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RECONSTRUCTION OF THE STURM-LIOUVILLE DIFFERENTIAL OPERATORS WITH TWO CONSTANT DELAYS

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Abstract. In this manuscript, we study the Sturm–Liouville differential operator with two constant delays. We investigate the properties of the asymptotic form of solutions, eigenvalues, and eigenfunctions of the operator. An inverse spectral problem is studied of recovering the potential functions and delay points from four boundary value problems. Also, we construct the Fourier coefficients. So, we construct the potential functions by using the Fourier series.

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1. INTRODUCTION

We consider the Sturm–Liouville differential equations

$$\ell_i y := -y''(x) + q_1(x)y(x-a_1) + (-1)^i q_2(x)y(x-a_2) = \lambda y(x), \quad x \in (0,\pi), \tag{1.1}$$

subject to the boundary conditions

$$y(0) = y^{(j)}(\pi) = 0, \tag{1.2}$$

where $q_1(x) \in L(a_1,\pi)$, $q_2(x) \in L(a_2,\pi)$, $q_1(x) = 0$ for $x < a_1$, and $q_2(x) = 0$ for $x < a_2$, are real functions. In what follows, we always take $i, j = 0, 1$. The coefficient $a_1, a_2 \in [0,\pi)$ are real and assumed to be known a priori and fixed and $a_1 < a_2$. For simplicity we use the notation $L_{i,j} := L_{i,j}(q_1(x); q_2(x); a_1; a_2)$, for the problems (1.1)–(1.2).

For the Sturm–Liouville problems, we have three types of problems: *Direct problems*, *Isospectral problems* and *Inverse problems*. In direct problems, the eigenvalues, eigenfunctions and some properties of the problem are estimated from the known coefficients. In isospectral problems, for a given problem, we want to obtain different problems of the same form, which have the same eigenvalues of the initial problem. Isospectral Sturm–Liouville problems are studied in [6, 9, 10]. The third type of problems related to the Sturm–Liouville problems are inverse problem. The inverse spectral Sturm–Liouville problem can be regarded as three aspects: existence, uniqueness and reconstruction of the coefficients with specific properties of eigenvalues and eigenfunctions, (see [4,8,12,16–19,21,22,26] and the references therein).

In the seminal paper [5], G. Freiling and V. A. Yurko motivated by the inverse Sturm–Liouville problem with a constant delays inside the interval. In this paper, they proved that if the spectra of the problems $L_j(q),\ j=0,1$, coincide with the spectra of $L_j(0),\ j=0,1$, respectively, then $q(x)=0$ a.e. on $(0,\pi)$. So, they proved a spacial uniqueness theorem in the case of one constant delay. This uniqueness theorem was later extended by me to the cases of two and finite number of constant delays [20,21]. In [23], M. Shahriari, B. N. Saray, and J. Manafian studied the inverse delay Sturm–Liouville problems with a transmission conditions inside the interval. We constructed delay point and the potential function by using the coefficients of the Fourier series of the Sturm–Liouville differential operator. Moreover, in [11], S. Mosazadeh were studied an inverse Sturm–Liouville problem with a delay and eigenparameter–dependent boundary conditions.

Although other effective methods have been created and some aspects of the direct and inverse problems for operators with a delay can be found in [1–3,7,14,15,24,27]. For general background on the delay differential equations we refer (e.g.) to the monographs [13,25].

In the present paper, we study an inverse problem of Sturm–Liouville differential operators. In this note, we discuss the reconstruction of the potential functions $q_1(x)$, $q_2(x)$ and constant delays a_1, a_2 with four boundary spectral problems (1.1)–(1.2). For this purpose, we study the asymptotic form of eigenvalues and eigenfunctions of the problems. So, we investigate the inverse spectral problem of recovering operators from their four spectra in the Dirichlet–Dirichlet and Dirichlet–Neumann boundary conditions with two constant delays inside the interval. Finally, we construct the potential functions by using the Fourier series.

2. ASYMPTOTIC FORM OF SOLUTIONS AND EIGENVALUES

Let $\varphi^i(x,\lambda)$ be the solution of Eq. (1.1) with $a_1N<\pi\le a_1(N+1)$ and $a_2M<\pi\le a_2(M+1)$ under the initial conditions $\varphi^i(0,\lambda)=0$, $\varphi^{i'}(0,\lambda)=1$. For each fixed x, the functions $\varphi^{i(j)}(x,\lambda)$ are entire in λ of order $1/2$. The function $\varphi^i(x,\lambda)$ is the unique solution of the integral equation

$$\varphi^i(x,\lambda)=\frac{\sin\rho x}{\rho}+\int_0^x\frac{\sin\rho(x-t)}{\rho}\left(q_1(t)\varphi^i(t-a_1,\lambda)+(-1)^iq_2(t)\varphi^i(t-a_2,\lambda)\right)\mathrm{d}t, \tag{2.1}$$

with $\rho^2=\lambda$ and $\rho=\sigma+i\tau$. Solving (2.1) by the method of successive approximations, we get

$$\varphi^i(x,\lambda)=\varphi^i_0(x,\lambda)+\varphi^i_1(x,\lambda)+\cdots+\varphi^i_N(x,\lambda). \tag{2.2}$$

So, we have

$$\varphi_0^i(x,\lambda)=\frac{\sin\rho x}{\rho}, \tag{2.3}$$

$$\varphi_k^i(x,\lambda)=\begin{cases}0, & x\le ka_1,\\ \int_{ka_1}^x \frac{\sin\rho(x-t)}{\rho}q_1(t)\varphi_{k-1}^i(t-a_1,\lambda)\mathrm{d}t, & ka_1\le x\le ka_2,\\ \int_{ka_1}^x \frac{\sin\rho(x-t)}{\rho}q_1(t)\varphi_{k-1}^i(t-a_1,\lambda)\mathrm{d}t & \\ \quad +(-1)^i\int_{ka_2}^x \frac{\sin\rho(x-t)}{\rho}q_2(t)\varphi_{k-1}^i(t-a_2,\lambda)\mathrm{d}t, & x\ge ka_2,\end{cases} \tag{2.4}$$

$$\varphi_k^{i\,\prime}(x,\lambda)=\begin{cases}0, & x\le ka_1,\\ \int_{ka_1}^x \cos\rho(x-t)q_1(t)\varphi_{k-1}^i(t-a_1,\lambda)\mathrm{d}t, & ka_1\le x\le ka_2,\\ \int_{ka_1}^x \cos\rho(x-t)q_1(t)\varphi_{k-1}^i(t-a_1,\lambda)\mathrm{d}t & \\ \quad +(-1)^i\int_{ka_2}^x \cos\rho(x-t)q_2(t)\varphi_{k-1}^i(t-a_2,\lambda)\mathrm{d}t, & x\ge ka_2,\end{cases} \tag{2.5}$$

for $k=1,2,\dots,M$ and

$$\varphi_k^i(x,\lambda)=\begin{cases}0, & x\le ka_1,\\ \int_{ka_1}^x \frac{\sin\rho(x-t)}{\rho}q_1(t)\varphi_{k-1}^i(t-a_1,\lambda)\mathrm{d}t, & x\ge ka_1,\end{cases} \tag{2.6}$$

$$\varphi_k^{i\,\prime}(x,\lambda)=\begin{cases}0, & x\le ka_1,\\ \int_{ka_1}^x \cos\rho(x-t)q_1(t)\varphi_{k-1}^i(t-a_1,\lambda)\mathrm{d}t, & x\ge ka_1,\end{cases} \tag{2.7}$$

for $k=M+1,M+2,\dots,N$. Using the formulas (2.3)–(2.5) we calculate

$$\varphi_1^i(x,\lambda)=\begin{cases}0, & x\le a_1,\\ \int_{a_1}^x \frac{\sin\rho(x-t)}{\rho}q_1(t)\varphi_0^i(t-a_1,\lambda)\mathrm{d}t, & a_1\le x\le a_2,\\ \int_{a_1}^x \frac{\sin\rho(x-t)}{\rho}q_1(t)\varphi_0^i(t-a_1,\lambda)\mathrm{d}t & \\ \quad +(-1)^i\int_{a_2}^x \frac{\sin\rho(x-t)}{\rho}q_2(t)\varphi_0^i(t-a_2,\lambda)\mathrm{d}t, & x\ge a_2,\end{cases}$$

$$= \begin{cases} 0, & x \le a_1, \\ \int_{a_1}^{x} \frac{\sin\rho(x-t)}{\rho} q_1(t) \frac{\sin\rho(t-a_1)}{\rho} \mathrm{d}t, & a_1 \le x \le a_2, \\ \int_{a_1}^{x} \frac{\sin\rho(x-t)}{\rho} q_1(t) \frac{\sin\rho(t-a_1)}{\rho} \mathrm{d}t \\ \quad +(-1)^i \int_{a_2}^{x} \frac{\sin\rho(x-t)}{\rho} q_2(t) \frac{\sin\rho(t-a_2)}{\rho} \mathrm{d}t, & x \ge a_2, \end{cases}$$

$$= \begin{cases} 0, & x \le a_1, \\ \frac{1}{2\rho^2} \big(-\cos\rho(x-a_1) \int_{a_1}^{x} q_1(t)\mathrm{d}t \\ \quad + \int_{a_1}^{x} \cos\rho(2t-x-a_1) q_1(t)\mathrm{d}t \big), & a_1 \le x \le a_2, \\ \frac{1}{2\rho^2} \big(-\cos\rho(x-a_1) \int_{a_1}^{x} q_1(t)\mathrm{d}t \\ \quad -(-1)^i \cos\rho(x-a_2) \int_{a_2}^{x} q_2(t)\mathrm{d}t \\ \quad + \int_{a_1}^{x} \cos\rho(2t-x-a_1) q_1(t)\mathrm{d}t \\ \quad +(-1)^i \int_{a_2}^{x} \cos\rho(2t-x-a_2) q_2(t)\mathrm{d}t \big), & x \ge a_2, \end{cases}$$

$$= \begin{cases} 0, & x \le a_1, \\ \frac{1}{2\rho^2} \big(-\cos\rho(x-a_1) \int_{a_1}^{x} q_1(t)\mathrm{d}t \\ \quad + \cos\rho(x+a_1) \int_{a_1}^{x} \cos(2\rho t) q_1(t)\mathrm{d}t \\ \quad + \sin\rho(x+a_1) \int_{a_1}^{x} \cos(2\rho t) q_1(t)\mathrm{d}t \big), & a_1 \le x \le a_2, \\ \frac{1}{2\rho^2} \big(-\cos\rho(x-a_1) \int_{a_1}^{x} q_1(t)\mathrm{d}t \\ \quad -(-1)^i \cos\rho(x-a_2) \int_{a_2}^{x} q_2(t)\mathrm{d}t \\ \quad + \cos\rho(x+a_1) \int_{a_1}^{x} \cos(2\rho t) q_1(t)\mathrm{d}t \\ \quad + \sin\rho(x+a_1) \int_{a_1}^{x} \sin(2\rho t) q_1(t)\mathrm{d}t \\ \quad +(-1)^i \cos\rho(x+a_2) \int_{a_2}^{x} \cos(2\rho t) q_2(t)\mathrm{d}t \\ \quad + \sin\rho(x+a_2) \int_{a_2}^{x} \sin(2\rho t) q_2(t)\mathrm{d}t \big), & x \ge a_2, \end{cases} \tag{2.8}$$

and

$\varphi_1^{i\,\prime}(x,\lambda)$

$$
= \begin{cases} 0, & x \le a_1, \\ \frac{1}{2\rho}\left(\sin\rho(x-a_1)\int_{a_1}^{x} q_1(t)\mathrm{d}t + \int_{a_1}^{x}\sin\rho(2t-x-a_1)q_1(t)\mathrm{d}t\right), & a_1 \le x \le a_2, \\ \frac{1}{2\rho}\left(\sin\rho(x-a_1)\int_{a_1}^{x} q_1(t)\mathrm{d}t + (-1)^i\sin\rho(x-a_2)\int_{a_2}^{x} q_2(t)\mathrm{d}t\right. \\ \quad + \int_{a_1}^{x}\sin\rho(2t-x-a_1)q_1(t)\mathrm{d}t \\ \quad \left. +(-1)^i\int_{a_2}^{x}\sin\rho(2t-x-a_2)q_2(t)\mathrm{d}t\right), & x \ge a_2, \end{cases}
$$

$$
= \begin{cases} 0, & x \le a_1, \\ \frac{1}{2\rho^2}\left(\sin\rho(x-a_1)\int_{a_1}^{x} q_1(t)\mathrm{d}t - \sin\rho(x+a_1)\int_{a_1}^{x}\cos(2\rho t)q_1(t)\mathrm{d}t\right. \\ \quad \left. +\sin\rho(x+a_1)\int_{a_1}^{x}\cos(2\rho t)q_1(t)\mathrm{d}t\right), & a_1 \le x \le a_2, \\ \frac{1}{2\rho^2}\left(-\cos\rho(x-a_1)\int_{a_1}^{x} q_1(t)\mathrm{d}t - (-1)^i\cos\rho(x-a_2)\int_{a_2}^{x} q_2(t)\mathrm{d}t\right. \\ \quad +\cos\rho(x+a_1)\int_{a_1}^{x}\sin(2\rho t)q_1(t)\mathrm{d}t \\ \quad -\sin\rho(x+a_1)\int_{a_1}^{x}\cos(2\rho t)q_1(t)\mathrm{d}t \\ \quad +(-1)^i\cos\rho(x+a_2)\int_{a_2}^{x}\sin(2\rho t)q_2(t)\mathrm{d}t \\ \quad \left. -(-1)^i\sin\rho(x+a_2)\int_{a_2}^{x}\cos(2\rho t)q_2(t)\mathrm{d}t\right), & x \ge a_2. \end{cases} \tag{2.9}
$$

For $k = 2$ from (2.5)–(2.9) we get

$$
\varphi_2^i(x,\lambda) = \begin{cases} 0, & x \le 2a_1, \\ \int_{2a_1}^{x}\frac{\sin\rho(x-t)}{\rho}q_1(t)\varphi_1(t-a_1,\lambda)\mathrm{d}t, & 2a_1 \le x \le 2a_2, \\ \int_{2a_1}^{x}\frac{\sin\rho(x-t)}{\rho}q_1(t)\varphi_1(t-a_1,\lambda)\mathrm{d}t \\ \quad +(-1)^i\int_{2a_2}^{x}\frac{\sin\rho(x-t)}{\rho}q_2(t)\varphi_1(t-a_2,\lambda)\mathrm{d}t, & x \ge 2a_2, \end{cases} \tag{2.10}
$$

and

$$
\varphi_2^{i\,\prime}(x,\lambda) = \begin{cases} 0, & x \le 2a_1, \\ \int_{2a_1}^{x}\cos\rho(x-t)q_1(t)\varphi_1(t-a_1,\lambda)\mathrm{d}t, & 2a_1 \le x \le 2a_2, \\ \int_{2a_1}^{x}\cos\rho(x-t)q_1(t)\varphi_1(t-a_1,\lambda)\mathrm{d}t \\ \quad +(-1)^i\int_{2a_2}^{x}\cos\rho(x-t)q_2(t)\varphi_1(t-a_2,\lambda)\mathrm{d}t, & x \ge 2a_2. \end{cases} \tag{2.11}
$$

From Eqs. (2.8)–(2.11) with a simple calculation we obtain

$$\varphi_2^{i\,(j)}(x,\lambda)=\begin{cases}0, & x\le 2a_1,\\ O(\rho^{j-3}\exp(|\tau|(x-2a_1))), & x\ge 2a_1,\end{cases}\qquad |\rho|\to\infty,$$

where $\tau=\operatorname{Im}\rho$. Using Eqs. (2.3)–(2.9) by induction one can easily show that

$$\varphi_k^{i\,(j)}(x,\lambda)=\begin{cases}0, & x\le ka_1,\\ O(\rho^{j-k-1}\exp(|\tau|(x-ka_1))), & x\ge ka_1,\end{cases}\qquad |\rho|\to\infty.$$

Denote $\Delta_j^i(\lambda):=\varphi^{i(j)}(\pi,\lambda)$. The functions $\Delta_j^i(\lambda)$ are entire functions in λ of order $\frac{1}{2}$ and the zeroes of $\Delta_j^i(\lambda)$ coincide with the eigenvalues λ_n^i of $L_{i,0}$ and μ_n^i of $L_{i,1}$. So, the function $\Delta_j^i(\lambda)$ is called the characteristic function for $L_{i,j}$. From Eqs. (2.1)–(2.2), (2.8)–(2.9), and (2.11) we calculate the following asymptotic formula for $|\rho|\to\infty$,

$$\begin{aligned}\Delta_0^i(\lambda)&=\varphi^i(\pi,\lambda) \qquad (2.12)\\ &=\frac{\sin\rho\pi}{\rho}+\frac{1}{2\rho^2}\Bigg[-\cos\rho(\pi-a_1)w_1-(-1)^i\cos\rho(\pi-a_2)w_2\\ &\quad+\int_{a_1}^{\pi}\cos\rho(2t-\pi-a_1)q_1(t)\mathrm{d}t+(-1)^i\int_{a_2}^{\pi}\cos\rho(2t-\pi-a_2)q_2(t)\mathrm{d}t\Bigg]\\ &\quad+O\left(\frac{\exp(|\tau|(\pi-a_1))}{\rho^3}\right),\end{aligned}$$

and

$$\begin{aligned}\Delta_1^i(\lambda)&=\varphi^{i\prime}(\pi,\lambda) \qquad (2.13)\\ &=\cos\rho\pi+\frac{1}{2\rho}\Bigg[\sin\rho(\pi-a_1)w_1+(-1)^i\sin\rho(\pi-a_2)w_2\\ &\quad+\int_{a_1}^{\pi}\sin\rho(2t-\pi-a_1)q_1(t)\mathrm{d}t+(-1)^i\int_{a_2}^{\pi}\sin\rho(2t-\pi-a_2)q_2(t)\mathrm{d}t\Bigg]\\ &\quad+O\left(\frac{\exp(|\tau|(\pi-a_1))}{\rho^2}\right),\end{aligned}$$

where $w_1:=\int_{a_1}^{\pi}q_1(t)\mathrm{d}t$ and $w_2:=\int_{a_2}^{\pi}q_2(t)\mathrm{d}t$.

Lemma 1 ([20, Sec. 2]). *The asymptotic formula for the eigenvalues* $\lambda_n^i={\rho_{n0}^i}^2$ *and* $\mu_n^i={\rho_{n1}^i}^2$ *as* $n\to\infty$ *are of the following forms:*

$$\rho_{n0}^i=n+\frac{1}{2\pi n}\left[w_1\cos na_1+(-1)^i w_2\cos na_2\right]+o\left(\frac{1}{n}\right), \qquad (2.14)$$

$$\rho_{n1}^i=n-\frac{1}{2}+\frac{1}{2\pi n}\left[w_1\cos\left(n-\frac{1}{2}\right)a_1+(-1)^i w_2\cos\left(n-\frac{1}{2}\right)a_2\right]+o\left(\frac{1}{n}\right).$$

Lemma 2 ([20, Lemma 2.1], [23, Lemma 2.1])**.** *The specification of the spectrum* $\{\lambda_n^i\}$*,* $n \geq 1$ *and* $j = 0, 1$*, uniquely determines the characteristic function* $\Delta_j^i(\lambda)$ *by the formulaes*

$$\Delta_0^i(\lambda) = \pi \prod_{n=1}^{\infty} \frac{\lambda_{n0}^i - \lambda}{n^2}, \qquad \Delta_1^i(\lambda) = \prod_{n=1}^{\infty} \frac{\lambda_{n1}^i - \lambda}{(n - \frac{1}{2})^2}.$$

3. RECONSTRUCTION OF POTENTIAL FUNCTION

In this section, we study the Sturm–Liouville differential operator with two constant delays. We solve the inverse spectral problems of these operators when $a_1 \in (\frac{\pi}{2}, \pi)$ and $a_2 \in (\frac{\pi}{2}, \pi)$. We will first show that the delay points a_1, a_2 and the values of w_1 and w_2 are uniquely determined by the spectrum. So, we prove our main theorem.

Lemma 3. *If* $\{\lambda_n^i\}_{n \geq 1}$ *are the spectrum of* $L_{i,0}$ *then the delay points* a_1 *and* a_2 *are uniquely determined.*

Proof. Let us consider the sequence

$$\tilde{\lambda}_n = \frac{\lambda_n^0 + \lambda_n^1}{2}$$

and

$$\hat{\lambda}_n = \frac{\lambda_n^0 - \lambda_n^1}{2}.$$

From (2.14) we get the asymptotic formulas:

$$\tilde{\lambda}_n = n^2 + \frac{w_1}{\pi} \cos n a_1 + o(1) \tag{3.1}$$

and

$$\hat{\lambda}_n = \frac{w_2}{\pi} \cos n a_2 + o(1). \tag{3.2}$$

There are infinitely many numbers $k, l \in \mathbb{N}$ with $\sin(k a_2) \neq 0$ and $\sin(l a_1) \neq 0$. From (3.1) and (3.2) we get

$$\begin{aligned} \lim_{k \to \infty} \frac{\hat{\lambda}_{k+2} - \hat{\lambda}_{k-2}}{\hat{\lambda}_{k+1} - \hat{\lambda}_{k-1}} &= \lim_{k \to \infty} \frac{w_2(\cos a_2(k+2) - \cos a_2(k-2)) + o(1)}{w_2(\cos a_2(k+1) - \cos a_2(k-1)) + o(1)} \\ &= \lim_{k \to \infty} \frac{\sin k a_2 \sin 2a_2 + o(1)}{\sin k a_2 \sin a_2 + o(1)} \\ &= \frac{\sin 2a_2}{\sin a_2} = 2 \cos a_2 \end{aligned} \tag{3.3}$$

and

$$\lim_{k \to \infty} \frac{\tilde{\lambda}_{l+2} - (l+2)^2 - \tilde{\lambda}_{l-2} + (l-2)^2}{\tilde{\lambda}_{l+1} - (l+1)^2 - \tilde{\lambda}_{l-1} + (l-1)^2}$$

$$
\begin{aligned}
&= \lim_{l\to\infty} \frac{\cos a_1(l+2)-\cos a_1(l-2)+o(1)}{\cos a_1(l+1)-\cos a_1(l-1)+o(1)}\\
&= \lim_{k\to\infty} \frac{\sin la_1 \sin 2a_1+o(1)}{\sin la_1 \sin a_1+o(1)}\\
&= \frac{\sin 2a_1}{\sin a_1}\\
&= 2\cos a_1. \qquad (3.4)
\end{aligned}
$$

Then, we get the delay points a_1 and a_2. □

Lemma 4. *If $\left\{\lambda_n^i\right\}_{n\geq 1}$ are the spectrum of $L_{i,0}$ then the values of w_1 and w_2 are uniquely determined.*

Proof. There are infinitely many $k\in\mathbb{N}$ satisfying

$$\det\begin{bmatrix} \cos ka_1 & \cos ka_2 \\ \cos ka_1 & -\cos ka_2 \end{bmatrix} \neq 0. \tag{3.5}$$

From (2.13), we get

$$w_1\cos na_1+(-1)^i w_2 \cos na_2 = \pi(\lambda_n^i-n^2)+o(1), \tag{3.6}$$

So, from (3.5) and (3.6), we calculate

$$
\begin{aligned}
w_1 &= \lim_{k\to\infty}\frac{\det\begin{bmatrix} \pi(\lambda_k^0-k^2)+o(1) & \cos ka_2 \\ \pi(\lambda_k^1-k^2)+o(1) & -\cos ka_2 \end{bmatrix}}{\det\begin{bmatrix} \cos ka_1 & \cos ka_2 \\ \cos ka_1 & -\cos ka_2 \end{bmatrix}}\\
w_2 &= \lim_{k\to\infty}\frac{\det\begin{bmatrix} \cos ka_1 & \pi(\lambda_k^0-k^2)+o(1) \\ \cos ka_1 & \pi(\lambda_k^1-k^2)+o(1) \end{bmatrix}}{\det\begin{bmatrix} \cos ka_1 & \cos ka_2 \\ \cos ka_1 & -\cos ka_2 \end{bmatrix}}
\end{aligned}
\tag{3.7}
$$

□

Applying $q_1(x)=0$ for $x\in[0,a_1)$ and $q_2(x)=0$ for $x\in[0,a_2)$, we obtain that $\int_0^\pi q_1(t)\mathrm{d}t=\int_{a_1}^\pi q_1(t)\mathrm{d}t$ and $\int_0^\pi q_2(t)\mathrm{d}t=\int_{a_2}^\pi q_2(t)\mathrm{d}t$. By using Lemma 4, we get the first Fourier coefficient of the potential q_1 and q_2 on $[0,\,\pi]$.

Now, we will obtain the Fourier coefficients of the potential function $q_1(x)$ and $q_2(x)$. So, we will prove that these coefficients are uniquely determined from the spectrum of $L_{i,j}$. Denote the Fourier coefficients of q_1 and q_2 by

$$
\begin{aligned}
a_n &= \int_0^\pi q_1(t)\cos 2nt\,\mathrm{d}t, & b_n &= \int_0^\pi q_1(t)\sin 2nt\,\mathrm{d}t,\\
c_n &= \int_0^\pi q_2(t)\cos 2nt\,\mathrm{d}t, & d_n &= \int_0^\pi q_2(t)\sin 2nt\,\mathrm{d}t.
\end{aligned}
$$

So, we finally come to our main result.

Theorem 1. *If* $\{\lambda_n^i\}_{n\geq 1}$ *and* $\{\mu_n^i\}_{n\geq 1}$ *be the eigenvalues of the boundary value problems* $L_{i,0}$ *and* $L_{i,1}$*, respectively, then the Fourier coefficient* a_n*,* b_n*,* c_n*, and* d_n *of the potential functions* q_1 *and* q_2 *uniquely determined for all* $n \in \mathbb{N}$*.*

Proof. From the Eq. (2.12), (2.13) and $a_1, a_2 \in (\frac{\pi}{2}, \pi)$ we obtain

$$\begin{aligned}
F_{i,0}(\rho) &= \Delta_0^i(\lambda) \\
&= \varphi^i(\pi,\lambda) \\
&= \frac{\sin\rho\pi}{\rho} + \frac{1}{2\rho^2}\Bigg[-\cos\rho(\pi-a_1)w_1 - (-1)^i\cos\rho(\pi-a_2)w_2 \\
&\quad + \int_{a_1}^{\pi}\cos\rho(2t-\pi-a_1)q_1(t)\mathrm{d}t + (-1)^i\int_{a_2}^{\pi}\cos\rho(2t-\pi-a_2)q_2(t)\mathrm{d}t\Bigg] \\
&= \frac{\sin\rho\pi}{\rho} + \frac{1}{2\rho^2}\Bigg[-\cos\rho(\pi-a_1)w_1 - (-1)^i\cos\rho(\pi-a_2)w_2 \\
&\quad + \cos\rho(\pi+a_1)\int_{a_1}^{\pi}\cos(2\rho t)q_1(t)\mathrm{d}t + \sin\rho(\pi+a_1)\int_{a_1}^{\pi}\sin(2\rho t)q_1(t)\mathrm{d}t \\
&\quad + (-1)^i\cos\rho(\pi+a_2)\int_{a_2}^{\pi}\cos(2\rho t)q_2(t)\mathrm{d}t \\
&\quad + (-1)^i\sin\rho(\pi+a_2)\int_{a_2}^{\pi}\sin(2\rho t)q_2(t)\mathrm{d}t\Bigg],
\end{aligned}$$

and

$$\begin{aligned}
F_{i,1}(\rho) &= \Delta_1^i(\lambda) = \varphi^{i\prime}(\pi,\lambda) \\
&= \cos\rho\pi + \frac{1}{2\rho}\Bigg[\sin\rho(\pi-a_1)w_1 + (-1)^i\sin\rho(\pi-a_2)w_2 \\
&\quad + \int_{a_1}^{\pi}\sin\rho(2t-\pi-a_1)q_1(t)\mathrm{d}t + (-1)^i\int_{a_2}^{\pi}\sin\rho(2t-\pi-a_2)q_2(t)\mathrm{d}t\Bigg] \\
&= \cos\rho\pi + \frac{1}{2\rho}\Bigg[\sin\rho(\pi-a_1)w_1 + (-1)^i\sin\rho(\pi-a_2)w_2 \\
&\quad + \cos\rho(\pi+a_1)\int_{a_1}^{\pi}\sin(2\rho t)q_1(t)\mathrm{d}t - \sin\rho(\pi+a_1)\int_{a_1}^{\pi}\cos(2\rho t)q_1(t)\mathrm{d}t \\
&\quad + (-1)^i\cos\rho(\pi+a_2)\int_{a_2}^{\pi}\sin(2\rho t)q_2(t)\mathrm{d}t
\end{aligned}$$

$$-(-1)^i \sin\rho(\pi+a_2)\int_{a_2}^{\pi}\cos(2\rho t)q_2(t)\mathrm{d}t\Bigg].$$

Now by putting $\rho = n$ (for $n \in \mathbb{N}$), we obtain

$$A_i(n) = A_n + (-1)^i C_n \tag{3.8}$$

$$B_i(n) = B_n + (-1)^i D_n \tag{3.9}$$

where

$$\begin{aligned}
&A_i(n) := 2n^2 F_{i,0}(n) + \cos n(\pi-a_1)w_1 + (-1)^i \cos n(\pi-a_2)w_2\\
&B_i(n) := 2n(F_{i,1}(n) - (-1)^n) - \sin n(\pi-a_1)w_1 - (-1)^i \sin n(\pi-a_2)w_2\\
&A_n := \int_{a_1}^{\pi}\cos n(2t-\pi-a_1)q_1(t)\mathrm{d}t, \qquad B_n := \int_{a_1}^{\pi}\sin n(2t-\pi-a_1)q_1(t)\mathrm{d}t\\
&C_n := \int_{a_2}^{\pi}\cos n(2t-\pi-a_2)q_2(t)\mathrm{d}t, \qquad D_n := \int_{a_2}^{\pi}\sin n(2t-\pi-a_2)q_2(t)\mathrm{d}t
\end{aligned}$$

Applying Lemmas 2, 3, and 4, we can compute $A_i(n)$ and $B_i(n)$. Using the formulas (3.8) and (3.9), we determine the coefficients A_n, B_n, C_n and D_n of the following forms:

$$A_n = \frac{A_0(n)+A_1(n)}{2}, \qquad C_n = \frac{A_0(n)-A_1(n)}{2},$$
$$B_n = \frac{B_0(n)+B_1(n)}{2}, \qquad D_n = \frac{B_0(n)-B_1(n)}{2}.$$

So, with a simple calculation we obtain the coefficients of the Fourier series

$$\begin{aligned}
a_n &= A_n\cos n(\pi+a_1) + B_n\sin n(\pi+a_1),\\
b_n &= A_n\sin n(\pi+a_1) - B_n\cos n(\pi+a_1),\\
c_n &= D_n\sin n(\pi+a_2) + C_n\cos n(\pi+a_2),\\
d_n &= C_n\sin n(\pi+a_2) - D_n\cos n(\pi+a_2).
\end{aligned}$$

Finally, we use the Fourier series for computing the potential functions $q_1(x)$ and $q_2(x)$. ☐

Algorithm 1.

(i) *Using Eqs.* (3.3) *and* (3.4), *for recovering the delay point a_1 and a_2.*
(ii) *Using* (3.7) *for constructing the values w_0 and w_1.*
(iii) *Applying Lemma 2 for computing $\Delta_0^i(n)$ and $\Delta_1^i(n)$.*
(iv) *Compute the value of a_n, b_n, c_n, and d_n the coefficients of Fourier series by applying Theorem 1.*
(v) *Applying the Fourier series for reconstruction the unknown potentials.*

3.1. *Example*

In this section, an example is presented.

Example 1. Suppose that in Eqs. (1.1)–(1.2), $a_1 = 2$, $a_2 = 2/5$, and the potential functions are

$$q_1(x) = \begin{cases} 7(x-2)(\pi - x), & x \geq 2, \\ 0, & x < 2. \end{cases}$$

and

$$q_2(x) = \begin{cases} 20(x-2.5)\sin(\pi - x), & x \geq 2/5, \\ 0, & x < 2/5, \end{cases}$$

Figure 1 shows that the reconstruction of $q_1(x)$ and $q_2(x)$.

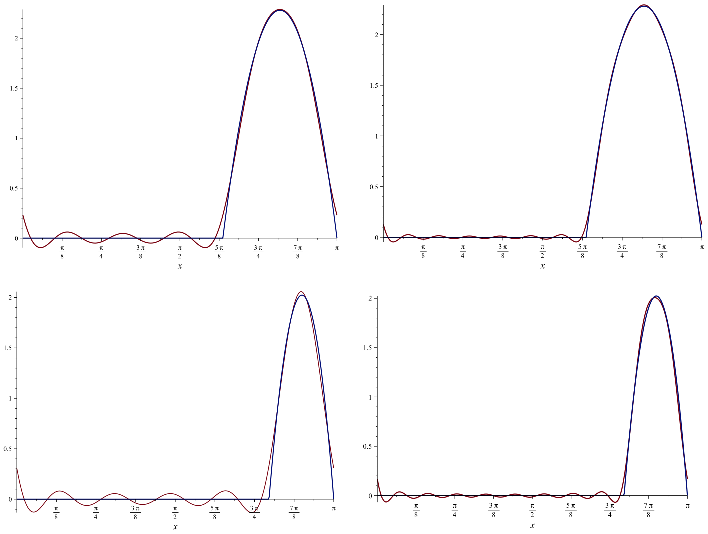

FIGURE 1. Reconstruction of potential functions $q_1(x)$ ((a) $n = 5$, (b) $n = 10$) and $q_2(x)$ ((c) $n = 5$, (d) $n = 10$).

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