APPLICATIONS OF HORADAM POLYNOMIALS ON A NEW FAMILY OF BI-PRESTARLIKE FUNCTIONS

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Abstract. In this article, we introduce and investigate a new family of analytic and bi-prestarlike functions by using the Horadam polynomials defined in the open unit disk $U$. We determine upper bounds for the first two coefficients $|a_2|$ and $|a_3|$ and solve Fekete-Szegő problem of functions that belong to this family. Also, we point out several certain special cases for our results.

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1. INTRODUCTION AND PRELIMINARIES

Indicate by $\mathcal{A}$ the collection of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ that have the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

Further, let $S$ stand for the subclass of $\mathcal{A}$ containing of functions in $U$ satisfying (1.1) which are univalent in $U$.

A function $f \in \mathcal{A}$ is called starlike of order $\theta$ $(0 \leq \theta < 1)$, if

$$\text{Re}\left\{zf'(z)/f(z)\right\} > \theta, \quad (z \in U).$$

For $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$
the Hadamard product of $f$ and $g$ is defined (as usual) by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in U).$$

Ruscheweyh [14] defined and considered the family of prestarlike functions of order $\theta$, which are the functions $f$ such that $f \ast I_\theta$ is a starlike function of order $\theta$, where

$$I_\theta(z) = \frac{z}{(1 - z)^{2(1 - \theta)}}, \quad (0 \leq \theta < 1, z \in U).$$

The function $I_\theta$ can be written in the form:

$$I_\theta(z) = z + \sum_{n=2}^{\infty} \varphi_n(\theta) z^n,$$

where

$$\varphi_n(\theta) = \frac{\prod_{i=2}^{n} (i - 2\theta)}{(n - 1)!}, \quad n \geq 2.$$ We note that $\varphi_n(\theta)$ is a decreasing function in $\theta$ and satisfies

$$\lim_{n \to \infty} \varphi_n(\theta) = \begin{cases} 
\infty, & \text{if } \theta < \frac{1}{2} \\
1, & \text{if } \theta = \frac{1}{2} \\
0, & \text{if } \theta > \frac{1}{2} 
\end{cases}$$

According to the Koebe one-quarter theorem (see [6]) every function $f \in S$ has an inverse $f^{-1}$ which satisfies $f^{-1}(f(z)) = z, \quad (z \in U)$ and $f(f^{-1}(w)) = w, \quad (|w| < r_0(f), r_0(f) \geq \frac{1}{4})$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^3 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots \quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $U$ if both $f$ and $f^{-1}$ are univalent in $U$. Let $\Sigma$ stands for the class of bi-univalent functions in $U$ given by (1.1). Srivastava et al. [19] revived the study of analytic and bi-univalent functions in recent years, was followed by such works as those by Bulut [4], Adegani and et al. [1], Caglar et al. [19] and others (see, for example [15, 17, 18, 20]). We notice that the class $\Sigma$ is not empty. For example, the functions $z, \frac{z}{1 - z}, -\log(1 - z)$ and $\frac{1}{2} \log \frac{1 + z}{1 - z}$ are members of $\Sigma$. However, the Koebe function is not a member of $\Sigma$. Until now, the coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|, \quad (n = 3, 4, \cdots)$ for functions $f \in \Sigma$ is still an open problem.

Let the functions $f$ and $g$ be analytic in $U$. We say that the function $f$ is said to be subordinate to $g$, if there exists a Schwarz function $w$ analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$ such that $f(z) = g(w(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z) (z \in U)$. It is well known (see [13]) that if the function $g$ is univalent in $U$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$. 
The Horadam polynomials $h_n(r)$ are defined by the following repetition relation (see [8]):

$$h_n(r) = prh_{n-1}(r) + qh_{n-2}(r) \quad (r \in \mathbb{R}, n \in \mathbb{N} = \{1, 2, 3, \ldots \}),$$

(1.3)

with $h_1(r) = a$ and $h_2(r) = br$, for some real constant $a, b, p$ and $q$. The characteristic equation of repetition relation (1.3) is $t^2 - pr - q = 0$. This equation has two real roots $x = \frac{pr + \sqrt{p^2r^2 + 4q}}{2}$ and $y = \frac{pr - \sqrt{p^2r^2 + 4q}}{2}$.

**Remark 1.** For particular values of $a, b, p$ and $q$, the Horadam polynomial $h_n(r)$ reduces to several known polynomials listed below:

1. the Fibonacci polynomials $F_n(r)$ ($a = b = p = q = 1$).
2. the Lucas polynomials $L_n(r)$ ($a = 2$ and $b = p = q = 1$).
3. the Pell polynomials $P_n(r)$ ($a = q = 1$ and $b = p = 2$).
4. the Pell-Lucas polynomials $Q_n(r)$ ($a = b = p = 2$ and $q = 1$).
5. the Chebyshev polynomials $T_n(r)$ of the first kind ($a = b = 1$, $p = 2$ and $q = -1$).
6. the Chebyshev polynomials $U_n(r)$ of the second kind ($a = 1$, $b = p = 2$ and $q = -1$).

These polynomials, the families of orthogonal polynomials and other special polynomials as well as their generalizations are potentially important in a variety of disciplines in many of sciences, specially in mathematics, statistics and physics. For more information associated with these polynomials see [7, 8, 10, 11, 21].

The generating function of the Horadam polynomials $h_n(r)$ (see [9]) is given by

$$\Pi(r, z) = \sum_{n=1}^{\infty} h_n(r)z^{n-1} = \frac{a + (b - ap)rz}{1 - prz - qz^2}.$$  

(1.4)

2. MAIN RESULTS

We begin this section by defining the subclass $\mathcal{H}_c(\delta, \lambda, \theta, r)$ as follows:

**Definition 1.** For $\delta \geq 0$, $0 \leq \lambda \leq 1$, $0 \leq \theta < 1$ and $r \in \mathbb{R}$, a function $f \in \Sigma$ is said to be in the class $\mathcal{H}_c(\delta, \lambda, \theta, r)$ if it satisfies the subordinations

$$(1 - \delta) \left[ (1 - \lambda) \frac{z(f*I_0)'(z)}{(f*I_0)(z)} + \lambda \left( 1 + \frac{z(f*I_0)''(z)}{(f*I_0)'(z)} \right) \right]$$

$$+ \delta \frac{\lambda z^2 (f*I_0)'(z) + z(f*I_0)'(z)}{\lambda z (f*I_0)'(z) + (1 - \lambda) (f*I_0)(z)} \prec \Pi(r, z) + 1 - a$$

and

$$(1 - \delta) \left[ (1 - \lambda) \frac{w(g*I_0)'(w)}{(g*I_0)(w)} + \lambda \left( 1 + \frac{w(g*I_0)''(w)}{(g*I_0)'(w)} \right) \right]$$
where $a$ is a real constant and the function $g = f^{-1}$ is given by (1.2).

**Theorem 1.** For $\delta \geq 0$, $0 \leq \lambda \leq 1$, $0 \leq \theta < 1$ and $r \in \mathbb{R}$, let $f \in A$ be in the class $A_\delta(\delta, \lambda, \theta, r)$. Then

\[
|a_2| \leq \frac{|br|}{\sqrt{2[(1-\theta)(2\lambda \delta(1-\theta)(1-\lambda) + 2\theta + 1) + 2(1-\theta)\delta^2(\lambda + 1)^2]br^2 - 2qa(1-\theta)^2(\lambda + 1)^2}}
\]

and

\[
|a_3| \leq \frac{|br|}{2(1-\theta)(3-2\theta)(2\lambda + 1)} + \frac{b^2r^2}{4(1-\theta)^2(\lambda + 1)^2}.
\]

**Proof.** Let $f \in A_\delta(\delta, \lambda, \theta, r)$. Then there are two analytic functions $u, v : U \rightarrow U$ given by

\[
u(w) = v_1w + v_2w^2 + v_3w^3 + \cdots \quad (w \in U),
\]

with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$, $z, w \in U$ such that

\[
(1 - \delta) \left[ (1 - \lambda) \frac{z(f * I_\delta)'(z)}{(f * I_\delta)(z)} + \lambda \left( 1 + \frac{z(f * I_\delta)''(z)}{(f * I_\delta)'(z)} \right) \right] + \delta \frac{\lambda z^2(f * I_\delta)''(z) + z(f * I_\delta)'(z)}{\lambda z(f * I_\delta)'(z) + (1 - \lambda)(f * I_\delta)(z)} = \Pi(r, u(z)) + 1 - a
\]

and

\[
(1 - \delta) \left[ (1 - \lambda) \frac{w(g * I_\delta)'(w)}{(g * I_\delta)(w)} + \lambda \left( 1 + \frac{w(g * I_\delta)''(w)}{(g * I_\delta)'(w)} \right) \right] + \delta \frac{\lambda w^2(g * I_\delta)''(w) + w(g * I_\delta)'(w)}{\lambda w(g * I_\delta)'(w) + (1 - \lambda)(g * I_\delta)(w)} = \Pi(r, v(w)) + 1 - a.
\]

Or, equivalently

\[
(1 - \delta) \left[ (1 - \lambda) \frac{z(f * I_\delta)'(z)}{(f * I_\delta)(z)} + \lambda \left( 1 + \frac{z(f * I_\delta)''(z)}{(f * I_\delta)'(z)} \right) \right] + \delta \frac{\lambda z^2(f * I_\delta)''(z) + z(f * I_\delta)'(z)}{\lambda z(f * I_\delta)'(z) + (1 - \lambda)(f * I_\delta)(z)} = 1 + h_1(r) + h_2(r)u(z) + h_3(r)u^2(z) + \cdots
\]

and

\[
(1 - \delta) \left[ (1 - \lambda) \frac{w(g * I_\delta)'(w)}{(g * I_\delta)(w)} + \lambda \left( 1 + \frac{w(g * I_\delta)''(w)}{(g * I_\delta)'(w)} \right) \right]
\]
Combining (2.1), (2.2), (2.3) and (2.4) yields

\[
\left(1-\delta\right) \left[ (1-\lambda) \frac{z (f * I_0)'(z)}{(f * I_0)'(z)} + \lambda \left( 1 + \frac{z (f * I_0)''(z)}{(f * I_0)'(z)} \right) \right]
\]

\[
+ \delta \lambda w^2 (g * I_0)'(w) + w^2 (g * I_0)''(w) = 1 + h_2(r) u_{1} z + [h_2(r) u_{2} + h_3(r) u_{1}^2] z^2 + \cdots
\]

and

\[
(1 - \delta) \left[ (1 - \lambda) \frac{w (g * I_0)'(w)}{(g * I_0)'(w)} + \lambda \left( 1 + \frac{w (g * I_0)''(w)}{(g * I_0)'(w)} \right) \right]
\]

\[
+ \delta \lambda w^2 (g * I_0)'(w) + w^2 (g * I_0)''(w) = 1 + h_2(r) v_{1} w + [h_2(r) v_{2} + h_3(r) v_{1}^2] w^2 + \cdots
\]

It is quite well-known that if \(|u(z)| < 1\) and \(|v(w)| < 1\), then

\[|u_i| \leq 1 \quad \text{and} \quad |v_i| \leq 1 \quad \text{for all} \quad i \in \mathbb{N}.\]

Comparing the corresponding coefficients in (2.5) and (2.6), after simplifying, we have

\[2(1 - \theta)(\lambda + 1)a_2 = h_2(r) u_1, \quad (2.8)\]

\[2(1 - \theta)(3 - 2\theta)(2\lambda + 1)a_3 - 4(1 - \theta)^2(\lambda \delta(\lambda - 1) + 3\lambda + 1)a_2^2
\]

\[= h_2(r) u_2 + h_3(r) u_1^2, \quad (2.9)\]

\[-2(1 - \theta)(\lambda + 1)a_2 = h_2(r) v_1 \quad (2.10)\]

and

\[2(1 - \theta)(3 - 2\theta)(2\lambda + 1)(2a_2^2 - a_3) - 4(1 - \theta)^2(\lambda \delta(\lambda - 1) + 3\lambda + 1)a_2^2
\]

\[= h_2(r) v_2 + h_3(r) v_1^2. \quad (2.11)\]

It follows from (2.8) and (2.10) that

\[u_1 = -v_1 \quad (2.12)\]

and

\[8(1 - \theta)^2(\lambda + 1)a_2^2 = h_2(r)(u_1^2 + v_1^2). \quad (2.13)\]

If we add (2.9) to (2.11), we find that

\[4(1 - \theta)[2\lambda \delta(1 - \theta)(1 - \lambda) + 2\theta \lambda + 1]a_2^2 = h_2(r)(u_2 + v_2) + h_3(r)(u_1^2 + v_1^2). \quad (2.14)\]
Substituting the value of $u_1^2 + v_1^2$ from (2.13) in the right hand side of (2.14), we deduce that

$$a_2^2 = \frac{h_2(r)(u_2 + v_2)}{4 \left[ h_2^2(r)(1 - \theta)(1 - \lambda) + 2\lambda + 1 \right] - 2h_3(r)(1 - \theta)^2 (\lambda + 1)^2}.$$  \hspace{1cm} (2.15)

If we make further computations using (1.3), (2.7) and (2.15), we obtain

$$|a_2| \leq \frac{|br| \sqrt{|br|}}{\sqrt{2} \left[ (1 - \theta)(1 - \lambda) + 2\lambda + 1 \right] b - 2p(1 - \theta)^2 (\lambda + 1)^2}.$$

Next, if we subtract (2.11) from (2.9), we can easily see that

$$4(1 - \theta)(3 - 2\theta)(2\lambda + 1)(a_3 - a_2^2) = h_2(r)(u_2 - v_2) + h_3(r)(u_1^2 - v_1^2).$$  \hspace{1cm} (2.16)

In view of (2.12) and (2.13), we get from (2.16)

$$a_3 = \frac{h_2(r)(u_2 - v_2)}{4(1 - \theta)(3 - 2\theta)(2\lambda + 1)} + \frac{h_2^2(r)(u_1^2 + v_1^2)}{8(1 - \theta)^2 (\lambda + 1)^2}.$$

Thus applying (1.3), we obtain

$$|a_3| \leq \frac{|br|}{2(1 - \theta)(3 - 2\theta)(2\lambda + 1)} + \frac{b^2r^2}{4(1 - \theta)^2 (\lambda + 1)^2}.$$  \hspace{1cm} (2.17)

This completes the proof of Theorem 1.

In the next theorem, we discuss the Fekete-Szegő problem for the subclass \(\mathcal{A}(\delta, \lambda, \theta, r).\)

**Theorem 2.** For \(\delta \geq 0, 0 \leq \lambda \leq 1, 0 \leq \theta < 1\) and \(r, \mu \in \mathbb{R},\) let \(f \in \mathcal{A}\) be in the class \(\mathcal{A}(\delta, \lambda, \theta, r).\) Then \(|a_3 - \mu a_2^2| \leq \)

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|br|}{2(1 - \theta)(3 - 2\theta)(2\lambda + 1)}, \\
\left[ (1 - \theta)(1 - \lambda) + 2\lambda + 1 \right] b - 2p(1 - \theta)^2 (\lambda + 1)^2, \\
\frac{|br|}{b^2r^2(1 - \theta)(3 - 2\theta)(2\lambda + 1)}, \\
\left[ (1 - \theta)(1 - \lambda) + 2\lambda + 1 \right] b - 2p(1 - \theta)^2 (\lambda + 1)^2, \\
\frac{|br|}{b^2r^2(1 - \theta)(3 - 2\theta)(2\lambda + 1)}. 
\end{cases}$$

**Proof.** It follows from (2.15) and (2.16) that

$$a_3 - \mu a_2^2 = \frac{h_2(r)(u_2 - v_2)}{4(1 - \theta)(3 - 2\theta)(2\lambda + 1)} + (1 - \mu) a_2^2.$$

If we make further computations using (1.3), (2.7) and (2.15), we obtain

$$|a_3| \leq \frac{|br|}{2(1 - \theta)(3 - 2\theta)(2\lambda + 1)} + \frac{b^2r^2}{4(1 - \theta)^2 (\lambda + 1)^2}.$$  \hspace{1cm} (2.17)

This completes the proof of Theorem 1.
After some computations, we obtain

\[
\frac{h^2_2(r)(u_2 + v_2)(1 - \mu)}{4} \left[ \frac{\psi(\mu, r) + \frac{1}{(1 - \theta)(3 - 2\theta)(2\lambda + 1)}}{4} u_2 + \left( \psi(\mu, r) - \frac{1}{(1 - \theta)(3 - 2\theta)(2\lambda + 1)} \right) v_2 \right],
\]

where

\[
\psi(\mu, r) = \frac{h^2_2(r)(1 - \mu)}{h^2_2(r)(1 - \theta)[2\lambda\delta(1 - \theta)(1 - \lambda) + 2\theta\lambda + 1] - 2h_3(r)(1 - \theta)^2(\lambda + 1)^2}.
\]

According to (1.3), we find that

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|br|}{2(1 - \theta)(3 - 2\theta)(2\lambda + 1)}, & 0 \leq |\psi(\mu, r)| \leq \frac{1}{(1 - \theta)(3 - 2\theta)(2\lambda + 1)}, \\
\frac{1}{2} |br| |\psi(\mu, r)|, & |\psi(\mu, r)| \geq \frac{1}{(1 - \theta)(3 - 2\theta)(2\lambda + 1)}.
\end{cases}
\]

After some computations, we obtain \[|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|br|}{2(1 - \theta)(3 - 2\theta)(2\lambda + 1)}; & \text{for } |\mu - 1| \leq \\
\frac{|br|^3 |\mu - 1|}{3(1 - \theta)[2\lambda\delta(1 - \theta)(1 - \lambda) + 2\theta\lambda + 1] b - 2p(1 - \theta)^2(\lambda + 1)^2} b^2 - 2qa(1 - \theta)^2(\lambda + 1)^2; & \text{for } |\mu - 1| \geq \\
\frac{|br|^3 |\mu - 1|}{3(1 - \theta)[2\lambda\delta(1 - \theta)(1 - \lambda) + 2\theta\lambda + 1] b - 2p(1 - \theta)^2(\lambda + 1)^2} b^2 - 2qa(1 - \theta)^2(\lambda + 1)^2.
\end{cases}\]

Putting \(\mu = 1\) in Theorem 2, we obtain the following result:

**Corollary 1.** For \(\delta \geq 0, 0 \leq \lambda \leq 1, 0 \leq \theta < 1\) and \(r \in \mathbb{R}\), let \(f \in \mathcal{A}\) be in the class \(\mathcal{N}_E(\delta, \lambda, \theta, r)\). Then

\[
|a_3 - a_2^2| \leq \frac{|br|}{2(1 - \theta)(3 - 2\theta)(2\lambda + 1)}.
\]

**Remark 2.** Special cases are shown below:
If we put \( \lambda = 0 \) and \( \theta = \frac{1}{2} \) in our Theorems, we have the results for well-known class \( S^\ast_{\Sigma}(r) \) of bi-starlike functions which was studied recently by Srivastava et al. [16].

If we put \( \lambda = 1 \) and \( \theta = \frac{1}{2} \) in our Theorems, we have the results for the class \( \mathcal{K}_{\Sigma}(r) \) which was considered recently by Magesh et al. [12].

If we put \( \delta = 0 \) and \( \theta = \frac{1}{2} \) in our Theorems, we have the results for the class \( M^\ast_{\Sigma}(\lambda, r) \) which was investigated recently by Magesh et al. [12].

If we put \( \lambda = 0, \theta = \frac{1}{2}, a = 1, b = p = 2, q = -1 \) and \( r \rightarrow t \) in our Theorems, we obtain the results for the class \( S^\ast_{\Sigma}(t) \) of bi-starlike functions which was introduced recently by Altınkaya and Yalçın [3].

If we put \( \delta = 0, \theta = \frac{1}{2}, a = 1, b = p = 2, q = -1 \) and \( r \rightarrow t \) in our Theorems, we have the results for the class \( M^\ast_{\Sigma}(\alpha, t) \) which was considered recently by Altınkaya and Yalçın [2].

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