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ESSENTIAL SUPPLEMENTED LATTICES

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Abstract. Let L be a complete modular lattice. If every essential element of L has a supplement in L, then L is called an essential supplemented (or briefly e-supplemented) lattice. In this work some properties of these lattices are investigated. Let L be a complete modular lattice and $1 = a_1 \lor a_2 \lor ... \lor a_n$ with $a_i \in L$ ($1 \le i \le n$). If $a_i/0$ is e-supplemented for every i = 1, 2, ..., n, then L is also e-supplemented. If L is e-supplemented, then 1/a is also e-supplemented for every $a \in L$.

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1. Introduction

In this paper, every lattice is a complete modular lattice with the smallest element 0 and the greatest element 1. Let L be a lattice, $x,y \in L$ and $x \leq y$. A sublattice $\{a \in L | x \le a \le y\}$ is called a *quotient sublattice* and denoted by y/x. An element a of a lattice L is called a *complement* of b in L if $a \lor b = 1$ and $a \land b = 0$, in this case we denote $1 = a \oplus b$ (here we also call a and b are direct summands of L). L is said to be *complemented* if each element has at least one complement in L. An element x of L is said to be *small* or *superfluous* and denoted by $x \ll L$ if y = 1 for every $y \in L$ such that $x \lor y = 1$. Let *L* be a lattice and $k \in L$. If t = 0 for very $t \in L$ with $k \land t = 0$, then k is called an *essential element* of L and denoted by $k \leq L$. The meet of all maximal $(\neq 1)$ elements of a lattice L is called the *radical* of L and denoted by r(L). If $a \ll L$, then $a \le r(L)$ holds. An element a of L is called a supplement of b in L if it is minimal for $1 = b \lor a$. a is a supplement of b in a lattice L if and only if $1 = b \lor a$ and $b \wedge a \ll a/0$. A lattice L is called a *supplemented lattice* if every element of L has a supplement in L. We say that an element y of L lies above an element x of L if $x \le y$ and $y \ll 1/x$. L is said to be hollow if every element distinct from 1 is superfluous in L, and L is said to be *local* if L has the greatest element $(\neq 1)$. We say an element $x \in L$ has ample supplements in L if for every $y \in L$ with $x \lor y = 1$, x has a supplement z in L with $z \le y$. L is said to be amply supplemented if every element of L has ample supplements in L. Let L be a lattice. It is defined β_* relation on the elements of L by

 $a\beta_*b$ with $a,b \in L$ if and only if for each $t \in L$ such that $a \lor t = 1$ then $b \lor t = 1$ and for each $k \in L$ such that $b \lor k = 1$ then $a \lor k = 1$.

More informations about (amply) supplemented lattices are in [1, 2, 5, 9]. More results about (amply) supplemented modules are in [8, 12]. The definition of β_* relation on lattices and some properties of this relation are in [10]. This relation is a generalization of β^* relation on modules. The definition of β^* relation on modules and some properties of this relation are in [4].

Lemma 1. Let L be a lattice and $a,b,c \in L$ with $a \le b$. If c is a supplement of b in L, then $a \lor c$ is a supplement of b in 1/a.

Proof. See [3, Lemma 1].

Definition 1. Let L be a lattice and k be a maximal $(\neq 1)$ element of L. If $k \leq L$, then k is called a g-maximal element of L. The meet of all g-maximal elements of L is called the g-radical of L and denoted by $r_g(L)$. If L have not any g-maximal elements, then we call $r_g(L) = 1$.

The g-radical of any lattice is a generalization of the generalized radical of any module. The definition of the generalized radical of any module and some properties of this concept are in [6,7].

Corollary 1. *Let* L *be a lattice. Then* $r(L) \leq r_g(L)$.

Proof. Clear from definitions.

Lemma 2. Let L be a lattice and $x \in L$. Then $r_g(x/0) \le r_g(L)$.

Proof. Let k be any g-maximal element of L. If $x \le k$, then $r_g(x/0) \le k$. If $x \ne k$, we can easily see that $x \land k$ is a g-maximal element of x/0 and hence $r_g(x/0) \le k$. Therefore, $r_g(x/0) \le r_g(L)$.

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Definition 2. Let L be a lattice. If every essential element of L has a supplement in L, then L is called an essential supplemented (or briefly e-supplemented) lattice.

This concept is a generalization of an essential supplemented module. The definition of essential supplemented modules and some properties of these modules are in [11].

It is clear that every supplemented lattice is e-supplemented. Hollow and local lattices are e-supplemented.

Proposition 1. Let L be an e-supplemented lattice. If every element of L with distinct from 0 is essential in L, then L is supplemented.

Proof. Clear from definitions.

Definition 3. Let L be a lattice and $a \in L$. If a is a supplement of an essential element in L, then a is called an e-supplement element in L.

Proposition 2. Let L be a lattice. If every essential element of L is e-supplement in L, then L is e-supplemented.

Proof. Let $x \subseteq L$. By hypothesis x is an e-supplement element in L and there exists $y \subseteq L$ such that x is a supplement of y in L. Here $1 = y \lor x$ and $y \land x \ll x/0$. Since $y \subseteq L$, by hypothesis, y is an e-supplement element in L. Since $x \land y \ll L$ and y is a supplement element in L, by [9, Lemma 10], $x \land y \ll y/0$ and y is a supplement of x in L. Hence L is e-supplemented.

Proposition 3. Let L be a lattice and a be an e-supplement element in L. Then $r_g(a/0) = a \wedge r_g(L)$.

Proof. By Lemma 2, $r_g(a/0) \le a \land r_g(L)$ holds. Since a is a e-supplement element in L, there exists an essential element x of L such that a is a supplement of x in L. Here $1 = x \lor a$ and $x \land a \ll a/0$. By Corollary 1, $x \land a \le r(a/0) \le r_g(a/0)$ hold. Let k be any g-maximal element of a/0. Here $x \land a \le r_g(a/0) \le k$. Since $\frac{1}{x \lor k} = \frac{x \lor k \lor a}{x \lor k} \cong \frac{a}{a \land (x \lor k)} = \frac{a}{(x \land a) \lor k} = \frac{a}{k}$ and $x \lor k \le L$, $x \lor k$ is a g-maximal element of L and hence $r_g(L) \le x \lor k$. This case $a \land r_g(L) \le a \land (x \lor k) = (x \land a) \lor k = k$. Therefore, $a \land r_g(L) \le r_g(a/0)$ and since $r_g(a/0) \le a \land r_g(L)$, $r_g(a/0) = a \land r_g(L)$.

Proposition 4. Let L be an essential supplemented lattice. Then 1/r(L) doesn"t have any essential element with distinct from 1.

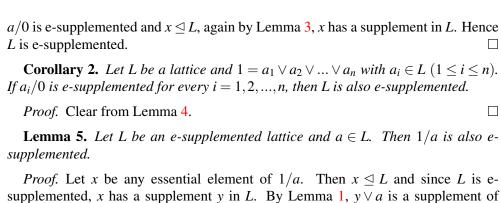
Proof. Let x be any essential element of 1/r(L). Then we clearly see that x is an essential element of L. Since L is essential supplemented, x has a supplement y in L. Here $1 = x \lor y$ and $x \land y \ll y/0$. Since $x \land y \ll y/0$, $x \land y \le r(L)$. Here $1 = x \lor y = x \lor y \lor r(L)$ and $x \land (y \lor r(L)) = (x \land y) \lor r(L) = r(L)$. Hence $1 = x \oplus (y \lor r(L))$ in 1/r(L) and since $x \le 1/r(L)$, x = 1.

Lemma 3. Let L be a lattice, $x \le L$ and $a \in L$. If $x \lor a$ has a supplement in L and a/0 essential supplemented, then x has a supplement in L.

Proof. Let y be a supplement of $x \lor a$ in L. Then $1 = x \lor a \lor y$ and $(x \lor a) \land y \ll y/0$. Since $x \le L$, $x \lor y \le L$ and hence $(x \lor y) \land a \le a/0$. Since a/0 is essential supplemented, $(x \lor y) \land a$ has a supplement z in a/0. Here $a = [(x \lor y) \land a] \lor z$ and $(x \lor y) \land z = (x \lor y) \land a \land z \ll z/0$. Following these we have $1 = x \lor a \lor y = x \lor y \lor [(x \lor y) \land a] \lor z = x \lor y \lor z$ and $x \land (y \lor z) \le [(x \lor y) \land z] \lor [(x \lor z) \land y] \le [(x \lor y) \land z] \lor [(x \lor a) \land y] \ll (y \lor z)/0$. Hence $y \lor z$ is a supplement of x in L.

Lemma 4. Let L be a lattice and $1 = a \lor b$ with $a,b \in L$. If a/0 and b/0 are e-supplemented, then L is also e-supplemented.

Proof. Let $x \le L$. Then 0 is a supplement of $x \lor a \lor b$ in L. Since b/0 is esupplemented and $x \lor a \le L$, by Lemma 3, $x \lor a$ has a supplement in L and since



Corollary 3. Let L be an e-supplemented lattice. Then a/0 is e-supplemented for every direct summand a of L.

x in 1/a. Hence 1/a is e-supplemented.

Proof. Since a is a direct summand of L, there exists $b \in L$ such that $1 = a \oplus b$. By Lemma 5, 1/b is e-supplemented. Then by $\frac{1}{b} = \frac{a \lor b}{b} \cong \frac{a}{a \land b} = \frac{a}{0}$, a/0 is also esupplemented.

Lemma 6. Let L be a lattice. If every essential element of L is β_* equivalent to an e-supplement element in L, then L is e-supplemented.

Proof. Let $x \subseteq L$. By hypothesis, there exists an e-supplement element y in L such that $x\beta_*y$. Since y is an e-supplement element in L, there exists $z \subseteq L$ such that y is a supplement of z in L. By hypothesis, there exists an e-supplement element a in L such that $z\beta_*a$. Since y is a supplement of z in L and $z\beta_*a$, by [10, Theorem 4 (1)], y is a supplement of a in L. Here $1 = a \lor y$ and $a \land y \ll y/0$. Since a is a supplement element in L, by [9, Lemma 10], $a \land y \ll a/0$ and a is a supplement of y in L. Since $x\beta_*y$, by [10, Theorem 4 (1)], a is a supplement of x in x. Hence x is e-supplemented.

Corollary 4. Let L be a lattice. If every essential element of L lies above an e-supplement element in L, then L is e-supplemented.

Proof. Clear from [10, Theorem 3] and Lemma 6.

Definition 4. Let L be a lattice. If every essential element of L with distinct from 1 is small in L or L have no essential elements with distinct from 1, then L is called a ehollow lattice. If L has an essential element $c \neq 1$ such that $k \leq c$ for every $1 \neq k \leq L$, then L is called an e-local lattice (here k is called the greatest essential element $(\neq 1)$ of L).

Clearly we can see that every hollow lattice is e-hollow. But the converse is not true in general (See Example 1).

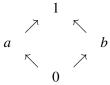
Proposition 5. Every e-hollow lattice is e-supplemented.

Proof. Let L be an e-hollow lattice. Then 1 is a supplement of every essential element of L with distinct from 1 and 0 is a supplement of 1 in L. Hence L is esupplemented.

Proposition 6. Let L be a lattice with $r(L) \leq L$ and $r(L) \neq 1$. If L is e-hollow, then L is e-local.

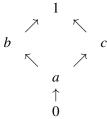
Proof. Let $x \le L$ and $x \ne 1$. Since L is e-hollow, $x \ll L$ and $x \le r(L)$. By hypothesis, $r(L) \ne 1$. Hence r(L) is the greatest essential element $(\ne 1)$ of L and L is e-local.

Example 1. Consider the lattice $L = \{0, a, b, 1\}$ given by the following diagram.



Then *L* is e-hollow but not hollow. Here $r(L) = 0 \neq 1 = r_g(L)$ hold.

Example 2. Consider the lattice $L = \{0, a, b, c, 1\}$ given by the following diagram.



Then *L* is e-supplemented but not e-hollow.

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