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OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS OF TWO VARIABLES VIA GENERALIZED FRACTIONAL INTEGRALS

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Abstract. In this paper, we establish some Ostrowski type integral inequalities for functions of two variables involving generalized fractional integrals. The results presented here provide extensions of those given in earlier works.

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1. INTRODUCTION

The study of various types of integral inequalities has been the focus of great attention for well over a century by a number of mathematicians, interested both in pure and applied mathematics. One of the many fundamental mathematical discoveries of A. M. Ostrowski [20] is the following classical integral inequality associated with the differentiable mappings:

Theorem 1. *Let $f:\ [a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':\ (a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty=\sup_{t\in(a,b)}|f'(t)|<+\infty$. Then, we have the following inequality:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty,$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

The Ostrowski inequality has applications in quadrature, probability and optimization theory, stochastic, statistics, information and integral operator theory. Until now, a large number of research papers and books have been written on Ostrowski inequalities and their numerous applications, see [1–3, 5–15, 17–19, 21–23].

Definition 1. Let $f \in L_1[a,b]$. The Riemann–Liouville fractional integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

Hadamard fractional integrals given by as follows:

Definition 2 ([16, Page 110])**.** Let $f \in L_1([a,b])$. The Hadamard fractional integrals $\mathbf{H}_{a+}^{\alpha} f$ and $\mathbf{H}_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$\mathbf{H}_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t} dt, \quad x > a$$

and

$$\mathbf{H}_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t} dt, \quad x < b,$$

respectively.

Definition 3 ([16, Page 99-100])**.** Let $g\colon [a,b] \to \mathbb{R}$ be a positive increasing function on $(a,b]$, having a continuous derivative $g'(x)$ on (a,b). The left–side ($I_{a^+;g}^{\alpha} f(x)$) and right–side ($I_{b^-;g}^{\alpha} f(x)$) fractional integral of f with respect to the function g on $[a,b]$ of order $\alpha > 0$ are defined by

$$I_{a^+;g}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t)}{[g(x)-g(t)]^{1-\alpha}} dt, \quad x > a$$

and

$$I_{b^-;g}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t)}{[g(t)-g(x)]^{1-\alpha}} dt, \quad x < b,$$

respectively.

Hadamard fractional integrals of a function with two variables can be given as follows:

Definition 4. Let $\Delta = [a,b]\times[c,d]$ and $f \in L_1(\Delta)$. The Hadamard fractional integrals $\mathbf{J}^{\alpha,\beta}_{a+,c+}f$, $\mathbf{J}^{\alpha,\beta}_{a+,d-}f$, $\mathbf{J}^{\alpha,\beta}_{b-,c+}f$ and $\mathbf{J}^{\alpha,\beta}_{b-,d-}f$ of order $\alpha,\beta > 0$ with $a,c \geq 0$ are defined by

$$\mathbf{J}^{\alpha,\beta}_{a+,c+}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y \left(\ln\frac{x}{t}\right)^{\alpha-1}\left(\ln\frac{y}{s}\right)^{\beta-1}\frac{f(t,s)}{ts}dsdt, \quad x>a, \quad y>c,$$

$$\mathbf{J}^{\alpha,\beta}_{a+,d-}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_y^d \left(\ln\frac{x}{t}\right)^{\alpha-1}\left(\ln\frac{s}{y}\right)^{\beta-1}\frac{f(t,s)}{ts}dsdt, \quad x>a, \quad y<d,$$

$$\mathbf{J}^{\alpha,\beta}_{b-,c+}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_c^y \left(\ln\frac{t}{x}\right)^{\alpha-1}\left(\ln\frac{y}{s}\right)^{\beta-1}\frac{f(t,s)}{ts}dsdt, \quad x<b, \quad y>c$$

and

$$\mathbf{J}^{\alpha,\beta}_{b-,d-}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d \left(\ln\frac{t}{x}\right)^{\alpha-1}\left(\ln\frac{y}{s}\right)^{\beta-1}\frac{f(t,s)}{ts}dsdt, \quad x<b, \quad y<d,$$

respectively.

Now, we give the following generalized fractional integral operators:

Definition 5 ([4, Definition 6])**.** Let $g\colon [a,b]\to\mathbb{R}$ be a positive increasing function on $(a,b]$, having a continuous derivative $g'(x)$ on (a,b) and let $w\colon [c,d]\to\mathbb{R}$ be a positive increasing function on $(c,d]$, having a continuous derivative $w'(y)$ on (c,d). If $f\in L_1(\Delta)$, then for $\alpha,\beta>0$, the generalized fractional integral operators for functions of two variables are defined by

$$\mathcal{J}^{\alpha,\beta}_{a+,c+;g,w}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y \frac{g'(t)}{[g(x)-g(t)]^{1-\alpha}}\frac{w'(s)}{[w(y)-w(s)]^{1-\beta}}f(t,s)dsdt,$$
$$x>a, \quad y>c;$$

$$\mathcal{J}^{\alpha,\beta}_{a+,d-;g,w}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_y^d \frac{g'(t)}{[g(x)-g(t)]^{1-\alpha}}\frac{w'(s)}{[w(s)-w(y)]^{1-\beta}}f(t,s)dsdt,$$
$$x>a, \quad y<d;$$

$$\mathcal{J}^{\alpha,\beta}_{b-,c+;g,w}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_c^y \frac{g'(t)}{[g(t)-g(x)]^{1-\alpha}}\frac{w'(s)}{[w(y)-w(s)]^{1-\beta}}f(t,s)dsdt,$$
$$x<b, \quad y>c$$

and

$$\mathcal{J}^{\alpha,\beta}_{b-,d-;g,w}f(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d\frac{g'(t)}{[g(t)-g(x)]^{1-\alpha}}\frac{w'(s)}{[w(s)-w(y)]^{1-\beta}}f(t,s)dsdt,$$
$$x<b,\quad y<d$$

respectively.

The aim of this study is to establish Ostrowski type integral inequalities for functions of two variables involving generalized fractional integrals. The results presented in this paper provide extensions of those given in earlier works.

2. Some Ostrowski type inequalities for generalized fractional integrals

Firstly, we define the following functions which will be used frequently:

$$M^{\alpha}_{g}(a,b;x)=\frac{[g(x)-g(a)]^{\alpha}+[g(b)-g(x)]^{\alpha}}{\Gamma(\alpha+1)}$$

and

$$N^{\beta}_{w}(c,d;y)=\frac{[w(y)-w(c)]^{\beta}+[w(d)-w(y)]^{\beta}}{\Gamma(\beta+1)},$$

for $(x,y)\in\Delta$. Introduce $M^{\alpha}_{g}(a,b)$ and $N^{\beta}_{w}(c,d)$ by actting

$$M^{\alpha}_{g}(a,b)=M^{\alpha}_{g}(a,b;a)=M^{\alpha}_{g}(a,b;b)=\frac{[g(b)-g(a)]^{\alpha}}{\Gamma(\alpha+1)}$$

and

$$N^{\beta}_{w}(c,d)=N^{\beta}_{w}(c,d;c)=N^{\beta}_{w}(c,d;d)=\frac{[w(d)-w(c)]^{\beta}}{\Gamma(\beta+1)}.$$

By choosing $g(t)=\ln t$, $t\in[a,b]$ and $w(s)=\ln s$, $s\in[c,d]$, we get representations

$$M^{\alpha}_{\ln}(a,b;x)=\frac{\left[\ln\frac{x}{a}\right]^{\alpha}+\left[\ln\frac{b}{x}\right]^{\alpha}}{\Gamma(\alpha+1)}\quad\text{and}\quad N^{\beta}_{\ln}(c,d;y)=\frac{\left[\ln\frac{y}{c}\right]^{\beta}+\left[\ln\frac{d}{y}\right]^{\beta}}{\Gamma(\beta+1)}$$

and

$$M^{\alpha}_{\ln}(a,b)=\frac{\left[\ln\frac{b}{a}\right]^{\alpha}}{\Gamma(\alpha+1)}\quad\text{and}\quad N^{\beta}_{\ln}(c,d)=\frac{\left[\ln\frac{d}{c}\right]^{\beta}}{\Gamma(\beta+1)}.$$

Throughout this paper, we denote the second partial derivative $\frac{\partial^2 f}{\partial t\partial s}$ by f_{ts}. We assume also that $g\colon[a,b]\to\mathbb{R}$ is a positive increasing function on $(a,b]$, having a continuous derivative $g'(x)$ on (a,b) and $w\colon[c,d]\to\mathbb{R}$ is a positive increasing function on $(c,d]$, having a continuous derivative $w'(y)$ on (c,d).

Now, we are in position to prove the following identity:

Lemma 1. *Let* $f\colon \Delta \to \mathbb{R}$ *be a twice partial differentiable mapping on* Δ° *with* $a<b$, $c<d$. *If* $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta)$, *then for* $\alpha, \beta > 0$ *we have the following equality:*

$$\begin{aligned}
& f(x,y) - \frac{1}{M_g^\alpha(a,b;x)}\left[J^\alpha_{x^-;g} f(a,y) + J^\alpha_{x^+;g} f(b,y)\right] \\
& - \frac{1}{N_w^\beta(c,d;y)}\left[J^\beta_{y^-;w} f(x,c) + J^\beta_{y^+;w} f(x,d)\right] + \frac{1}{M_g^\alpha(a,b;x)N_w^\beta(c,d;y)} \\
& \times \left[J^{\alpha,\beta}_{x^-y^-;g,w} f(a,c) + J^{\alpha,\beta}_{x^-y^+;g,w} f(a,d) + J^{\alpha,\beta}_{x^+y^-;g,w} f(b,c) + J^{\alpha,\beta}_{x^+y^+;g,w} f(b,d)\right] \\
& = \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)M_g^\alpha(a,b;x)N_w^\beta(c,d;y)}\left[I_1 - I_2 - I_3 + I_4\right],
\end{aligned}$$

where

$$I_1 = \int_a^x \int_c^y (g(t)-g(a))^\alpha (w(s)-w(c))^\beta f_{ts}(t,s)\,dsdt,$$

$$I_2 = \int_a^x \int_y^d (g(t)-g(a))^\alpha (w(d)-w(s))^\beta f_{ts}(t,s)\,dsdt,$$

$$I_3 = \int_x^b \int_c^y (g(b)-g(t))^\alpha (w(s)-w(c))^\beta f_{ts}(t,s)\,dsdt$$

and

$$I_4 = \int_x^b \int_y^d (g(b)-g(t))^\alpha (w(d)-w(s))^\beta f_{ts}(t,s)\,dsdt.$$

Proof. By integration by parts, we have

$$\begin{aligned}
I_1 &= \int_a^x \int_c^y (g(t)-g(a))^\alpha (w(s)-w(c))^\beta f_{t,s}(t,s)\,dsdt \qquad (2.1)\\
&= \int_a^x (g(t)-g(a))^\alpha \Bigg[(w(s)-w(c))^\beta f_t(t,s)\Big|_c^y \\
&\qquad\qquad - \beta \int_c^y (w(s)-w(c))^{\beta-1} w'(s) f_t(t,s)\,ds \Bigg] dt \\
&= \int_a^x (g(t)-g(a))^\alpha (w(y)-w(c))^\beta f_t(t,y)\,dt
\end{aligned}$$

$$-\beta\int_a^x\int_c^y (g(t)-g(a))^{\alpha}(w(s)-w(c))^{\beta-1}w'(s)f_t(t,s)dsdt$$

$$= (w(y)-w(c))^{\beta}$$

$$\times\left[(g(t)-g(a))^{\alpha}f(t,y)-\alpha\int_a^x (g(t)-g(a))^{\alpha-1}g'(t)f(t,y)dt\right]$$

$$-\beta\int_c^y (w(s)-w(c))^{\beta-1}w'(s)$$

$$\times\left[(g(t)-g(a))^{\alpha}f(t,s)|_a^x-\alpha\int_a^x (g(t)-g(a))^{\alpha-1}g'(t)f(t,s)dt\right]ds$$

$$= (w(y)-w(c))^{\beta}(g(x)-g(a))^{\alpha}f(x,y)-\Gamma(\alpha+1)(w(y)-w(c))^{\beta}J^{\alpha}_{x^-;g}f(a,y)$$

$$-\Gamma(\beta+1)(g(x)-g(a))^{\alpha}J^{\beta}_{y^-;w}f(x,c)+\Gamma(\alpha+1)\Gamma(\beta+1)J^{\alpha,\beta}_{x^-y^-;g,w}f(a,c).$$

Similarly, we get

$$\begin{aligned} I_2 = &-(g(x)-g(a))^{\alpha}(w(d)-w(y))^{\beta}f(x,y) \\ &+\Gamma(\alpha+1)(w(d)-w(y))^{\beta}J^{\alpha}_{x^-;g}f(a,y) \\ &+\Gamma(\beta+1)(g(x)-g(a))^{\alpha}J^{\beta}_{y^+;w}f(x,d) \\ &-\Gamma(\alpha+1)\Gamma(\beta+1)J^{\alpha,\beta}_{x^-y^+;g,w}f(a,d), \end{aligned} \tag{2.2}$$

$$\begin{aligned} I_3 = &-(g(b)-g(x))^{\alpha}(w(y)-w(c))^{\beta}f(x,y) \\ &+\Gamma(\alpha+1)(w(y)-w(c))^{\beta}J^{\alpha}_{x^+;g}f(b,y) \\ &+\Gamma(\beta+1)(g(b)-g(x))^{\alpha}J^{\beta}_{y^-;w}f(x,c) \\ &-\Gamma(\alpha+1)\Gamma(\beta+1)J^{\alpha,\beta}_{x^+y^-;g,w}f(b,c) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} I_4 = &(g(b)-g(x))^{\alpha}(w(d)-w(y))^{\beta}f(x,y) \\ &-\Gamma(\alpha+1)(w(d)-w(y))^{\beta}J^{\alpha}_{x^+;g}f(b,y) \\ &-\Gamma(\beta+1)(g(b)-g(x))^{\alpha}J^{\beta}_{y^+;w}f(x,d) \\ &+\Gamma(\alpha+1)\Gamma(\beta+1)J^{\alpha,\beta}_{x^+y^+;g,w}f(b,d). \end{aligned} \tag{2.4}$$

By equalities (2.1)–(2.4), we establish

$$I_1 - I_2 - I_3 + I_4 = \Gamma(\alpha+1)\Gamma(\beta+1) M_g^{\alpha}(a,b;x) N_w^{\beta}(c,d;y) f(x,y)$$
$$-\Gamma(\alpha+1)\Gamma(\beta+1) N_w^{\beta}(c,d;y)\left[J_{x^-;g}^{\alpha} f(a,y) + J_{x^+;g}^{\alpha} f(b,y)\right]$$
$$-\Gamma(\alpha+1)\Gamma(\beta+1) M_g^{\alpha}(a,b;x)\left[J_{y^-;w}^{\beta} f(x,c) + J_{y^+;w}^{\beta} f(b,y)\right]$$
$$+\Gamma(\alpha+1)\Gamma(\beta+1)$$
$$\times\left[J_{x^-y^-;g,w}^{\alpha,\beta} f(a,c) + J_{x^-y^+;g,w}^{\alpha,\beta} f(a,d) + J_{x^+y^-;g,w}^{\alpha,\beta} f(b,c) + J_{x^+y^+;g,w}^{\alpha,\beta} f(b,d)\right],$$

which gives the desired result. □

Theorem 2. *Under assumptions of Lemma 1, if the function* $\frac{f_{ts}}{g'w'} \in L_\infty(\Delta)$*, then we have the following Ostrowski type inequality for the generalized fractional integrals:*

$$\Bigg| f(x,y) - \frac{1}{M_g^{\alpha}(a,b;x)}\left[J_{x^-;g}^{\alpha} f(a,y) + J_{x^+;g}^{\alpha} f(b,y)\right]$$
$$-\frac{1}{N_w^{\beta}(c,d;y)}\left[J_{y^-;w}^{\beta} f(x,c) + J_{y^+;w}^{\beta} f(x,d)\right] + \frac{1}{M_g^{\alpha}(a,b;x) N_w^{\beta}(c,d;y)}$$
$$\times\left[J_{x^-y^-;g,w}^{\alpha,\beta} f(a,c) + J_{x^-y^+;g,w}^{\alpha,\beta} f(a,d) + J_{x^+y^-;g,w}^{\alpha,\beta} f(b,c) + J_{x^+y^+;g,w}^{\alpha,\beta} f(b,d)\right]\Bigg|$$
$$\leq \frac{M_g^{\alpha+1}(a,b;x) N_w^{\beta+1}(c,d;y)}{M_g^{\alpha}(a,b;x) N_w^{\beta}(c,d;y)} \left\| \frac{f_{ts}}{g'w'} \right\|_{\Delta,\infty}$$

where

$$\left\| \frac{f_{ts}}{g'w'} \right\|_{\Delta,\infty} = \sup_{(t,s)\in\Delta} \left| \frac{f_{ts}(t,s)}{g'(t)w'(s)} \right|.$$

Proof. Using Lemma 1, we get

$$\Bigg| f(x,y) - \frac{1}{M_g^{\alpha}(a,b;x)}\left[J_{x^-;g}^{\alpha} f(a,y) + J_{x^+;g}^{\alpha} f(b,y)\right] \tag{2.5}$$
$$-\frac{1}{N_w^{\beta}(c,d;y)}\left[J_{y^-;w}^{\beta} f(x,c) + J_{y^+;w}^{\beta} f(x,d)\right] + \frac{1}{M_g^{\alpha}(a,b;x) N_w^{\beta}(c,d;y)}$$
$$\times\left[J_{x^-y^-;g,w}^{\alpha,\beta} f(a,c) + J_{x^-y^+;g,w}^{\alpha,\beta} f(a,d) + J_{x^+y^-;g,w}^{\alpha,\beta} f(b,c) + J_{x^+y^+;g,w}^{\alpha,\beta} f(b,d)\right]\Bigg|$$
$$\leq \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1) M_g^{\alpha}(a,b;x) N_w^{\beta}(c,d;y)} \left[|I_1| + |I_2| + |I_3| + |I_4|\right].$$

By the assumptions of Theorem 2, we have

$$\begin{aligned}
|I_1| &= \left| \int_a^x \int_c^y (g(t)-g(a))^{\alpha} (w(s)-w(c))^{\beta} f_{ts}(t,s)\,dsdt \right| \qquad (2.6)\\
&\leq \int_a^x \int_c^y (g(t)-g(a))^{\alpha} (w(s)-w(c))^{\beta} \left| f_{ts}(t,s) \right| dsdt \\
&= \int_a^x \int_c^y (g(t)-g(a))^{\alpha} (w(s)-w(c))^{\beta} \left| \frac{f_{ts}(t,s)}{g'(t)w'(s)} \right| g'(t)w'(s)\,dsdt \\
&\leq \left\| \frac{f_{ts}}{g'w'} \right\|_{[a,x]\times[c,y],\infty} \int_a^x \int_c^y (g(t)-g(a))^{\alpha} (w(s)-w(c))^{\beta} g'(t)w'(s)\,dsdt \\
&= \frac{(g(x)-g(a))^{\alpha+1} (w(y)-w(c))^{\beta+1}}{(\alpha+1)(\beta+1)} \left\| \frac{f_{ts}}{g'w'} \right\|_{[a,x]\times[c,y],\infty}
\end{aligned}$$

and

$$\begin{aligned}
|I_2| &= \left| \int_a^x \int_y^d (g(t)-g(a))^{\alpha} (w(d)-w(s))^{\beta} f_{ts}(t,s)\,dsdt \right| \\
&\leq \int_a^x \int_y^d (g(t)-g(a))^{\alpha} (w(d)-w(s))^{\beta} \left| f_{ts}(t,s) \right| dsdt \\
&= \int_a^x \int_y^d (g(t)-g(a))^{\alpha} (w(d)-w(s))^{\beta} \left| f_{ts}(t,s) \right| dsdt \\
&= \int_a^x \int_y^d (g(t)-g(a))^{\alpha} (w(d)-w(s))^{\beta} \left| \frac{f_{ts}(t,s)}{g'(t)w'(s)} \right| g'(t)w'(s)\,dsdt \\
&\leq \left\| \frac{f_{ts}}{g'w'} \right\|_{[a,x]\times[y,d],\infty} \int_a^x \int_y^d (g(t)-g(a))^{\alpha} (w(d)-w(s))^{\beta} g'(t)w'(s)\,dsdt \\
&= \frac{(g(x)-g(a))^{\alpha+1} (w(d)-w(y))^{\beta+1}}{(\alpha+1)(\beta+1)} \left\| \frac{f_{ts}}{g'w'} \right\|_{[a,x]\times[y,d],\infty}.
\end{aligned}$$

In a similar way, we obtain

$$|I_3| = \left| \int_x^b \int_c^y (g(b)-g(t))^{\alpha}(w(s)-w(c))^{\beta} f_{t,s}(t,s)\,dsdt \right| \\ \leq \frac{(g(b)-g(x))^{\alpha+1}(w(y)-w(c))^{\beta+1}}{(\alpha+1)(\beta+1)} \left\| \frac{f_{ts}}{g'w'} \right\|_{[x,b]\times[c,y],\infty} \tag{2.7}$$

and

$$|I_4| = \left| \int_x^b \int_y^d (g(b)-g(t))^{\alpha}(w(d)-w(s))^{\beta} f_{t,s}(t,s)\,dsdt \right| \\ \leq \frac{(g(b)-g(x))^{\alpha+1}(w(d)-w(y))^{\beta+1}}{(\alpha+1)(\beta+1)} \left\| \frac{f_{ts}}{g'w'} \right\|_{[x,b]\times[y,d],\infty}. \tag{2.8}$$

By using the inequalities (2.6)–(2.8) in (2.5), we get

$$\begin{aligned}
&\Bigg| f(x,y) - \frac{1}{M_g^{\alpha}(a,b;x)} \left[J_{x^-;g}^{\alpha} f(a,y) + J_{x^+;g}^{\alpha} f(b,y) \right] \\
&- \frac{1}{N_w^{\beta}(c,d;y)} \left[J_{y^-;w}^{\beta} f(x,c) + J_{y^+;w}^{\beta} f(x,d) \right] + \frac{1}{M_g^{\alpha}(a,b;x) N_w^{\beta}(c,d;y)} \\
&\times \left[J_{x^-y^-;g,w}^{\alpha,\beta} f(a,c) + J_{x^-y^+;g,w}^{\alpha,\beta} f(a,d) + J_{x^+y^-;g,w}^{\alpha,\beta} f(b,c) + J_{x^+y^+;g,w}^{\alpha,\beta} f(b,d) \right] \Bigg| \\
&\leq \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1) M_g^{\alpha}(a,b;x) N_w^{\beta}(c,d;y)} \\
&\quad \times \Bigg[\frac{(g(x)-g(a))^{\alpha+1}(w(y)-w(c))^{\beta+1}}{(\alpha+1)(\beta+1)} \left\| \frac{f_{ts}}{g'w'} \right\|_{[a,x]\times[c,y],\infty} \\
&\quad + \frac{(g(x)-g(a))^{\alpha+1}(w(d)-w(y))^{\beta+1}}{(\alpha+1)(\beta+1)} \left\| \frac{f_{ts}}{g'w'} \right\|_{[a,x]\times[y,d],\infty} \\
&\quad + \frac{(g(b)-g(x))^{\alpha+1}(w(y)-w(c))^{\beta+1}}{(\alpha+1)(\beta+1)} \left\| \frac{f_{ts}}{g'w'} \right\|_{[x,b]\times[c,y],\infty} \\
&\quad \frac{(g(b)-g(x))^{\alpha+1}(w(d)-w(y))^{\beta+1}}{(\alpha+1)(\beta+1)} \left\| \frac{f_{ts}}{g'w'} \right\|_{[x,b]\times[y,d],\infty} \Bigg] \\
&\leq \frac{1}{\Gamma(\alpha+2)\Gamma(\beta+2) M_g^{\alpha}(a,b;x) N_w^{\beta}(c,d;y)} \left((g(x)-g(a))^{\alpha+1} + (g(b)-g(x))^{\alpha+1} \right) \\
&\quad \times \left((w(y)-w(c))^{\beta+1} + (w(d)-w(y))^{\beta+1} \right) \left\| \frac{f_{ts}}{g'w'} \right\|_{[a,b]\times[b,d],\infty},
\end{aligned}$$

which completes the proof. □

By choosing $g(t)=t$, $t\in[a,b]$ and $w(s)=s$, $s\in[c,d]$ in Theorem 2, we get the following Ostrowski type inequality for Riemann–Liouville fractional integrals.

Corollary 1. *The following inequality holds:*

$$\begin{aligned}&\Bigg|f(x,y)-\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}+(b-x)^{\alpha}}\left[J_{x^-}^{\alpha}f(a,y)+J_{x^+}^{\alpha}f(b,y)\right]\\&-\frac{\Gamma(\beta+1)}{(y-c)^{\beta}+(d-y)^{\beta}}\left[J_{y^-}^{\beta}f(x,c)+J_{y^+}^{\beta}f(x,d)\right]\\&+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\left[(y-c)^{\beta}+(d-y)^{\beta}\right]}\\&\times\left[J_{x^-y^-}^{\alpha,\beta}f(a,c)+J_{x^-y^+}^{\alpha,\beta}f(a,d)+J_{x^+y^-}^{\alpha,\beta}f(b,c)+J_{x^+y^+}^{\alpha,\beta}f(b,d)\right]\Bigg|\\&\le\frac{\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]\left[(y-c)^{\beta+1}+(d-y)^{\beta+1}\right]}{(\alpha+1)(\beta+1)\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\left[(y-c)^{\beta}+(d-y)^{\beta}\right]}\|f_{ts}\|_{\Delta,\infty}.\end{aligned}\tag{2.9}$$

Remark 1. If we take $\alpha=\beta=1$ in Corollary 1, then inequality (2.9) reduces to the inequality established in [3, Theorem 2.1].

By choosing $g(t)=\ln t$, $t\in[a,b]$ and $w(s)=\ln s$, $s\in[c,d]$ in Theorem 2, we get the following Ostrowski type inequalities for Hadamard fractional integrals.

Corollary 2. *The following inequality holds:*

$$\begin{aligned}&\Bigg|f(x,y)-\frac{1}{M_{\ln}^{\alpha}(a,b;x)}\left[\mathbf{J}_{x^-;g}^{\alpha}f(a,y)+\mathbf{J}_{x^+;g}^{\alpha}f(b,y)\right]\\&-\frac{1}{N_{\ln}^{\beta}(c,d;y)}\left[\mathbf{J}_{y^-;w}^{\beta}f(x,c)+\mathbf{J}_{y^+;w}^{\beta}f(x,d)\right]+\frac{1}{M_{\ln}^{\alpha}(a,b;x)N_{\ln}^{\beta}(c,d;y)}\\&\times\left[\mathbf{J}_{x^-y^-;g,w}^{\alpha,\beta}f(a,c)+\mathbf{J}_{x^-y^+;g,w}^{\alpha,\beta}f(a,d)+\mathbf{J}_{x^+y^-;g,w}^{\alpha,\beta}f(b,c)+\mathbf{J}_{x^+y^+;g,w}^{\alpha,\beta}f(b,d)\right]\Bigg|\\&\le\frac{M_{\ln}^{\alpha+1}(a,b;x)N_{\ln}^{\beta+1}(c,d;y)}{M_{\ln}^{\alpha}(a,b;x)N_{\ln}^{\beta}(c,d;y)}\left\|\frac{f_{ts}}{g'w'}\right\|_{\Delta,\infty}.\end{aligned}$$

Theorem 3. *Under assumptions of Lemma 1, if the function $\frac{f_{ts}}{g'w'}\in L_p(\Delta)$, then we have the following Ostrowski type inequality for generalized fractional integrals:*

$$\begin{aligned}&\Bigg|f(x,y)-\frac{1}{M_{g}^{\alpha}(a,b;x)}\left[J_{x^-;g}^{\alpha}f(a,y)+J_{x^+;g}^{\alpha}f(b,y)\right]\\&-\frac{1}{N_{w}^{\beta}(c,d;y)}\left[J_{y^-;w}^{\beta}f(x,c)+J_{y^+;w}^{\beta}f(x,d)\right]+\frac{1}{M_{g}^{\alpha}(a,b;x)N_{w}^{\beta}(c,d;y)}\end{aligned}$$

$$\times \left[J^{\alpha,\beta}_{x^-y^-;g,w} f(a,c) + J^{\alpha,\beta}_{x^-y^+;g,w} f(a,d) + J^{\alpha,\beta}_{x^+y^-;g,w} f(b,c) + J^{\alpha,\beta}_{x^+y^+;g,w} f(b,d) \right] \Bigg|$$

$$\leq \frac{\left[(g(x)-g(a))^{\alpha+\frac{1}{q}} + (g(b)-g(x))^{\alpha+\frac{1}{q}} \right]}{(\alpha q+1)^{\frac{1}{q}} \Gamma(\alpha+1) M^{\alpha}_{g}(a,b;x)}$$

$$\times \frac{\left[(w(y)-w(c))^{\beta+\frac{1}{q}} + (w(d)-w(y))^{\beta+\frac{1}{q}} \right]}{(\beta q+1)^{\frac{1}{q}} \Gamma(\beta+1) N^{\beta}_{w}(c,d;y)} \left\| \frac{f_{ts}}{(g'w')^{\frac{1}{q}}} \right\|_{[a,b]\times[c,d],p},$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\|F\|_{\Delta,p} = \left(\int_a^b \int_c^d |F(t,s)|^p \, dsdt \right)^{\frac{1}{p}} < +\infty.$$

Proof. By Hölder inequality and properties of the modulus, we have

$$|I_1| \leq \int_a^x \int_c^y (g(t)-g(a))^{\alpha} (w(s)-w(c))^{\beta} |f_{ts}(t,s)| \, dsdt \tag{2.10}$$

$$= \int_a^x \int_c^y (g(t)-g(a))^{\alpha} (w(s)-w(c))^{\beta} \left| \frac{f_{ts}(t,s)}{g'(t)w'(s)} \right| (g'(t))^{\frac{1}{p}+\frac{1}{q}} (w'(s))^{\frac{1}{p}+\frac{1}{q}} dsdt$$

$$\leq \left(\int_a^x \int_c^y \left| \frac{f_{ts}(t,s)}{g'(t)w'(s)} \right|^p g'(t)w'(s) dsdt \right)^{\frac{1}{p}}$$

$$\times \left(\int_a^x \int_c^y (g(t)-g(a))^{\alpha q} (w(s)-w(c))^{\beta q} g'(t)w'(s) dsdt \right)^{\frac{1}{q}}$$

$$\leq \left\| \frac{f_{ts}}{(g'w')^{\frac{1}{q}}} \right\|_{[a,x]\times[c,y],p} \left(\int_a^x (g(t)-g(a))^{\alpha q} g'(t) dt \right)^{\frac{1}{q}}$$

$$\times \left(\int_c^y (w(s)-w(c))^{\beta q} w'(s) ds \right)^{\frac{1}{q}}$$

$$= \left(\frac{(g(x)-g(a))^{\alpha q+1}}{\alpha q+1} \right)^{\frac{1}{q}} \left(\frac{(w(y)-w(c))^{\beta q+1}}{\beta q+1} \right)^{\frac{1}{q}} \left\| \frac{f_{ts}}{(g'w')^{\frac{1}{q}}} \right\|_{[a,x]\times[c,y],p}$$

$$= \frac{(g(x)-g(a))^{\alpha+\frac{1}{q}}(w(y)-w(c))^{\beta+\frac{1}{q}}}{(\alpha q+1)^{\frac{1}{q}}(\beta q+1)^{\frac{1}{q}}}\left\|\frac{f_{ts}}{(g'w')^{\frac{1}{q}}}\right\|_{[a,x]\times[c,y],p}$$

Similarly, we get

$$|I_2| \leq \frac{(g(x)-g(a))^{\alpha+\frac{1}{q}}(w(d)-w(y))^{\beta+\frac{1}{q}}}{(\alpha q+1)^{\frac{1}{q}}(\beta q+1)^{\frac{1}{q}}}\left\|\frac{f_{ts}}{(g'w')^{\frac{1}{q}}}\right\|_{[a,x]\times[y,d],p}, \tag{2.11}$$

$$|I_3| \leq \frac{(g(b)-g(x))^{\alpha+\frac{1}{q}}(w(y)-w(c))^{\beta+\frac{1}{q}}}{(\alpha q+1)^{\frac{1}{q}}(\beta q+1)^{\frac{1}{q}}}\left\|\frac{f_{ts}}{(g'w')^{\frac{1}{q}}}\right\|_{[x,b]\times[c,y],p} \tag{2.12}$$

and

$$|I_4| \leq \frac{(g(b)-g(x))^{\alpha+\frac{1}{q}}(w(d)-w(y))^{\beta+\frac{1}{q}}}{(\alpha q+1)^{\frac{1}{q}}(\beta q+1)^{\frac{1}{q}}}\left\|\frac{f_{ts}}{(g'w')^{\frac{1}{q}}}\right\|_{[x,b]\times[y,d],p}. \tag{2.13}$$

By substituting the inequalities (2.10)–(2.13) in (2.5), we establish

$$\begin{aligned}
&\Bigg| f(x,y) - \frac{1}{M_g^{\alpha}(a,b;x)}\left[J^{\alpha}_{x^-;g}f(a,y)+J^{\alpha}_{x^+;g}f(b,y)\right] \\
&- \frac{1}{N_w^{\beta}(c,d;y)}\left[J^{\beta}_{y^-;w}f(x,c)+J^{\beta}_{y^+;w}f(b,y)\right] + \frac{1}{M_g^{\alpha}(a,b;x)N_w^{\beta}(c,d;y)} \\
&\times \left[J^{\alpha,\beta}_{x^-y^-;g,w}f(a,c)+J^{\alpha,\beta}_{x^-y^+;g,w}f(a,d)+J^{\alpha,\beta}_{x^+y^-;g,w}f(b,c)+J^{\alpha,\beta}_{x^+y^+;g,w}f(b,d)\right]\Bigg| \\
&\leq \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)M_g^{\alpha}(a,b;x)N_w^{\beta}(c,d;y)} \\
&\quad\times \Bigg[\frac{(g(x)-g(a))^{\alpha+\frac{1}{q}}(w(y)-w(c))^{\beta+\frac{1}{q}}}{(\alpha q+1)^{\frac{1}{q}}(\beta q+1)^{\frac{1}{q}}}\left\|\frac{f_{ts}}{(g'w')^{\frac{1}{q}}}\right\|_{[a,x]\times[c,y],p} \\
&\quad+\frac{(g(x)-g(a))^{\alpha+\frac{1}{q}}(w(d)-w(y))^{\beta+\frac{1}{q}}}{(\alpha q+1)^{\frac{1}{q}}(\beta q+1)^{\frac{1}{q}}}\left\|\frac{f_{ts}}{(g'w')^{\frac{1}{q}}}\right\|_{[a,x]\times[y,d],p} \\
&\quad+\frac{(g(b)-g(x))^{\alpha+\frac{1}{q}}(w(y)-w(c))^{\beta+\frac{1}{q}}}{(\alpha q+1)^{\frac{1}{q}}(\beta q+1)^{\frac{1}{q}}}\left\|\frac{f_{ts}}{(g'w')^{\frac{1}{q}}}\right\|_{[x,b]\times[c,y],p} \\
&\quad+\frac{(g(b)-g(x))^{\alpha+\frac{1}{q}}(w(d)-w(y))^{\beta+\frac{1}{q}}}{(\alpha q+1)^{\frac{1}{q}}(\beta q+1)^{\frac{1}{q}}}\left\|\frac{f_{ts}}{(g'w')^{\frac{1}{q}}}\right\|_{[x,b]\times[y,d],p}\Bigg] \\
&\leq \frac{1}{(\alpha q+1)^{\frac{1}{q}}(\beta q+1)^{\frac{1}{q}}\Gamma(\alpha+1)\Gamma(\beta+1)M_g^{\alpha}(a,b;x)N_w^{\beta}(c,d;y)}\left\|\frac{f_{ts}}{(g'w')^{\frac{1}{q}}}\right\|_{[a,b]\times[c,d],p}
\end{aligned}$$

$$\times\left[(g(x)-g(a))^{\alpha+\frac{1}{q}}+(g(b)-g(x))^{\alpha+\frac{1}{q}}\right]$$
$$\times\left[(w(y)-w(c))^{\beta+\frac{1}{q}}+(w(d)-w(y))^{\beta+\frac{1}{q}}\right].$$

This completes the proof. □

By coosing $g(t)=t$, $t\in[a,b]$ and $w(s)=s$, $s\in[c,d]$ in Theorem 3, we get the following Ostrowski type inequality for Riemann–Liouville fractional integrals.

Corollary 3. *The following inequality holds:*

$$\Bigg|f(x,y)-\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}+(b-x)^{\alpha}}\left[J^{\alpha}_{x^-}f(a,y)+J^{\alpha}_{x^+}f(b,y)\right] \tag{2.14}$$
$$-\frac{\Gamma(\beta+1)}{(y-c)^{\beta}+(d-y)^{\beta}}\left[J^{\beta}_{y^-}f(x,c)+J^{\beta}_{y^+}f(x,d)\right]$$
$$+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\left[(y-c)^{\beta}+(d-y)^{\beta}\right]}$$
$$\times\left[J^{\alpha,\beta}_{x^-y^-}f(a,c)+J^{\alpha,\beta}_{x^-y^+}f(a,d)+J^{\alpha,\beta}_{x^+y^-}f(b,c)+J^{\alpha,\beta}_{x^+y^+}f(b,d)\right]\Bigg|$$
$$\le\frac{\left[(x-a)^{\alpha+\frac{1}{q}}+(b-x)^{\alpha+\frac{1}{q}}\right]\left[(y-c)^{\beta+\frac{1}{q}}+(d-y)^{\beta+\frac{1}{q}}\right]}{(\alpha q+1)^{\frac{1}{q}}(\beta q+1)^{\frac{1}{q}}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]\left[(y-c)^{\beta}+(d-y)^{\beta}\right]}\left\|f_{ts}\right\|_{\Delta,p}.$$

Remark 2. If we take $\alpha=\beta=1$ in Corollary 3, then we have

$$\Bigg|f(x,y)-\frac{1}{b-a}\int_a^b f(t,y)dt-\frac{1}{d-c}\int_c^d f(x,s)ds$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\Bigg|$$
$$\le\frac{\left[(x-a)^{1+\frac{1}{q}}+(b-x)^{1+\frac{1}{q}}\right]\left[(y-c)^{1+\frac{1}{q}}+(d-y)^{1+\frac{1}{q}}\right]}{(b-a)(d-c)(q+1)^{\frac{2}{q}}}\left\|f_{ts}\right\|_{\Delta,p}.$$

By choosing $g(t)=\ln t$, $t\in[a,b]$ and $w(s)=\ln s$, $s\in[c,d]$ in Theorem 3, we the following Ostrowski type inequality for Hadamard fractional integrals.

Corollary 4. *The following inequality holds:*

$$\Bigg|f(x,y)-\frac{1}{M^{\alpha}_{\ln}(a,b;x)}\left[\mathbf{J}^{\alpha}_{x^-;g}f(a,y)+\mathbf{J}^{\alpha}_{x^+;g}f(b,y)\right]$$
$$-\frac{1}{N^{\beta}_{\ln}(c,d;y)}\left[\mathbf{J}^{\beta}_{y^-;w}f(x,c)+\mathbf{J}^{\beta}_{y^+;w}f(x,d)\right]+\frac{1}{M^{\alpha}_{\ln}(a,b;x)N^{\beta}_{\ln}(c,d;y)}$$

$$\times\left[\mathbf{J}^{\alpha,\beta}_{x^-y^-;g,w}f(a,c)+\mathbf{J}^{\alpha,\beta}_{x^-y^+;g,w}f(a,d)+\mathbf{J}^{\alpha,\beta}_{x^+y^-;g,w}f(b,c)+\mathbf{J}^{\alpha,\beta}_{x^+y^+;g,w}f(b,d)\right]\Bigg|$$
$$\leq\frac{\left[(\ln\frac{x}{a})^{\alpha+\frac{1}{q}}+(\ln\frac{b}{x})^{\alpha+\frac{1}{q}}\right]\left[(\ln\frac{y}{c})^{\beta+\frac{1}{q}}+(\ln\frac{d}{y})^{\beta+\frac{1}{q}}\right]}{(\alpha q+1)^{\frac{1}{q}}(\beta q+1)^{\frac{1}{q}}\Gamma(\alpha+1)\Gamma(\beta+1)M^{\alpha}_{\ln}(a,b;x)N^{\beta}_{\ln}(c,d;y)}\left\|\frac{f_{ts}}{(h)^{\frac{1}{q}}}\right\|_{[a,b]\times[c,d],p},$$

where

$$h(t,s)=\frac{1}{st}.$$

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