^{s} a(\eta)d\eta}\,ds$$
$$\le |\gamma|\int_0^t e^{-\alpha\xi}d\xi$$

$$+\int_0^t [|f(\xi)|+|b(\xi)|\left|u'(\xi-r)\right|+|c(\xi)|\,|u(\xi-r)|]\int_\xi^t e^{-\alpha(s-\xi)}ds$$

$$\leq \alpha^{-1}|\gamma|(1-e^{-\alpha\xi})$$

$$+\alpha^{-1}\int_0^t [|f(\xi)|+|b(\xi)|\left|u'(\xi-r)\right|+|c(\xi)\,|u(\xi-r)||](1-e^{-\alpha(t-\xi)})d\xi.$$

So, we get

$$|u(t)|\leq \alpha^{-1}[|\gamma|+\|f\|_{1,1}+\|b\|_{1,1}\left\|\varphi'\right\|_{\infty,0}+\|c\|_{1,1}\|\varphi\|_{\infty,0}]\equiv C_1.$$

Next, from (2.3), we have

$$\left|u'(t)\right|\leq \left|u'(0)\right|e^{-\alpha t}+\int_0^t [|f(\xi)|+|b(\xi)|\left|u'(\xi-r)\right|+|c(\xi)\,|u(\xi-r)||]e^{-\alpha(t-\xi)}d\xi$$

$$\leq |\gamma|+\alpha^{-1}[\|f\|_{\infty,1}+\|b\|_{\infty,1}\left\|\varphi'\right\|_{\infty,0}+\|c\|_{\infty,1}\|\varphi\|_{\infty,0}](1-e^{-\alpha t})\equiv \overline{C}_1,$$

thus the inequalities (2.1) and (2.2) are valid for $p=1$. And now, let the inequalities (2.1) and (2.2) be true for $p=k$. That is,

$$C_k=\alpha^{-1}\left[\overline{C}_{k-1}+\|f\|_{1,k}+\|b\|_{1,k}\overline{C}_{k-1}+\|c\|_{1,k}C_{k-1}\right],$$

$$\overline{C}_k=\overline{C}_{k-1}+\alpha^{-1}[\|f\|_{\infty,k}+\|b\|_{\infty,k}\overline{C}_{k-1}+\|c\|_{\infty,k}C_{k-1}].$$

For $t\in I_{k+1}$, because of (2.4) we get

$$|u(t)|\leq \left|u'(r_k)\right|\int_{r_k}^t e^{-\alpha(\xi-r_k)}d\xi+\int_{r_k}^t d\xi\,|F(\xi)|\int_\xi^t e^{-\alpha(s-\xi)}ds$$

$$\leq \alpha^{-1}\left|u'(r_k)\right|(1-e^{-\alpha(t-r_k)})$$

$$+\alpha^{-1}\int_{r_k}^t (|f(\xi)|+|b(\xi)|\left|u'(\xi-r)\right|+|c(\xi)|\,|u(\xi-r)|)(1-e^{-\alpha(t-\xi)})d\xi$$

$$\leq \alpha^{-1}\overline{C}_k+\alpha^{-1}[\|f\|_{1,k+1}+\|b\|_{1,k+1}\overline{C}_k+\|c\|_{1,k+1}C_k]$$

and from (2.3) we have

$$\left|u'(t)\right|\leq \left|u'(r_k)\right|e^{-\alpha(t-r_k)}$$

$$+\int_{r_k}^t (|f(\xi)|+|b(\xi)|\left|u'(\xi-r)\right|+|c(\xi)|\,|u(\xi-r)|)e^{-\alpha(t-\xi)}d\xi$$

$$\leq \overline{C}_k+\alpha^{-1}[\|f\|_{\infty,k+1}+\|b\|_{\infty,k+1}\overline{C}_k+\|c\|_{\infty,k+1}C_k](1-e^{-\alpha(t-r_k)})\equiv \overline{C}_{k+1}.$$

Therefore the inequalities (2.1) and (2.2) hold for $p = k+1$. □

3. THE DIFFERENCE SCHEME AND MESH

We further denote by ω_{N_0} the uniform mesh on $\bar{I}$:

$$\omega_{N_0} = \{t_i = ih,\ i = 1,2,...,N_0,\ h = T/N_0 = r/N\}$$

which contains N mesh points at each subinterval I_p $(1 \le p \le m)$:

$$\omega_{N_p} = \{t_i : (p-1)N + 1 \le i \le pN\}, \qquad 1 \le p \le m,$$

and consequently

$$\omega_{N_0} = \bigcup_{p=1}^{m} \omega_{N_p}.$$

For any mesh function $g(t)$, we set $g_i = g(t_i)$ and y_i denote by an approximation of $u(t)$ at t_i. We use the notation

$$g_{\bar{t},i} = \frac{g_i - g_{i-1}}{h}, \qquad g_{t,i} = \frac{g_{i+1} - g_i}{h},$$

$$g_{\overset{0}{t},i} = \frac{g_{i+1} - g_{i-1}}{2h}, \qquad g_{\bar{t}t,i} = \frac{g_{i+1} - 2g_i + g_{i-1}}{h^2},$$

$$\|g\|_{\infty,p} \equiv \|g\|_{\infty,\omega_{N_p}} := \max_{1 \le i \le N} |g_i|, \qquad 1 \le p \le m.$$

To construct the numerical method, we begin with the identity

$$h^{-1} \int_{t_{i-1}}^{t_{i+1}} Lu(t)\psi_i(t)dt = h^{-1} \int_{t_{i-1}}^{t_{i+1}} f(t)\psi_i(t)dt,\ 1 \le i \le N_0, \tag{3.1}$$

with basis functions

$$\psi_i(t) = \begin{cases} \psi_i^{(1)}(t), & t_{i-1} < t < t_i, \\ \psi_i^{(2)}(t), & t_i < t < t_{i+1}, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\psi_i^{(1)} = \frac{\int_{t_{i-1}}^{t} e^{\int_s^t a(\eta)d\eta} ds}{\int_{t_{i-1}}^{t_i} e^{\int_s^{t_i} a(\eta)d\eta} ds},$$

$$\psi_i^{(2)}(t) = \left(1 - \frac{\int_{t_i}^{t} e^{\int_s^{t_{i+1}} a(\eta)d\eta} ds}{\int_{t_i}^{t_{i+1}} e^{\int_s^{t_{i+1}} a(\eta)d\eta} ds}\right) e^{\int_{t_i}^{t} a(\tau)d\tau},$$

are the solutions of the following problems, respectively:

$$\begin{cases} \psi_i''(t) - (a(t)\psi_i(t))' = 0, & t_{i-1} < t < t_i, \\ \psi_i(t_{i-1}) = 0,\ \psi_i(t_i) = 1, \end{cases}$$

$$\begin{cases} \psi_i''(t)-(a(t)\psi_i(t))'=0, & t_i<t<t_{i+1}, \\ \psi_i(t_i)=1,\ \psi_i(t_{i+1})=0. \end{cases}$$

The relation (3.1) is rewritten as

$$\begin{aligned} -h^{-1}\int_{t_{i-1}}^{t_{i+1}} u'(t)\psi_i'(t)dt + h^{-1}\int_{t_{i-1}}^{t_{i+1}} a(t)u'(t)\psi_i(t)dt \\ + h^{-1}\int_{t_{i-1}}^{t_{i+1}} b(t)u'(t-r)\psi_i(t)dt + h^{-1}\int_{t_{i-1}}^{t_{i+1}} c(t)u(t-r)\psi_i(t)dt \\ = h^{-1}\int_{t_{i-1}}^{t_{i+1}} f(t)\psi_i(t)dt \end{aligned} \tag{3.2}$$

for $t \in (t_{i-1}, t_{i+1})$. Using the formulas (2.1) and (2.2) from [1] on each interval (t_{i-1}, t_i) and (t_i, t_{i+1}) taking into account (3.2) we have the following precise relation

$$\ell u_i \equiv A_i u_{\bar{t}t,i} + B_i u_{\bar{t}t,i-N} + C_i u_{\overset{0}{t},i} + D_i u_{\overset{0}{t},i-N} + E_i u_{i-N} = F_i + R_i,\ 1 \le i \le N_0,$$

with

$$\begin{aligned} A_i &= 1 - \frac{1}{2}\int_{t_{i-1}}^{t_i} a(t)\psi_i^{(1)}(t)dt + \frac{1}{2}\int_{t_i}^{t_{i+1}} a(t)\psi_i^{(2)}(t)dt, \\ B_i &= -\frac{1}{2}\int_{t_{i-1}}^{t_i} [b(t)+(t-t_i)c(t)]\,\psi_i^{(1)}(t)dt \\ &\quad + \frac{1}{2}\int_{t_i}^{t_{i+1}} [b(t)+(t-t_i)c(t)]\,\psi_i^{(2)}(t)dt, \\ C_i &= h^{-1}\int_{t_{i-1}}^{t_i} a(t)\psi_i^{(1)}(t)dt + h^{-1}\int_{t_i}^{t_{i+1}} a(t)\psi_i^{(2)}(t)dt, \\ D_i &= h^{-1}\int_{t_{i-1}}^{t_i} [b(t)+(t-t_i)c(t)]\,\psi_i^{(1)}(t)dt \\ &\quad + h^{-1}\int_{t_i}^{t_{i+1}} [b(t)+(t-t_i)c(t)]\,\psi_i^{(2)}(t)dt, \\ E_i &= h^{-1}\int_{t_{i-1}}^{t_i} c(t)\psi_i^{(1)}(t)dt + h^{-1}\int_{t_i}^{t_{i+1}} c(t)\psi_i^{(2)}(t)dt, \\ F_i &= h^{-1}\int_{t_{i-1}}^{t_i} f(t)\psi_i^{(1)}(t)dt + h^{-1}\int_{t_i}^{t_{i+1}} f(t)\psi_i^{(2)}(t)dt, \\ R_i &= h^{-1}\int_{t_{i-1}}^{t_{i+1}} dt\left[c(t)\psi_i(t) - \frac{d}{dt}(b(t)\psi_i(t))\right]\int_{t_{i-1}}^{t_{i+1}} u'(\xi-r)K_0(t,\xi)d\xi, \end{aligned} \tag{3.3}$$

where

$$K_0(t,\xi) = T_0(t-\xi) - h^{-1}(t-t_{i-1}),$$
$$T_0(\lambda)=1, \quad \lambda \ge 0; \qquad T_0(\lambda)=0, \quad \lambda<0.$$

On the other hand, in order to define an approximation for the initial condition (1.2), we begin with the identity

$$h^{-1}\int_{t_0}^{t_1} Lu(t)\psi^{(0)}(t)dt = h^{-1}\int_{t_0}^{t_1} f(t)\psi^{(0)}(t)dt,$$

with

$$\psi^{(0)}(t) = \left(1 - \frac{\int_{t_0}^{t} e^{\int_s^{t_1} a(\eta)d\eta}ds}{\int_{t_0}^{t_1} e^{\int_s^{t_1} a(\eta)d\eta}ds}\right) e^{\int_{t_0}^{t} a(\tau)d\tau},$$

which is the solution of the problem

$$\psi^{(0)}(t) = \begin{cases} \psi^{(0)\prime\prime}(t) - (a(t)\psi^{(0)}(t))' = 0, & t_0 < t < t_1, \\ \psi^{(0)}(t_0) = 1, & \psi^{(0)}(t_1) = 0. \end{cases}$$

In a similar way as above, we have

$$A_0 u_{t,0} = B_0 + r^{(0)},$$

where

$$\begin{aligned} A_0 &= h^{-1}\left(1 + \int_{t_0}^{t_1} a(t)\psi^{(0)}(t)dt\right), \\ B_0 &= h^{-1}\gamma + h^{-1}\int_{t_0}^{t_1} f(t)\psi^{(0)}(t)dt - h^{-1}u_{-N}\int_{t_0}^{t_1} c(t)\psi^{(0)}(t)dt \\ &\quad - h^{-1}u_{t,-N}\int_{t_0}^{t_1}[b(t) + tc(t)]\psi^{(0)}(t)dt, \\ r^{(0)} &= -h^{-1}\int_{t_0}^{t_1} dt(b(t)\psi^{(0)}(t))'\int_{t_0}^{t_1} u'(\xi - r)K_0(t,\xi)d\xi \\ &\quad + h^{-1}\int_{t_0}^{t_1} dt\, c(t)\psi^{(0)}(t)\int_{t_0}^{t_1} u'(\xi - r)K_0(t,\xi)d\xi. \end{aligned} \tag{3.4}$$

Consequently, we propose the following difference scheme for approximating the problem (1.1)–(1.2):

$$\ell y_i \equiv A_i y_{\bar{t}t,i} + B_i y_{\bar{t}t,i-N} + C_i y_{\overset{0}{t},i} + D_i y_{\overset{0}{t},i-N} + E_i y_{i-N} = F_i, \qquad 1 \le i \le N_0, \tag{3.5}$$

$$y_i = \varphi_i, \qquad -N \le i \le 0; \qquad \ell y_0 \equiv A_0 y_{t,0} = B_0. \tag{3.6}$$

Moreover, we can easily propose the implicit Euler method as an alternative to the approximate solution of the problem (1.1)–(1.2):

$$\ell y_i \equiv y_{\bar{t}t,i} + a_i y_{\overset{0}{t},i} + b_i y_{\overset{0}{t},i-N} + c_i y_{i-N} = f_i, \qquad 1 \le i \le N_0, \tag{3.7}$$

$$y_i = \varphi_i, \qquad -N \le i \le 0; \qquad \ell y_0 \equiv y_{t,0} = \gamma. \tag{3.8}$$

4. Convergence analysis of the method

To analyze the convergence of the method, we define the error function $z_i = y_i - u_i$, $0 \le i \le N_0$, which is the solution of the following discrete problem

$$\ell z_i = R_i, \qquad 1 \le i \le N_0, \tag{4.1}$$

$$z_i = 0, \qquad -N_0 \le i \le 0; \qquad \ell z_0 = -r^{(0)}, \tag{4.2}$$

where the truncation errors R_i and $r^{(0)}$ are given by (3.3) and (3.4), respectively.

Lemma 2. *If a, c, $f \in C(\bar{I})$, $b \in C^1(\bar{I})$, and $\varphi \in C^1(I_0)$, then for the truncation errors R_i and $r^{(0)}$, we have*

$$\|R\|_{\infty,N_0} \le CN^{-1}, \tag{4.3}$$

$$\left|r^{(0)}\right| \le CN^{-1}. \tag{4.4}$$

Proof. From (3.3), if we rewrite R_i,

$$\begin{aligned} |R_i| &\le Ch^{-1}\int_{t_{i-1}}^{t_{i+1}} dt[(|c(t)| + \left|b'(t)\right| + |a(t)|\,|b(t)|)\psi_i(t)]\int_{t_{i-1}}^{t_{i+1}} \left|u'(\xi - r)\right| d\xi \\ &\le Ch^{-1}\int_{t_{i-1}}^{t_{i+1}} dt\psi_i(t)\int_{t_{i-1}}^{t_{i+1}} \left|u'(\xi - r)\right| d\xi \end{aligned}$$

and, by virtue of Lemma 1 and $0 < \psi_i(t) \le C$,

$$|R_i| \le Ch,$$

which implies (4.3). Now, we have to prove (4.4). From (3.4), we get

$$\left|r^{(0)}\right| \le Ch^{-1}\int_{t_0}^{t_1} dt[|c(t)| + \left|b'(t)\right| + |a(t)|\,|b(t)|]\psi^{(0)}(t)\int_{t_0}^{t_1} \left|u'(\xi - r)\right| d\xi$$

and also, due to Lemma 1 and $0 < \psi_0(t) \le C$, we have

$$\left|r^{(0)}\right| \le Ch.$$

□

Lemma 3. *For z_i, which is the solution of the problem (4.1)–(4.2), the following estimate holds:*

$$|z_k| \le \lambda_1 \left|r^{(0)}\right| + \lambda_2 \sum_{j=1}^{k-1} \|R\|_{\infty,j}\,,\ 1 \le k \le N_0, \tag{4.5}$$

where

$$\lambda_1 = TA_0^{-1}e^{(\|b\|_\infty + \|c\|_\infty)T}, \qquad \lambda_2 = e^{(\|b\|_\infty + \|c\|_\infty)T}.$$

Proof. If $v_i = z_{t,i}$, then we can write the equation (4.1) as follows,

$$A_i z_{\bar{t}t,i} + B_i z_{\bar{t}t,i-N} + C_i z_{\overset{0}{t},i} + D_i z_{\overset{0}{t},i-N} + E_i z_{i-N} = R_i$$

$$A_i v_{\bar{t},i} + \frac{C_i}{2}(v_i + v_{i-1}) = G_i$$

where

$$G_i = R_i - B_i z_{\bar{t}t,i-N} - D_i z_{\overset{0}{t},i-N} - E_i z_{i-N}.$$

Solving this difference equation with respect to v_i, we obtain

$$v_i = v_0 Q_i + \sum_{k=1}^{i} \phi_k Q_{i,k} \tag{4.6}$$

where

$$Q_{i,k} = \begin{cases} 1, & k = i, \\ \prod_{j=k+1}^{i} \frac{2A_i - hC_i}{2A_i + hC_i}, & 1 \leq k \leq i-1, \end{cases} \qquad \phi_k = \frac{2hG_i}{2A_i + hC_i}.$$

Taking into account $v_i = z_{t,i}$ in (4.6), we write

$$z_{i+1} = z_i + hv_0 Q_i + h\sum_{k=1}^{i} \phi_k Q_{i,k}. \tag{4.7}$$

Solving the first-order difference equation (4.7), we obtain

$$z_i = z_1 + h\sum_{k=1}^{i-1} z_{t,0} Q_k + h\sum_{k=1}^{i-1}\sum_{j=1}^{k} \phi_j Q_{k,j} = -\frac{hr^{(0)}}{A_0} - \frac{hr^{(0)}}{A_0}\sum_{k=1}^{i-1} Q_k + h\sum_{k=1}^{i-1}\sum_{j=1}^{k} \phi_j Q_{k,j}.$$

Next, since $2A_i + hC_i > 0$ and $0 < (2A_i - hC_i)/(2A_i + hC_i) < 1$ $(1 \leq i \leq N_0)$,

$$\begin{aligned} |z_i| &\leq \frac{h\left|r^{(0)}\right|}{A_0} + \frac{h\left|r^{(0)}\right|(i-1)}{A_0} \\ &\quad + h\sum_{k=1}^{i-1}\left[\|R\|_{\infty,k} + (\|b\|_\infty + \|c\|_\infty)(|z_{k-N+1}| + |z_{k-N}| + |z_{k-N-1}|)\right] \\ &\leq \frac{hN_0\left|r^{(0)}\right|}{A_0} + h\sum_{k=1}^{i-1}\left[\|R\|_{\infty,k} + (\|b\|_\infty + \|c\|_\infty)\,|z_k|\right]. \end{aligned}$$

Thus, we have

$$|z_i| \leq \frac{T\left|r^{(0)}\right|}{A_0} + h\sum_{k=1}^{i-1}\|R\|_{\infty,k} + h(\|b\|_\infty + \|c\|_\infty)\sum_{k=1}^{i-1}|z_k|.$$

Using difference analogue of Gronwall's inequality, we get

$$|z_i| \leq \left[TA_0^{-1}\left|r^{(0)}\right| + \sum_{k=1}^{i-1}\|R\|_{\infty,k}\right] e^{(\|b\|_\infty + \|c\|_\infty)t_i}$$

which gives (4.5). □

Now, we formulate the theorem, which expresses the main result of the paper.

Theorem 1. *For $a,\ c,\ f \in C(\bar{I})$, $b \in C^1(\bar{I})$, and $\varphi \in C^1(I_0)$, the solution of the difference problem (3.1)–(3.2) is first order convergent to the solution of (1.1)–(1.2):*

$$\|y-u\|_{\infty,\omega_{N_0}} \leq CN^{-1}.$$

Proof. It immediately follows from (4.5) by taking into consideration (4.3)–(4.4). □

5. Numerical results

In this section, we present the numerical results obtained not only by the proposed method (3.5)–(3.6) but also by the other method (3.7)–(3.8), confirming the theoretical results.

Example 1. We consider the particular problem:

$$u''(t)+2u'(t)+u'(t-1)-u(t-1)+e^{-t}=0, \qquad 0<t\leq 2 \tag{5.1}$$

$$u(t)=e^t, \qquad -1\leq t\leq 0, \qquad\qquad u'(0)=1. \tag{5.2}$$

whose the exact solution is

$$u(t)=\begin{cases} 1+e^{-t}-e^{-2t}, & t\in(0,1], \\ \frac{1}{2}(1+t)+(1-2e)e^{-t}+\frac{1}{2}[(1+3t)e^2-2]e^{-2t}, & t\in(1,2]. \end{cases}$$

Also, we define the exact error e_i^N and the computed maximum pointwise error e^N as follows:

$$e_i^N=|y_i-u_i|, \qquad e^N=\max_{1\leq i\leq N} e_i^N.$$

The computational results for solving the problem (5.1)–(5.2) obtained by using both the present method (PM) and the implicit Euler method (EM) are shown in the Tables 1–3.

6. Conclusion

In this paper, we have developed the finite difference method for solving the linear second order DDE. We have shown that the method has first order convergence in the discrete maximum norm. The numerical results for the test problem are computed for different values N in Tables 1–3. The graph comparing the numerical results obtained from both methods is shown in Fig. 1. The computational results with both methods in Table 3 and Fig. 1 show that the proposed method is more effective than the other method. The ideas presented method here can be used for the study of initial or boundary value problems for linear differential equations with delay as well as neutral type.

TABLE 1. The numerical results on $(0,2]$ for Example 1(PM).

t_i	u_i	$y_i(N=64)$	e_i^{64}	$y_i(N=128)$	e_i^{128}
0.125	1.1036961	1.1036953	$8.277E-7$	1.1036959	$2.069E-7$
0.250	1.1722701	1.1722687	$1.472E-6$	1.1722698	$3.681E-7$
0.375	1.2149227	1.2149208	$1.974E-6$	1.2149222	$4.936E-7$
0.500	1.2386512	1.2386489	$2.365E-6$	1.2386506	$5.914E-7$
0.625	1.2487566	1.2487540	$2.670E-6$	1.2487560	$6.675E-7$
0.750	1.2492364	1.2492335	$2.907E-6$	1.2492357	$7.268E-7$
0.875	1.2430881	1.2430850	$3.092E-6$	1.2430873	$7.730E-7$
1.000	1.2325442	1.2325409	$3.235E-6$	1.2325433	$8.089E-7$
1.125	1.2203862	1.2203905	$4.365E-6$	1.2203872	$1.091E-6$
1.250	1.2123285	1.2123368	$8.219E-6$	1.2123306	$2.055E-6$
1.375	1.2122725	1.2122821	$9.654E-6$	1.2122749	$2.413E-6$
1.500	1.2219502	1.2219598	$9.585E-6$	1.2219526	$2.396E-6$
1.625	1.2417225	1.2417311	$8.636E-6$	1.2417246	$2.159E-6$
1.750	1.2711252	1.2711324	$7.223E-6$	1.2711270	$1.806E-6$
1.875	1.3092394	1.3092450	$5.617E-6$	1.3092408	$1.404E-6$
2.000	1.3549343	1.3549382	$3.990E-6$	1.3549353	$9.974E-7$

TABLE 2. The numerical results on $(0,2]$ for Example 1 (EM).

t_i	u_i	$y_i(N=64)$	e_i^{64}	$y_i(N=128)$	e_i^{128}
0.125	1.1036961	1.1062957	$2.600E-3$	1.1049940	$1.298E-3$
0.250	1.1722701	1.1768911	$4.621E-3$	1.1745781	$2.308E-3$
0.375	1.2149227	1.2211157	$6.193E-3$	1.2180167	$3.094E-3$
0.500	1.2386512	1.2460666	$7.415E-3$	1.2423569	$3.706E-3$
0.625	1.2487566	1.2571224	$8.366E-3$	1.2529384	$4.182E-3$
0.750	1.2492364	1.2583412	$9.105E-3$	1.2537885	$4.552E-3$
0.875	1.2430881	1.2527673	$9.679E-3$	1.2479284	$4.840E-3$
1.000	1.2325442	1.2426699	$1.013E-2$	1.2376087	$5.065E-3$
1.125	1.2203862	1.2307176	$1.033E-2$	1.2255527	$5.167E-3$
1.250	1.2123285	1.2226009	$1.027E-2$	1.2174656	$5.137E-3$
1.375	1.2122725	1.2223646	$1.009E-2$	1.2173198	$5.047E-3$
1.500	1.2219502	1.2318340	$9.884E-3$	1.2268938	$4.944E-3$
1.625	1.2417225	1.2514273	$9.705E-3$	1.2465769	$4.854E-3$
1.750	1.2711252	1.2807138	$9.589E-3$	1.2759216	$4.796E-3$
1.875	1.3092394	1.3187915	$9.552E-3$	1.3140176	$4.778E-3$
2.000	1.3549343	1.3645361	$9.602E-3$	1.3597371	$4.803E-3$

TABLE 3. Comparison of e^N for both methods on $(0,2]$ for Example 1.

N	$e^N(\text{EM})$	$e^N(\text{PM})$	N	$e^N(\text{EM})$	$e^N(\text{PM})$
32	$2.066E-2$	$3.906E-5$	256	$2.584E-3$	$6.104E-7$
64	$1.034E-2$	$9.766E-6$	512	$1.292E-3$	$1.526E-7$
128	$5.169E-3$	$2.442E-6$	1024	$6.462E-4$	$3.817E-8$

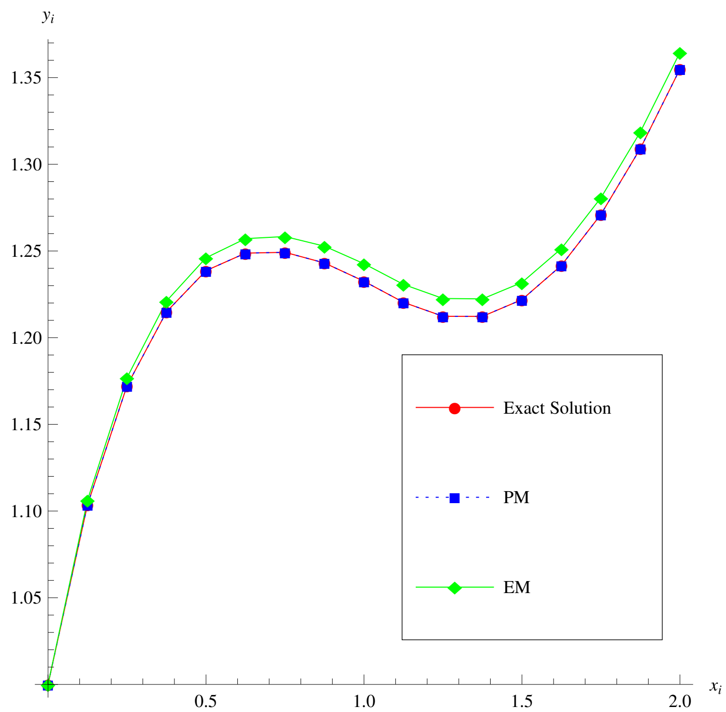

FIGURE 1. Computational results of Example 1 for $N = 64$.

REFERENCES

[1] G. M. Amiraliyev and Y. D. Mamedov, "Difference schemes on the uniform mesh for singular perturbed pseudo-parabolic equations," *Turkish J. Math.*, vol. 19, pp. 207–222, 1995.

[2] B. Balachandran, T. K. Nagy, and D. E. Gilsinn, *Delay Differential Equations*. New York: Springer, 2009. doi: 10.1007/978-0-387-85595-0.

[3] M. Benchohra and S. Abbas, *Advanced Functional Evolution Equations and Inclusions*. Switzerland: Springer, 2015. doi: 10.1007/978-3-319-17768-7.

[4] E. Cimen and G. M. Amiraliyev, "Uniform convergence method for a delay differential problem with layer behaviour," *Mediterr. J. Math.*, vol. 16:57, pp. 1–15, 2019, doi: 10.1007/s00009-019-1335-9.

[5] C. Corduneanu, Y. Li, and M. Mahdavi, *Functional Differential Equations: Advances and Applications*. New Jersey: John Wiley & Sons, Inc., 2016. doi: 10.1002/9781119189503.

[6] J. Diblik, M. Kudelcikova, and M. Ruzickova, "Positive solutions to delayed differential equations of the second-order," *Appl. Math. Lett.*, vol. 94, pp. 52–58, 2019, doi: 10.1016/j.aml.2019.02.022.

[7] C. Foley and M. C. Mackey, "Dynamic hematological disease: a review," *J. Math. Biol.*, vol. 58, pp. 285–322, 2009, doi: 10.1007/s00285-008-0165-3.

[8] P. Getto and M. Waurick, "A differential equation with state-dependent delay from cell population biology," *J. Differ. Equat.*, vol. 260, pp. 6176–6200, 2016, doi: 10.1016/j.jde.2015.12.038.

[9] D. E. Gilsinn, "Estimating critical Hopf bifurcation parameters for a second-order delay differential equation with application to machine tool chatter," *Nonlinear Dynamics*, vol. 30, pp. 103–154, 2002, doi: 10.1023/A:1020455821894.

[10] H. Y. Hu and Z. H. Wang, *Dynamics of Controlled Mechanical Systems with Delayed Feedback*. Berlin: Springer, 2002. doi: 10.1007/978-3-662-05030-9.

[11] V. A. Ilea and D. Otrocol, "Some properties of solutions of a functional-differential equation of second order with delay," *Sci. World J.*, vol. Volume 2014, p. 8 Pages, 2014, doi: 10.1155/2014/878395.

[12] A. Keane, B. Krauskopf, and C. M. Postlethwaite, "Climate models with delay differential equations," *Chaos*, vol. 27, pp. 1–15, 2017, doi: 10.1063/1.5006923.

[13] V. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*. Netherlands: Kluwer Academic Pub., 1999. doi: 10.1007/978-94-017-1965-0.

[14] Y. N. Kyrychko and S. J. Hogan, "On the use of delay equations in engineering applications," *J. Vib. Control*, vol. 16, pp. 943–960, 2010, doi: 10.1177/1077546309341100.

[15] H. Liu, A. Xiao, and L. Su, "Convergence of variational iteration method for second-order delay differential equations," *J. Appl. Math.*, vol. Volume 2013, p. 9 Pages, 2013, doi: 10.1155/2013/634670.

[16] E. Liz and G. Röst, "Global dynamics in a commodity market model," *J. Math. Anal. Appl.*, vol. 398, pp. 707–714, 2013, doi: 10.1016/j.jmaa.2012.09.024.

[17] F. A. Rihan, S. Lakshmanan, and H. Maurer, "Optimal control of tumour-immune model with time-delay and immuno-chemotherapy," *Appl. Math. Comput.*, vol. 353, pp. 147–165, 2019, doi: 10.1016/j.amc.2019.02.002.

[18] H. Y. Seong and Z. A. Majid, "Solving second order delay differential equations using direct two-point block method," *Ain Shams Engineering Journal*, vol. 8, pp. 59–66, 2017, doi: 10.1016/j.asej.2015.07.014.

[19] H. Y. Seong, Z. A. Majid, and F. Ismail, "Solving second-order delay differential equations by direct Adams-Moulton method," *Math. Probl. Eng.*, vol. Volume 2013, p. 7 Pages, 2013, doi: 10.1155/2013/261240.

[20] V. Subburayan, "An hybrid initial value method for singularly perturbed delay differential equations with interior layers and weak boundary layer," *Ain Shams Engineering Journal*, vol. 9, pp. 727–733, 2018, doi: 10.1016/j.asej.2016.03.018.

[21] V. Subburayan and M. Ramanujam, "Asymptotic initial value technique for singularly perturbed convection–diffusion delay problems with boundary and weak interior layers," *Appl. Math. Lett.*, vol. 25, pp. 2272–2278, 2012, doi: 10.1016/j.aml.2012.06.016.

[22] R. Vallee, M. Dubois, M. Cote, and C. Delisle, "Second-order differential-delay equation to describe a hybrid bistable device," *Phys. Rev. A*, vol. 36, pp. 1327–1332, 1987, doi: 10.1103/PhysRevA.36.1327.

[23] H. Wang and H. Hu, "Remarks on the perturbation methods in solving the second-order delay differential equations," *Nonlinear Dynamics*, vol. 33, pp. 379–398, 2003, doi: 10.1023/B:NODY.0000009957.42817.4f.

[24] J. Yan, “Existence of oscillatory solutions of forced second order delay differential equations,” *Appl. Math. Lett.*, vol. 24, pp. 1455–1460, 2011, doi: 10.1016/j.aml.2011.03.031.

[25] L. Yi and Z. Wang, “Legendre-Gauss spectral collocation method for second order nonlinear delay differential equations,” *Numer. Math. Theor. Meth. Appl.*, vol. 7, pp. 149–178, 2014, doi: 10.1017/S1004897900000921.

Authors' addresses

Erkan Cimen

(**Corresponding author**) Department of Mathematics, Faculty of Education, Van Yuzuncu Yil University, 65080 Van, Turkey

E-mail address: cimenerkan@hotmail.com

Sevket Uncu

Department of Mathematics, Institute of Pure and Applied Sciences, Van Yuzuncu Yil University, 65080 Van, Turkey

E-mail address: uncusevket@gmail.com