Miskolc Mathematical Notes
Vol. 21 (2020), No. 2, pp. 1013–1018

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2020.3246

ESSENTIAL SUPPLEMENTED LATTICES

HASAN HÜSEYIN ÖKTEN AND AYTEN PEKIN

Received 22 February, 2020

Abstract. Let L be a complete modular lattice. If every essential element of L has a supplement in L, then L is called an essential supplemented (or briefly e-supplemented) lattice. In this work some properties of these lattices are investigated. Let L be a complete modular lattice and $1 = a_1 \vee a_2 \vee ... \vee a_n$ with $a_i \in L$ $(1 \leq i \leq n)$. If $a_i/0$ is e-supplemented for every $i = 1, 2, ..., n$, then L is also e-supplemented. If L is e-supplemented, then $1/a$ is also e-supplemented for every $a \in L$.

2010 *Mathematics Subject Classification:* 06C05; 06C15

Keywords: lattices, essential elements, small elements, supplemented lattices

1. INTRODUCTION

In this paper, every lattice is a complete modular lattice with the smallest element 0 and the greatest element 1. Let L be a lattice, $x, y \in L$ and $x \leq y$. A sublattice $\{a \in L | x \leq a \leq y\}$ is called a *quotient sublattice* and denoted by y/x. An element a of a lattice L is called a *complement* of b in L if $a \vee b = 1$ and $a \wedge b = 0$, in this case we denote $1 = a \oplus b$ (here we also call a and b are *direct summands* of L). L is said to be *complemented* if each element has at least one complement in L. An element x of L is said to be *small* or *superfluous* and denoted by $x \ll L$ if $y = 1$ for every $y \in L$ such that $x \vee y = 1$. Let L be a lattice and $k \in L$. If $t = 0$ for very $t \in L$ with $k \wedge t = 0$, then k is called an *essential element* of L and denoted by $k \trianglelefteq L$. The meet of all maximal $(\neq 1)$ elements of a lattice L is called the *radical* of L and denoted by $r(L)$. If $a \ll L$, then $a \leq r(L)$ holds. An element a of L is called a *supplement* of b in L if it is minimal for $1 = b \vee a$. a is a supplement of b in a lattice L if and only if $1 = b \vee a$ and $b \wedge a \ll a/0$. A lattice L is called a *supplemented lattice* if every element of L has a supplement in L. We say that an element y of L *lies above* an element x of L if $x \leq y$ and $y \ll 1/x$. L is said to be *hollow* if every element distinct from 1 is superfluous in L, and L is said to be *local* if L has the greatest element $(\neq 1)$. We say an element $x \in L$ has *ample supplements* in L if for every $y \in L$ with $x \vee y = 1$, x has a supplement z in L with $z \leq y$. L is said to be *amply supplemented* if every element of L has ample supplements in L. Let L be a lattice. It is defined β_* relation on the elements of L by

$a\beta_* b$ with $a,b \in L$ if and only if for each $t \in L$ such that $a \vee t = 1$ then $b \vee t = 1$ and for each $k \in L$ such that $b \vee k = 1$ then $a \vee k = 1$.

More informations about (amply) supplemented lattices are in [1, 2, 5, 9]. More results about (amply) supplemented modules are in [8, 12]. The definition of β_* relation on lattices and some properties of this relation are in [10]. This relation is a generalization of β^* relation on modules. The definition of β^* relation on modules and some properties of this relation are in [4].

Lemma 1. *Let L be a lattice and $a,b,c \in L$ with $a \leq b$. If c is a supplement of b in L, then $a \vee c$ is a supplement of b in $1/a$.*

Proof. See [3, Lemma 1]. □

Definition 1. Let L be a lattice and k be a maximal $(\neq 1)$ element of L. If $k \trianglelefteq L$, then k is called a g-maximal element of L. The meet of all g-maximal elements of L is called the g-radical of L and denoted by $r_g(L)$. If L have not any g-maximal elements, then we call $r_g(L) = 1$.

The g-radical of any lattice is a generalization of the generalized radical of any module. The definition of the generalized radical of any module and some properties of this concept are in [6, 7].

Corollary 1. *Let L be a lattice. Then $r(L) \leq r_g(L)$.*

Proof. Clear from definitions. □

Lemma 2. *Let L be a lattice and $x \in L$. Then $r_g(x/0) \leq r_g(L)$.*

Proof. Let k be any g-maximal element of L. If $x \leq k$, then $r_g(x/0) \leq k$. If $x \nleq k$, we can easily see that $x \wedge k$ is a g-maximal element of $x/0$ and hence $r_g(x/0) \leq k$. Therefore, $r_g(x/0) \leq r_g(L)$. □

2. Essential supplemented lattices

Definition 2. Let L be a lattice. If every essential element of L has a supplement in L, then L is called an essential supplemented (or briefly e-supplemented) lattice.

This concept is a generalization of an essential supplemented module. The definition of essential supplemented modules and some properties of these modules are in [11].

It is clear that every supplemented lattice is e-supplemented. Hollow and local lattices are e-supplemented.

Proposition 1. *Let L be an e-supplemented lattice. If every element of L with distinct from 0 is essential in L, then L is supplemented.*

Proof. Clear from definitions. □

Definition 3. Let L be a lattice and $a \in L$. If a is a supplement of an essential element in L, then a is called an e-supplement element in L.

Proposition 2. *Let L be a lattice. If every essential element of L is e-supplement in L, then L is e-supplemented.*

Proof. Let $x \trianglelefteq L$. By hypothesis x is an e-supplement element in L and there exists $y \trianglelefteq L$ such that x is a supplement of y in L. Here $1 = y \vee x$ and $y \wedge x \ll x/0$. Since $y \trianglelefteq L$, by hypothesis, y is an e-supplement element in L. Since $x \wedge y \ll L$ and y is a supplement element in L, by [9, Lemma 10], $x \wedge y \ll y/0$ and y is a supplement of x in L. Hence L is e-supplemented. □

Proposition 3. *Let L be a lattice and a be an e-supplement element in L. Then* $r_g(a/0) = a \wedge r_g(L)$.

Proof. By Lemma 2, $r_g(a/0) \leq a \wedge r_g(L)$ holds. Since a is a e-supplement element in L, there exists an essential element x of L such that a is a supplement of x in L. Here $1 = x \vee a$ and $x \wedge a \ll a/0$. By Corollary 1, $x \wedge a \leq r(a/0) \leq r_g(a/0)$ hold. Let k be any g-maximal element of $a/0$. Here $x \wedge a \leq r_g(a/0) \leq k$. Since $\frac{1}{x\vee k} = \frac{x\vee k\vee a}{x\vee k} \cong \frac{a}{a\wedge(x\vee k)} = \frac{a}{(x\wedge a)\vee k} = \frac{a}{k}$ and $x \vee k \trianglelefteq L$, $x \vee k$ is a g-maximal element of L and hence $r_g(L) \leq x \vee k$. This case $a \wedge r_g(L) \leq a \wedge (x \vee k) = (x \wedge a) \vee k = k$. Therefore, $a \wedge r_g(L) \leq r_g(a/0)$ and since $r_g(a/0) \leq a \wedge r_g(L)$, $r_g(a/0) = a \wedge r_g(L)$. □

Proposition 4. *Let L be an essential supplemented lattice. Then* $1/r(L)$ *doesn"t have any essential element with distinct from* 1.

Proof. Let x be any essential element of $1/r(L)$. Then we clearly see that x is an essential element of L. Since L is essential supplemented, x has a supplement y in L. Here $1 = x \vee y$ and $x \wedge y \ll y/0$. Since $x \wedge y \ll y/0$, $x \wedge y \leq r(L)$. Here $1 = x \vee y = x \vee y \vee r(L)$ and $x \wedge (y \vee r(L)) = (x \wedge y) \vee r(L) = r(L)$. Hence $1 = x \oplus (y \vee r(L))$ in $1/r(L)$ and since $x \trianglelefteq 1/r(L)$, $x = 1$. □

Lemma 3. *Let L be a lattice,* $x \trianglelefteq L$ *and* $a \in L$*. If* $x \vee a$ *has a supplement in L and* $a/0$ *essential supplemented, then x has a supplement in L.*

Proof. Let y be a supplement of $x \vee a$ in L. Then $1 = x \vee a \vee y$ and $(x \vee a) \wedge y \ll y/0$. Since $x \trianglelefteq L$, $x \vee y \trianglelefteq L$ and hence $(x \vee y) \wedge a \trianglelefteq a/0$. Since $a/0$ is essential supplemented, $(x \vee y) \wedge a$ has a supplement z in $a/0$. Here $a = [(x \vee y) \wedge a] \vee z$ and $(x \vee y) \wedge z = (x \vee y) \wedge a \wedge z \ll z/0$. Following these we have $1 = x \vee a \vee y = x \vee y \vee [(x \vee y) \wedge a] \vee z = x \vee y \vee z$ and $x \wedge (y \vee z) \leq [(x \vee y) \wedge z] \vee [(x \vee z) \wedge y] \leq [(x \vee y) \wedge z] \vee [(x \vee a) \wedge y] \ll (y \vee z)/0$. Hence $y \vee z$ is a supplement of x in L. □

Lemma 4. *Let L be a lattice and* $1 = a \vee b$ *with* $a, b \in L$*. If* $a/0$ *and* $b/0$ *are e-supplemented, then L is also e-supplemented.*

Proof. Let $x \trianglelefteq L$. Then 0 is a supplement of $x \vee a \vee b$ in L. Since $b/0$ is e-supplemented and $x \vee a \trianglelefteq L$, by Lemma 3, $x \vee a$ has a supplement in L and since

$a/0$ is e-supplemented and $x \trianglelefteq L$, again by Lemma 3, x has a supplement in L. Hence L is e-supplemented. □

Corollary 2. *Let L be a lattice and* $1 = a_1 \vee a_2 \vee \ldots \vee a_n$ *with* $a_i \in L$ $(1 \leq i \leq n)$. *If* $a_i/0$ *is e-supplemented for every* $i = 1, 2, \ldots, n$, *then L is also e-supplemented.*

Proof. Clear from Lemma 4. □

Lemma 5. *Let L be an e-supplemented lattice and* $a \in L$. *Then* $1/a$ *is also e-supplemented.*

Proof. Let x be any essential element of $1/a$. Then $x \trianglelefteq L$ and since L is e-supplemented, x has a supplement y in L. By Lemma 1, $y \vee a$ is a supplement of x in $1/a$. Hence $1/a$ is e-supplemented. □

Corollary 3. *Let L be an e-supplemented lattice. Then* $a/0$ *is e-supplemented for every direct summand a of L.*

Proof. Since a is a direct summand of L, there exists $b \in L$ such that $1 = a \oplus b$. By Lemma 5, $1/b$ is e-supplemented. Then by $\frac{1}{b} = \frac{a \vee b}{b} \cong \frac{a}{a \wedge b} = \frac{a}{0}$, $a/0$ is also e-supplemented. □

Lemma 6. *Let L be a lattice. If every essential element of L is* β_* *equivalent to an e-supplement element in L, then L is e-supplemented.*

Proof. Let $x \trianglelefteq L$. By hypothesis, there exists an e-supplement element y in L such that $x\beta_* y$. Since y is an e-supplement element in L, there exists $z \trianglelefteq L$ such that y is a supplement of z in L. By hypothesis, there exists an e-supplement element a in L such that $z\beta_* a$. Since y is a supplement of z in L and $z\beta_* a$, by [10, Theorem 4 (1)], y is a supplement of a in L. Here $1 = a \vee y$ and $a \wedge y \ll y/0$. Since a is a supplement element in L, by [9, Lemma 10], $a \wedge y \ll a/0$ and a is a supplement of y in L. Since $x\beta_* y$, by [10, Theorem 4 (1)], a is a supplement of x in L. Hence L is e-supplemented. □

Corollary 4. *Let L be a lattice. If every essential element of L lies above an e-supplement element in L, then L is e-supplemented.*

Proof. Clear from [10, Theorem 3] and Lemma 6. □

Definition 4. Let L be a lattice. If every essential element of L with distinct from 1 is small in L or L have no essential elements with distinct from 1, then L is called a e-hollow lattice. If L has an essential element $c \neq 1$ such that $k \leq c$ for every $1 \neq k \trianglelefteq L$, then L is called an e-local lattice (here k is called the greatest essential element $(\neq 1)$ of L).

Clearly we can see that every hollow lattice is e-hollow. But the converse is not true in general (See Example 1).

Proposition 5. *Every e-hollow lattice is e-supplemented.*

Proof. Let L be an e-hollow lattice. Then 1 is a supplement of every essential element of L with distinct from 1 and 0 is a supplement of 1 in L. Hence L is e-supplemented. □

Proposition 6. *Let L be a lattice with $r(L) \trianglelefteq L$ and $r(L) \neq 1$. If L is e-hollow, then L is e-local.*

Proof. Let $x \trianglelefteq L$ and $x \neq 1$. Since L is e-hollow, $x \ll L$ and $x \leq r(L)$. By hypothesis, $r(L) \neq 1$. Hence $r(L)$ is the greatest essential element $(\neq 1)$ of L and L is e-local. □

Example 1. Consider the lattice $L = \{0, a, b, 1\}$ given by the following diagram.

$$\begin{array}{ccccc} & & 1 & & \\ & \nearrow & & \nwarrow & \\ a & & & & b \\ & \nwarrow & & \nearrow & \\ & & 0 & & \end{array}$$

Then L is e-hollow but not hollow. Here $r(L) = 0 \neq 1 = r_g(L)$ hold.

Example 2. Consider the lattice $L = \{0, a, b, c, 1\}$ given by the following diagram.

$$\begin{array}{ccccc} & & 1 & & \\ & \nearrow & & \nwarrow & \\ b & & & & c \\ & \nwarrow & & \nearrow & \\ & & a & & \\ & & \uparrow & & \\ & & 0 & & \end{array}$$

Then L is e-supplemented but not e-hollow.

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Authors' addresses

Hasan Hüseyin Ökten
Amasya University, Technical Sciences Vocational School, Amasya, Turkey
E-mail address: hokten@gmail.com

Ayten Pekin
Istanbul University, Department of Mathematics, Vezneciler, Istanbul, Turkey
E-mail address: aypekin@istanbul.edu.tr