Monotone iterations for differential problems

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MONOTONE ITERATIONS FOR DIFFERENTIAL PROBLEMS

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Abstract. The purpose of this survey paper is to formulate existence results for parameterized non-linear boundary value problems employing the method of upper and lower solutions and the method of quasilinearization.

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1. Introduction

Let us consider the differential equation of first order with a parameter of the form

\[
\begin{cases}
    x'(t) = f(t, x(t), \lambda), & t \in J = [0, b] \\
    x(0) = k_0 \in R & G(x(b), \lambda) = 0
\end{cases}
\]

(1.1)

where \( f \in C(J \times R \times R, R) \), and \( G \in C(R \times R, R) \). By a solution of (1.1) we mean a pair \((x, \lambda) \in C^1(J, R) \times R\) for which problem (1.1) is satisfied. Problems with a parameter have been considered for many years. Some of them appeared as mathematical models of physical systems (see, for example [16]).

An important area of research in the qualitative theory of differential equations is the study of existence of solutions. Existence/uniqueness theorems for problem (1.1) can be formulated under the assumption that \( f \) and \( G \) satisfy the Lipschitz condition with respect to the last two variables with suitable Lipschitz constants or Lipschitz functions (see [1, 2, 8, 15]). In the above mentioned papers the method of successive approximations and comparison technique are used to obtain sufficient conditions on existence/uniqueness solutions for problems with a parameter. An interesting technique for proving existence results is the method based on upper and lower solutions (for details, see [9]). The purpose of this paper is to formulate existence results for problem (1.1) employing the method of upper and lower solutions. As we see one-sided Lipschitz conditions are imposed on \( f \) and \( G \) (Theorems 1 and 2). If we apply the method of quasilinearization (for details, see [10]), then we can construct monotone sequences which converge quadratically to the unique solution of problem (1.1). We show that if we replace \( f \) and \( G \) by the sum of two corresponding functions,
2. Extremal solutions of problem (1.1)

A pair \((v, \alpha) \in C^1(J, R) \times R\) is said to be a lower solution of (1.1) if

\[
\begin{align*}
v'(t) &\leq f(t, v(t), \alpha), \quad t \in J \\
v(0) &= k_0 \quad 0 \leq G(v(b), \alpha),
\end{align*}
\]

and an upper solution of (1.1) if the above inequalities are reversed.

The next two theorems give constructive sufficient conditions when problem (1.1) has minimal and maximal solutions. Note that one-sided Lipschitz conditions are imposed on \(f\) and \(G\).

**Theorem 1** (see [4]). Assume that \(f \in C(J \times R \times R, R)\), \(G \in C(R \times R, R)\) and

1° \((y_0, \lambda_0), (z_0, \gamma_0) \in C^1(J, R) \times R\) are lower and upper solutions of problem (1.1) such that \(y_0(t) \leq z_0(t), \quad t \in J\) and \(\lambda_0 \leq \gamma_0\).

2° \(f\) is nondecreasing with respect to the last variable,

3° \(G(\bar{u}, \lambda) - G(u, \lambda) \geq -M(\bar{u} - u)\) for \(y_0 \leq u \leq \bar{u} \leq z_0\) with \(M \geq 0\),

4° \(G(\bar{u}, \lambda) - G(u, \lambda) \geq Q(\bar{u} - u)\) for \(y_0 \leq u \leq \bar{u} \leq z_0\), \(\lambda_0 \leq \lambda \leq \gamma_0\) with \(Q \geq 0\),

5° \(G(\bar{u}, \lambda) - G(u, \lambda) \geq -N(\bar{\lambda} - \lambda)\) for \(\lambda_0 \leq \lambda \leq \bar{\lambda} \leq \gamma_0\), \(y_0 \leq u \leq z_0\) with \(N > 0\).

Then there exist monotone sequences \(\{y_n, \lambda_n\}, \{z_n, \gamma_n\}\) such that \(y_n(t) \to y(t), \quad z_n(t) \to z(t), \quad t \in J\) and \(\lambda_n \to \lambda, \quad \gamma_n \to \gamma\) as \(n \to \infty\) and this convergence is uniform and monotone on \(J\). Moreover, \((y, \lambda)\) and \((z, \gamma)\) are minimal and maximal solutions of problem (1.1), respectively.

**Remark 1.** We observe that the special cases when \(f\) is monotone nondecreasing with respect to the second variable and \(G\) is monotone nondecreasing with respect to the first and second variables are covered by Theorem 1. To see this, it is enough to put \(M = 0\) in condition 3°, and \(Q = 0\) in condition 4°. If \(G\) is monotone nondecreasing with respect to the second variable, then there exists \(N > 0\) such that for \(\bar{\lambda} \geq \lambda\) we have

\[G(\bar{u}, \bar{\lambda}) - G(u, \lambda) \geq 0 \geq -N(\bar{\lambda} - \lambda),\]

which proves that condition 5° holds also.

**Example** (see [4]). Let \(J = [0, 1]\),

\[
\begin{align*}
x'(t) &= x^2(t) + \frac{1}{2}[\sin^2 x(t) + t^2 + g(\lambda)] \equiv f(t, x(t), \lambda), \quad t \in J, \quad x(0) = 0 \\
0 &= x(1.1) + \frac{1}{2} \sin[x(1.1) + \lambda] - \lambda \equiv G(x(1.1), \lambda)
\end{align*}
\]
where
\[ g(\lambda) = \begin{cases} 
0 & \text{if } \lambda \leq 0, \\
\lambda^2 & \text{if } 0 < \lambda < 1, \\
1 & \text{if } 1 \leq \lambda.
\end{cases} \]

Since
\[ \frac{1}{3}t^2 \leq f(t, x, \lambda) \leq 1 + x^2, \]
then
\[ y_0(t) = \frac{1}{9}t^3, \quad z_0(t) = \tan t, \quad t \in J. \]

Similarly,
\[ \frac{1}{9} - \frac{1}{10} - \lambda \leq x(1.1) - \frac{1}{10} - \lambda \leq G(x(1.1), \lambda) \leq x(1.1) + \frac{1}{10} - \lambda \leq \tan 1 + \frac{1}{10} - \lambda, \]
then
\[ \lambda_0 = \frac{1}{90} \approx 0.01, \quad \gamma_0 = \tan 1 + \frac{1}{10} \approx 1.66. \]

Note that \((y_0, \lambda_0), (z_0, \gamma_0)\) are lower and upper solutions of our problem, respectively, and \(y_0(t) \leq z_0(t), \quad t \in J, \quad \lambda_0 < \gamma_0.\) Condition 1° of Theorem 1 holds since \(g\) is nondecreasing in \(\lambda.\) Let \(\bar{u} \geq u\) and \(\bar{\lambda} \geq \lambda.\) Using the mean value theorem we see that
\[ f(t, \bar{u}, \lambda) - f(t, u, \lambda) = (\bar{u})^2 - u^2 + \frac{1}{3}[\sin^2 \bar{u} - \sin^2 u] \geq \frac{1}{3}[\sin^2 \bar{u} - \sin^2 u] \geq -\frac{1}{3}(\bar{u} - u), \]
\[ G(\bar{u}, \lambda) - G(u, \lambda) = \bar{u} - u + \frac{1}{10}[\sin(\bar{u} + \lambda) - \sin(u + \lambda)] \geq \frac{9}{10}(\bar{u} - u), \]
\[ G(u, \bar{\lambda}) - G(u, \lambda) = \frac{1}{10}[\sin(u + \bar{\lambda}) - \sin(u + \lambda)] - (\bar{\lambda} - \lambda) \geq -\frac{11}{9}(\bar{\lambda} - \lambda). \]

All assumptions of Theorem 1 hold, its assertion is satisfied for this problem.

**Theorem 2** (see [4]). Let \(f \in C(J \times R \times R, R), \ G \in C(R \times R, R)\) and }\]
\[ 1° \quad (y_0, \lambda_0), (z_0, \gamma_0) \in C^1(J, R) \times R \text{ are lower and upper solutions of problem (1.1)} \]
such that \(y_0(t) \leq z_0(t), \quad t \in J\) and \(\lambda_0 \leq \gamma_0,\)
\[ 2° \quad f(t, u, \bar{\lambda}) - f(t, u, \lambda) \geq Q(\bar{\lambda} - \lambda) \text{ for } \lambda_0 \leq \lambda \leq \bar{\lambda} \leq \gamma_0 \text{, } y_0 \leq u \leq z_0 \text{ with } Q \geq 0, \]
\[ 3° \quad f(t, \bar{u}, \lambda) - f(t, u, \lambda) \geq -M(\bar{u} - u) = \text{ for } y_0 \leq u \leq \bar{u} \leq z_0 \text{ with } M \geq 0, \]
\[ 4° \quad G \text{ is nondecreasing with respect to the first variable,} \]
\[ 5° \quad G(u, \bar{\lambda}) - G(u, \lambda) \geq -N(\bar{\lambda} - \lambda) \text{ for } \lambda_0 \leq \lambda \leq \bar{\lambda} \leq \gamma_0, \quad y_0 \leq u \leq z_0, \quad \text{with } N > 0. \]

Then there exist monotone sequences \(\{y_n, \lambda_n\}, \ {z_n, \gamma_n\} \text{ such that } y_n(t) \to y(t), \z_n(t) \to z(t), \quad t \in J\) and \(\lambda_n \to \lambda, \quad \gamma_n \to \gamma \text{ as } n \to \infty\) and this convergence is uniform and monotone on \(J.\) Moreover, \((y, \lambda)\) and \((z, \gamma)\) are minimal and maximal solutions of problem (1.1), respectively.

**Remark 2.** Our results may be extended to a finite system of differential equations of type (1.1). Although this extension follows similar ideas, such a case requires special
care since we need to split variables as it is evident in the technique of monotone iterations.

3. Quasilinearization method

Let \( y_0, z_0 \in C^1(J, R) \) and \( \lambda_0, \gamma_0 \in R \) such that \( y_0(t) \leq z_0(t) \) on \( J \) and \( \lambda_0 \leq \gamma_0 \). Define the closed sets:
\[
\Omega = \{(t, y, \lambda) : t \in J, \ y_0(t) \leq y \leq z_0(t), \ \lambda_0 \leq \lambda \leq \gamma_0 \},
\]
\[
\bar{\Omega} = \{(y, \lambda) : y_0(b) \leq y \leq z_0(b), \ \lambda_0 \leq \lambda \leq \gamma_0 \}.
\]

Now, instead of (1.1), we shall consider the following problem
\[
\begin{cases}
x'(t) = f(t, x(t), \lambda) + g(t, x(t), \lambda), & t \in J, \ x(0) = k_0 \in R, \\
G(x(b), \lambda) + H(x(b), \lambda) = 0,
\end{cases}
\]
where \( f, g \in C(J \times R \times R, R) \), \( G, H \in C(R \times R, R) \).

The method of quasilinearization offers monotone sequences of approximate solutions that converge quadratically to the unique solution. This problem is considered in the next theorems using less restrictive assumptions.

**Theorem 3** (see [5]). Let \( f, g, f_y, g_y, f_\lambda, g_\lambda \in C(\Omega, R) \), \( G, H, G_y, H_y, G_\lambda, H_\lambda \in C(\bar{\Omega}, R) \). Assume that:

1° \((y_0, \lambda_0), (z_0, \gamma_0) \in C^1(J, R) \times R\) are lower and upper solutions of problem (3.1), respectively, and such that \( y_0(t) \leq z_0(t) \), \( t \in J \) and \( \lambda_0 \leq \gamma_0 \),

2° \(-G_\lambda(u, v) \geq K_1, -H_\lambda(u, v) \geq K_2, K = K_1 + K_2 > 0, \bar{L}_1 \leq G_y(u, v) \leq L_1, \bar{L}_2 \leq H_y(u, v) \leq L_2, \bar{L} = L_1 + L_2 \geq 0, \quad M_1 \leq f_\lambda(t, \bar{u}, \bar{v}) \leq M_2, \quad M_2 \leq g_\lambda(t, \bar{u}, \bar{v}) \leq M_2, \quad M = M_1 + M_2 \geq 0\)

for \((u, v) \in \bar{\Omega}, (t, \bar{u}, \bar{v}) \in \bar{\Omega}\), and put \( L = L_1 + L_2, \quad M = M_1 + M_2 \),

3° \(|f_y(t, u, v)| \leq N_1, \quad |g_y(t, u, v)| \leq N_2 \) for \((t, u, v) \in \Omega\), and put \( N = N_1 + N_2 \),

4° \( S(b) < 1 \), where
\[
S(t) = \begin{cases} \frac{MLt}{KN} & \text{if } N = 0, \\
\frac{ML}{KN} \exp(NT) - 1 & \text{if } N > 0,
\end{cases}
\]

5° \( f_{yy}, f_{y\lambda}, f_{\lambda\lambda}, f_{y\gamma}, g_{yy}, g_{y\lambda}, g_{\lambda\lambda}, g_{y\gamma} \) exist, are continuous and satisfy the relations:
\[
f_{yy}(t, u, v) \geq 0, \quad f_{y\lambda}(t, u, v) \geq 0, \quad f_{\lambda\lambda}(t, u, v) \geq 0 \text{ for } (t, u, v) \in \Omega,
\]
\[
g_{yy}(t, u, v) \leq 0, \quad g_{y\lambda}(t, u, v) \leq 0, \quad g_{\lambda\lambda}(t, u, v) \leq 0 \text{ for } (t, u, v) \in \Omega,
\]

6° \( G_{yy}, G_{y\lambda}, G_{\lambda\lambda}, H_{yy}, H_{y\lambda}, H_{\lambda\lambda} \) exist, are continuous and satisfy the relations:
\[
G_{yy}(u, v) \geq 0, \quad G_{y\lambda}(u, v) \geq 0, \quad G_{\lambda\lambda}(u, v) \geq 0 \text{ for } (u, v) \in \bar{\Omega},
\]
Then there exist monotone sequences \( \{y_n\}, \{z_n\}, \{\lambda_n\}, \{\gamma_n\} \) which converge uniformly to the unique solution \((x, \lambda)\) of problem (3.1) and the convergence is quadratic, i.e.,

\[
\max_{t \in J} |x(t) - y_{n+1}(t)| \leq a_1 \max_{t \in J} |x(t) - y_n(t)|^2 + a_2 \max_{t \in J} |x(t) - z_n(t)|^2 + a_3 |\lambda - \lambda_n|^2 + a_4 |\lambda - \gamma_n|^2,
\]

\[
|\lambda - \lambda_{n+1}| \leq b_1 \max_{t \in J} |x(t) - y_n(t)|^2 + b_2 \max_{t \in J} |x(t) - z_n(t)|^2 + b_3 |\lambda - \lambda_n|^2 + b_4 |\lambda - \gamma_n|^2,
\]

\[
|\lambda - \gamma_{n+1}| \leq \bar{b}_1 \max_{t \in J} |x(t) - y_n(t)|^2 + \bar{b}_2 \max_{t \in J} |x(t) - z_n(t)|^2 + \bar{b}_3 |\lambda - \lambda_n|^2 + \bar{b}_4 |\lambda - \gamma_n|^2
\]

for some nonnegative constants \(a_i, b_i, \bar{a}_i, \bar{b}_i, i = 1, 2, 3, 4\).

### 4. Generalization of the quasilinearization method

We will use the following notation \( f \in C^{0,2,2}(\Omega, R) \) which means that \( f, f_y, f_\lambda, f_{yy}, f_{y\lambda}, f_{\lambda y}, f_{\lambda\lambda} \in C(\Omega, R) \). The next theorem generalizes the result of Theorem 3.

**Theorem 4** (see [6]). Let \( f, g \in C^{0,2,2}(\Omega, R) \), \( G, H \in C^2(\Omega, R) \). Assume that:

1. \((y_0, \lambda_0), (z_0, \gamma_0) \in C^1(J, R) \times R\) is lower and upper solutions of problem (3.1), respectively, and such that \( y_0(t) \leq z_0(t), t \in J \) and \( \lambda_0 \leq \gamma_0 \),
2. \(-G_\lambda(u, v) \geq K_1, -H_\lambda(u, v) \geq K_2\) and \( K = K_1 + K_2 > 0 \),
3. \( \bar{L}_1 \leq G_y(u, v) \leq L_1, \bar{L}_2 \leq H_y(u, v) \leq L_2, \bar{L} = \bar{L}_1 + \bar{L}_2 \),
4. \( \bar{M}_1 \leq f_\lambda(t, u, v) \leq M_1, \bar{M}_2 \leq g_\lambda(t, u, v) \leq M_2, \bar{M} = \bar{M}_1 + \bar{M}_2 \) for \((u, v) \in \Omega, (t, \bar{u}, \bar{v}) \in \Omega, \) and put \( \bar{L} = L_1 + L_2, \bar{M} = M_1 + M_2 \),
5. \( |f_y(t, u, v)| \leq N_1, |g_y(t, u, v)| \leq N_2\) for \((t, u, v) \in \Omega\), and put \( N = N_1 + N_2 \),
6. \( S(b) < 1\), where \( S \) is defined as in condition 4 of Theorem 3,
7. \( \Psi, \Phi \in C^{0,2,2}(\Omega, R) \) and satisfy the relations for \((t, u, v) \in \Omega\) with \( F = f + \Phi, \Psi = T = g + \Psi \)

\[
\Phi_{yy}(t, u, v) \geq 0, \Phi_{y\lambda}(t, u, v) \geq 0, \Phi_{\lambda\lambda}(t, u, v) \geq 0,
\]

\[
F_{yy}(t, u, v) \geq 0, F_{y\lambda}(t, u, v) \geq 0, F_{\lambda\lambda}(t, u, v) \geq 0,
\]

\[
\Psi_{yy}(t, u, v) \leq 0, \Psi_{y\lambda}(t, u, v) \leq 0, \Psi_{\lambda\lambda}(t, u, v) \leq 0,
\]
$T_{yy}(t,u,v) \leq 0$, $T_{y\lambda}(t,u,v) \leq 0$, $T_{\lambda\lambda}(t,u,v) \leq 0$

$6^o \Delta, \Gamma \in C^2(\bar{\Omega}, R)$ and satisfy the relations for $(u,v) \in \Omega$ with $P = G + \Delta, = Q = H + \Gamma$

$$\Delta_{yy}(u,v) \geq 0, \quad \Delta_{y\lambda}(u,v) \geq 0, \quad \Delta_{\lambda\lambda}(u,v) \geq 0,$$

$$P_{yy}(u,v) \geq 0, \quad P_{y\lambda}(u,v) \geq 0, \quad P_{\lambda\lambda}(u,v) \geq 0,$$

$$\Gamma_{yy}(u,v) \leq 0, \quad \Gamma_{y\lambda}(u,v) \leq 0, \quad \Gamma_{\lambda\lambda}(u,v) \leq 0,$$

$$Q_{yy}(u,v) \leq 0, \quad Q_{y\lambda}(u,v) \leq 0, \quad Q_{\lambda\lambda}(u,v) \leq 0,$$

$7^o \quad \eta \leq \Gamma_y(u,v) \leq \eta, \quad \bar{\zeta} \leq \Delta_y(u,v) \leq \zeta, \quad \bar{L} + \bar{\zeta} - \zeta + \bar{\eta} - \eta \geq 0, \quad \bar{\vartheta} \leq \Phi_\lambda(t, \bar{u}, \bar{v}) \leq \vartheta, \quad \bar{\mu} \leq \Psi_\lambda(t, \bar{u}, \bar{v}) \leq \mu, \quad \bar{M} + \bar{\vartheta} - \vartheta + \bar{\mu} - \mu \geq 0$ for $(u,v) \in \Omega, \quad (t, \bar{u}, \bar{v}) \in \bar{\Omega}.$

Then there exist monotone sequences $\{y_n\}, \{z_n\}, \{\lambda_n\}, \{\gamma_n\}$ which converge uniformly to the unique solution of problem (3.1) and the convergence is quadratic.

**Remark 3.** Note that Theorem 4 contains, as a special case, the result of Theorem 3 if we put $\Phi(t,u,v) = \Psi(t,u,v) = 0$ for $(t,u,v) \in \Omega$, and $\Delta(\bar{u}, \bar{v}) = \Gamma(\bar{u}, \bar{v}) = 0$ for $(\bar{u}, \bar{v}) \in \bar{\Omega}$. Finally, we study the situation when problem (3.1) is replaced by the following one

$$\{ \begin{array}{l} x'(t) = \bar{f}(t,x(t),\lambda), \quad t \in J, \quad x(0) = k_0, \\ 0 = \bar{G}(x(b),\lambda) \end{array} \quad (4.1)$$

with $\bar{f} = f + \bar{f} + g + \bar{g} = \bar{G} = G + \bar{G} + H + \bar{H}$. We get weak quadratic convergence only if we assume that $f,g \in C^{0,2,2}(\bar{\Omega}, R), \quad G,H \in C^2(\bar{\Omega}, R), \quad \bar{f}, \bar{g} \in C^{0,1,1}(\bar{\Omega}, R) = \bar{G}, \bar{H} \in C^1(\bar{\Omega}, R).$ This general case is considered in the next theorem.

**Theorem 5** (see [7]). Let $f,g \in C^{0,2,2}(\bar{\Omega}, R), \quad G,H \in C^2(\bar{\Omega}, R), \quad \bar{f}, \bar{g} \in C(\bar{\Omega}, R), = G, \bar{H} \in C(\bar{\Omega}, R).$ Assume that:

$1^o \quad (y_0, \lambda_0), \quad (z_0, \gamma_0) \in C^1(J, R) \times R$ are lower and upper solutions of problem (4.1), respectively, and such that $y_0(t) \leq z_0(t), \quad t \in J$ and $\lambda_0 \leq \gamma_0$,

$2^o$ conditions $5^o$ and $6^o$ of Theorem 4 hold,

$3^o \quad \bar{G}_{\lambda}(u,v) \geq \bar{K}_1, \quad \bar{H}_{\lambda}(u,v) \geq \bar{K}_2, \quad \bar{L}_1 \leq \bar{G}_y(u,v) \leq \bar{L}_1, \quad \bar{L}_2 \leq \bar{H}_y(u,v) \leq \bar{L}_2,
\quad M_1 \leq \bar{f}_\lambda(t,\bar{u},\bar{v}) \leq M_1, \quad \bar{M}_2 \leq \bar{g}_\lambda(t,\bar{u},\bar{v}) \leq \bar{M}_2 \text{ for } (u,v) \in \bar{\Omega}, \quad (t,\bar{u},\bar{v}) \in \bar{\Omega}; \quad |\bar{f}_y(t,u,v)| \leq N_1, \quad |\bar{g}_y(t,u,v)| \leq N_2 \text{ for } (t,u,v) \in \bar{\Omega},$

$5^o \quad \bar{F}_y, \bar{F}_\lambda, \bar{\Phi}_y, \bar{P}_y, \bar{P}_\lambda, \bar{\Delta}_y, \bar{\Delta}_\lambda \text{ exist, are nondecreasing in the last two variables, while } \bar{T}_y, \bar{T}_\lambda, \bar{\Psi}_y, \bar{Q}_y, \bar{Q}_\lambda, \bar{\Gamma}_y, \bar{\Gamma}_\lambda \text{ exist and are nonincreasing in the last two variables with } \bar{F} = \bar{f} + \Phi, \quad \bar{T} = \bar{g} + \Psi, \quad \bar{P} = \bar{G} + \Delta, = \bar{Q} = \bar{H} + \Gamma,
\quad \bar{6}^o \quad \bar{M}_3 \leq \bar{f}_\lambda(t,\bar{u},\bar{v}) \leq \bar{M}_3, \quad \bar{M}_4 \leq \bar{g}_\lambda(t,\bar{u},\bar{v}) \leq \bar{M}_4,$

$K_3 \leq -\bar{G}_{\lambda}(u,v) \leq K_3, \quad K_4 \leq -\bar{H}_{\lambda}(u,v) \leq K_4, \quad L_3 \leq \bar{G}_y(u,v) \leq L_3, \quad L_4 \leq \bar{H}_y(u,v) \leq L_4, \quad |\bar{f}_y(t,\bar{u},\bar{v})| \leq N_3, \quad |\bar{g}_y(t,\bar{u},\bar{v})| \leq N_4, \quad \bar{\eta}_1 \leq \bar{F}_y(u,v) \leq \eta_1, \quad \bar{\zeta}_1 \leq \bar{\Delta}_y(u,v) \leq \zeta_1, \quad \bar{\vartheta}_1 \leq \bar{\Phi}_\lambda(t,\bar{u},\bar{v}) \leq \vartheta_1, \quad \bar{\mu}_1 \leq \bar{\Psi}_\lambda(t,\bar{u},\bar{v}) \leq \mu_1 \text{ for } (u,v) \in \bar{\Omega}, \quad (t,\bar{u},\bar{v}) \in \bar{\Omega}.$
7° put \( L = L_1 + L_2 + L_3 + L_4 \), \( K = K_1 + K_2 + K_3 + K_4 > 0 \), \( M = M_1 + M_2 + M_3 + M_4 \), \( N = N_1 + N_2 + N_3 + N_4 \), \( \bar{L} = L_1 + L_2 + L_3 + L_4 + \zeta_1 - \zeta_t - \eta_1 - \eta_t \), \( \bar{M} = M_1 + M_2 + M_3 + M_4 + \bar{\vartheta}_1 - \bar{\vartheta}_t + \bar{\mu}_1 - \mu_t \),

8° condition 7° of Theorem 4 holds,

9° \( S(b) < 1 \), where \( S \) is defined as in condition 4° of Theorem 3.

Then there exist monotone sequences \( \{y_n\}, \{z_n\}, \{\lambda_n\}, \{\gamma_n\} \) which converge uniformly and monotonically to the unique solution of problem (4.1) and the convergence is semi–quadratic, i.e.,

\[
\max_{t \in J} |x(t) - y_{n+1}(t)| \leq a_1 \max_{t \in J} |x(t) - y_n(t)| + a_2 \max_{t \in J} |x(t) - z_n(t)|^2 + a_3 |\lambda - \lambda_n|^2 + a_4 |\lambda - \gamma_n|^2 + a_5 \max_{t \in J} |x(t) - y_n(t)| + a_6 |\lambda - \lambda_n|,
\]

\[
|\lambda - \lambda_{n+1}| \leq b_1 \max_{t \in J} |x(t) - y_n(t)|^2 + b_2 \max_{t \in J} |x(t) - z_n(t)|^2 + b_3 |\lambda - \lambda_n|^2 + b_4 |\lambda - \gamma_n|^2 + b_5 \max_{t \in J} |x(t) - y_n(t)| + b_6 |\lambda - \lambda_n|,
\]

\[
\max_{t \in J} |x(t) - z_{n+1}(t)| \leq \bar{a}_1 \max_{t \in J} |x(t) - y_n(t)|^2 + \bar{a}_2 \max_{t \in J} |x(t) - z_n(t)|^2 + \bar{a}_3 |\lambda - \lambda_n|^2 + \bar{a}_4 |\lambda - \gamma_n|^2 + \bar{a}_5 \max_{t \in J} |x(t) - z_n(t)| + \bar{a}_6 |\lambda - \lambda_n|,
\]

\[
|\lambda - \gamma_{n+1}| \leq \bar{b}_1 \max_{t \in J} |x(t) - y_n(t)|^2 + \bar{b}_2 \max_{t \in J} |x(t) - z_n(t)|^2 + \bar{b}_3 |\lambda - \lambda_n|^2 + \bar{b}_4 |\lambda - \gamma_n|^2 + \bar{b}_5 \max_{t \in J} |x(t) - z_n(t)| + \bar{b}_6 |\lambda - \lambda_n|,
\]

for some nonnegative constants \( a_i, b_i, \bar{a}_i, \bar{b}_i \), \( i = 1, 2, 3, 4, 5, 6 \).

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