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SS-LIFTING MODULES AND RINGS

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Abstract. A module M is called *ss-lifting* if for every submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $A \cap M_2 \subseteq Soc_s(M)$, where $Soc_s(M) = Soc(M) \cap Rad(M)$. In this paper, we provide the basic properties of *ss*-lifting modules. It is shown that: (1) a module M is *ss*-lifting iff it is amply *ss*-supplemented and its *ss*-supplement submodules are direct summand; (2) for a ring R, ${}_RR$ is *ss*-lifting if and only if it is *ss*-supplemented iff it is semiperfect and its radical is semisimple; (3) a ring R is a left and right artinian serial ring and $Rad(R) \subseteq Soc({}_RR)$ iff every left R-module is *ss*-lifting. We also study on factor modules of *ss*-lifting modules.

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1. Introduction

In this study R is used to show a ring which is associative and has an identity. All mentioned modules will be unital left R-module. Let M be an R-module. The notation $A \leq M$ means that A is a submodule of M. A proper submodule A of M is called *small* in M and showed by $A \ll M$ whenever $A + C \neq M$ for all proper submodule C of M. A module M is called *hollow* if every submodule of M is small in M. By $Rad(M)$, namely *radical*, we will denote the sum of all small submodules of M. Equivalently, $Rad(M)$ is the intersection of all maximal submodules of M. A hollow module M with maximal radical is *local*. As a dual notion of a small submodule, a submodule $E \subseteq M$ is called *essential* in M, denoted by $E \trianglelefteq M$, if $E \cap K \neq 0$ for every nonzero submodule K of M. The socle of M which is the sum of all simple submodules of M is denoted by $Soc(M)$. It is well known that $Soc(M)$ is the intersection of all essential submodules of M. The relation between radical and socle of a module M is not determined. In [8], the sum of all simple submodules of the module M that is small is denoted by $Soc_s(M)$. It is shown in [4, Lemma 2] that $Soc_s(M) = Soc(M) \cap Rad(M)$.

A module M is called *extending* if every submodule of M is essential in a direct summand of M [3]. Dually, a module M is *lifting* if for every submodule A of M lies over a direct summand, that is, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$, $A \cap M_2 \ll M_2$. A characterization of lifting modules is given with help of supplemented modules in [3]. Here a module M is *supplemented* if every submodule A of M has a supplement B in M, that is, $M = A + B$ and $A \cap \ll B$. M is called *amply supplemented* if whenever $M = A + B$, B contains a supplement of A in M. Clearly, direct summands are supplements. By [7], M is lifting if and only if M is amply supplemented and every supplement submodule of M is a direct summand of it.

Since $Soc_s(X) = Soc(X) \cap Rad(X) \ll X$ for any module X, the authors call a submodule V of a module M *ss-supplement* of a submodule U in M if $M = U + V$ and $U \cap V \subseteq Soc_s(V)$ (see [4]). It is shown in [4, Lemma 3] that a submodule V of M is *ss*-supplement of some submodule U in M if and only if V is a supplement of U in M and $U \cap V$ is semisimple. Following [4], a module M is said to be *ss-supplemented* if every submodule A of M has an *ss*-supplement B in M, and it is called *amply ss-supplemented* if whenever $M = A + B$, B contains an *ss*-supplement of A in M. Clearly, the class of *ss*-supplemented modules is between the class of semisimple modules and the class of supplemented modules. The basic properties and characterizations of *ss*-supplemented modules are given in the same paper.

Considering all of these definitions, we can define *ss*-lifting modules. A module M is called *ss-lifting* if for every submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$, $A \cap M_2 \ll M$ and $A \cap M_2$ is semisimple. In this paper, some fundamental properties of *ss*-lifting modules will be examined. It is proved that a module M is *ss*-lifting module if and only if it is amply *ss*-supplemented and every *ss*-supplement submodule of M is direct summand. It is shown that every π-projective and *ss*-supplemented module is *ss*-lifting. It is proved that for a ring R, $_RR$ is *ss*-lifting if and only if R is semiperfect and its radical is semisimple. Moreover, it is shown that R is a left and right artinian serial ring and $Rad(R) \subseteq Soc(_RR)$ if and only if every left R-module is *ss*-lifting. Nevertheless, it is proved that any factor module generated by submodule of a weakly distributive module is *ss*-lifting.

2. SS-LIFTING MODULES

In this section, we examine the basic properties of *ss*-lifting modules. In particular, we give characterizations of some ring classes via *ss*-lifting modules. Let us begin with the following definition.

Definition 1. Let M be a module. M is called *ss-lifting* if, for every submodule U of M, M has a decomposition $M = U^{'} \oplus V$ such that $U^{'} \subseteq U$ and $U \cap V \subseteq Soc_s(V)$.

It can be seen that a module M is *ss*-lifting if and only if, for every submodule U of M, M has a decomposition $M = U^{'} \oplus V$ such that $U^{'} \subseteq U$ and $U \cap V \subseteq Soc_s(M)$. Note that we shall freely use this fact in this paper. It is clear that every *ss*-lifting module is lifting. The following example shows that in general a lifting module need not be *ss*-lifting.

Example 1. Let R be a local Dedekind domain and K be the quotient field of R. Put $M =_R K$. Then M is hollow and so it is lifting. Since R is a commutative domain and

M is an injective module, it follows that $M = Rad(M)$ and $Soc(M) = 0$. Therefore $Soc_s(M) = Soc(M) \cap Rad(M) = 0$. So M is not ss-lifting.

Lemma 1. *Let M be an ss-lifting module. Then M is amply ss-supplemented.*

Proof. Let U be any submodule of M. By the hypothesis, there are a submodule V of M and $U' \leq U$ such that $M = U' \oplus V$ and $U \cap V \subseteq Soc_s(V)$. Therefore $M = U + V$. It means that V is an ss-supplement of U in M and so M is ss-supplemented. It follows from [4, Proposition 26] that U' is ss-supplemented as a direct summand of M. Now, by modularity law, we can write $U = U \cap M = U \cap (U' \oplus V) = = U' \oplus (U \cap V)$ and then U is ss-supplemented by [4, Corollary 24] since $U \cap V$ is semisimple. Hence M is amply ss-supplemented according to [4, Proposition 33]. □

In [4], a module M is said to be *strongly local* if M is local and its radical is semisimple. Using Lemma 1, we have the next result.

Corollary 1. *For the non-zero hollow module M, the following are equivalent:*

(1) *M is strongly local.*
(2) *M is ss-lifting.*
(3) *M is amply ss-supplemented.*

Proof. (1) $\Rightarrow$ (2) Let U be a proper submodule of M. Since M is strongly local, we have $U \subseteq Rad(M) \subseteq Soc(M)$ and then U is semisimple. Now, if $U' = 0$ and $V = M$ are taken, we obtain that $M = U' \oplus M$, $U' \leq U$ and $U \cap M = U$ is semisimple. Thus M is ss-lifting.

(2) $\Rightarrow$ (3) It is clear that by Lemma 1.

(3) $\Rightarrow$ (1) By [4, Proposition 15]. □

Observe from Corollary 1 that the local $\mathbb{Z}$-module $\mathbb{Z}_8$ is lifting which is not ss-lifting.

Lemma 2. *Let M be a module and $A \leq M$. The following conditions are equivalent:*

(1) *There is a direct summand X of M such that $X \leq A$ and $\frac{A}{X} \subseteq Soc_s\left(\frac{M}{X}\right)$.*
(2) *There are a direct summand X of M and a submodule Y of M such that $X \leq A$, $A = X + Y$ and $Y \subseteq Soc_s(M)$.*
(3) *There is a decomposition $M = X \oplus X'$ with $X \subseteq A$ and $X' \cap A \subseteq Soc_s(M)$.*
(4) *A has an ss-supplement X' in M such that $X' \cap A$ is a direct summand in A.*
(5) *There is a homomorphism $e\colon M \longrightarrow M$ with $e^2 = e$ such that $e(M) \leq A$ and $(1-e)(A) \subseteq Soc_s(1-e)(M)$.*

Proof. (1)$\Rightarrow$(2) Since X is a direct summand of M, there exists a submodule $X^{'}$ of M with $M = X \oplus X'$. If both sides of the equality are taken with A, we get that $A = X + (X' \cap A)$. Since $\frac{A}{X} \ll \frac{M}{X}$ and $\frac{A}{X}$ is semisimple, $\Psi\left(\frac{A}{X}\right) = X' \cap A \ll X$ and $X' \cap A$ is semisimple where $\Psi\colon M \to X$ is the canonical projection.

(2)⇒(3) By the hypothesis, we can write $M = X \oplus X'$ for some submodule $X^{'}$ of M. Then X' is a *ss*-supplement of X in M and so a *ss*-supplement of $A = X + Y$ in M by [4, Lemma 22]. Therefore $X' \cap A \subseteq Soc_s(X')$.

(3)⇒(4) If we take an intersection the equality $M = X \oplus X'$ with A, we can write $A = X \oplus (X' \cap A)$. Hence X' is a *ss*-supplement of A in M.

(4)⇒(5) From the hypothesis, we have $M = A + X'$, $X' \cap A \subseteq Soc_s(X')$ and $A = (X' \cap A) \oplus X$ for some $X \subseteq A$. Then $M = A + X' = (X' \cap A) + X + X' = X + X'$ and $(X' \cap A) \cap X = 0$ and $M = X + X'$. Let $e \colon M \to M$ be the projection such that $e(m) = x$, $m = x + x'$, $x \in X$, $x' \in X'$. Then $e(M) \subseteq X \subseteq U$. Since $(1-e)(M) = X'$, we get that $(1-e)(A) = X' \cap A \ll X' = (1-e)(M)$ and $X' \cap A$ is semisimple.

(5)⇒(1) Let $X = e(M)$. Since e is an idempotent, we have $M = e(M) \oplus (1-e)(M)$. Then $M = X \oplus (1-e)(M)$ with $X \subseteq A$. We will consider the isomorphism $\Phi \colon \frac{M}{X} \to (1-e)(M)$. From here, $\Phi\left(\frac{A}{X}\right) = (1-e)(A) \ll (1-e)(M) = \Phi\left(\frac{M}{X}\right)$. Since Φ^{-1} is an isomorphism, we can get $\frac{A}{X} \ll \frac{M}{X}$ and $\Phi^{-1}((1-e)(A)) = \frac{A}{X}$ is semisimple. Therefore $\frac{A}{X} \subseteq Soc_s\left(\frac{M}{X}\right)$. □

Note that every direct summand of a module is an *ss*-supplement submodule of the module and *ss*-supplement submodules are supplement.

Theorem 1. *For a module M, the following conditions are equivalent:*

(1) *M is ss-lifting.*
(2) *Every submodule A of M can be written as $A = N \oplus S$ with N is a direct summand of M and $S \subseteq Soc_s(M)$.*
(3) *M is an amply ss-supplemented module and every ss-supplement submodule of M is a direct summand.*

Proof. (1) ⇔ (2) By Lemma 2.

(1) ⇒ (3) It follows from Lemma 1 that M is an amply *ss*-supplemented module. Since every supplement submodule of a lifting module is a direct summand of the module, it follows from (1) that every every *ss*-supplement in M is a direct summand.

(3) ⇒ (1) Let A be a submodule of M. By the hypothesis, A has an *ss*-supplement X and X has an *ss*-supplement Y such that $Y \leq A$ and Y is a direct summand of M. Then there exists a submodule T of M with $M = Y \oplus T$. Hence we get that $A = Y \oplus (A \cap T)$ and $A = Y + (A \cap X)$. If we consider the projection $\pi \colon Y \oplus T \to T$, we can obtain that $\pi(A) = \pi(Y + (A \cap X)) = A \cap T$. In this way, we say that there is a decomposition $M = Y \oplus T$ such that $Y \leq A$, $A \cap T \ll M$ and $A \cap T \subseteq Soc_s(M)$ and so M is *ss*-lifting. □

Theorem 2. *Let M be a π-projective and ss-supplemented module. Then M is ss-lifting.*

Proof. By Proposition 37 of [4], M is amply *ss*-supplemented. Since M is *ss*-supplemented, there exists a submodule V of M such that $M = U + V$ and $U \cap V \subseteq Soc_s(V)$. On the other side, there exists a submodule U of M such that

$M = U' + V$, $U' \subseteq U$ and $U' \cap V \subseteq Soc_s(U')$ because M is amply *ss-supplemented. Hence* U' and V are mutual *ss-supplements*. By 41.14 (2) in [7], we can write $U' \cap V = 0$. It means that $M = U' \oplus V$. Thus M is *ss-lifting.* □

Theorem 3. *Let M be an ss-lifting module. Then every direct summand of M is ss-lifting.*

Proof. Let X be a direct summand of M with $M = X \oplus X'$ for some submodule X' of M and $U \leq X$. Since M is *ss*-lifting, there there exists a submodule V of M such that $M = U + V$, $U \cap V \subseteq Soc_s(V)$. Then we can write $X = U' \oplus (X \cap V)$ and $U \cap (X \cap V) = (U \cap X) \cap V = U \cap V$ is semisimple. It follows from $U \cap V \ll M$ and $U \cap V \ll X$ because X is a direct summand of M. Hence $X = U' \oplus (X \cap V)$ implies that $U \cap V \ll X \cap V$. Thus X is *ss*-lifting. □

Now, we will give necessary conditions for any lifting module to be *ss*-lifting.

Theorem 4. *Let M be a module with small radical. Then the following statements are equivalent:*

(1) *M is ss-lifting.*
(2) *M is lifting and $Rad(M) \subseteq Soc(M)$.*

Proof. (1) ⇒ (2) Since $Rad(M)$ is a small submodule of M and M is *ss*-lifting, M is an *ss*-supplement of $Rad(M)$ in M and so $Rad(M) \cap M = Rad(M)$ is semisimple.

(2) ⇒ (1) Let $U \leq M$. Since M is lifting, there is a decomposition for a submodule V of M, $M = U' \oplus V$, $U' \leq U$ and $U \cap V \ll V$. It follows that $U \cap V \subseteq Rad(V) \subseteq$ $\subseteq Rad(M)$ is semisimple. Thus M is *ss*-lifting. □

Since a projective supplemented module has small radical, we have the following fact as a result of Theorem 4.

Corollary 2. *Let M be a projective module. Then M is ss-lifting if and only if it is lifting and its radical is semisimple.*

Recall from [7, 43.9] that a ring whose all left modules are supplemented is *left perfect*. It follows from [7, 43.9] that a ring R is left perfect if and only if R is semilocal and $Rad(R)$ is right t-nilpotent if and only if every left R-module has a projective cover, that is, for any left R-module M, there exist a projective module P and an epimorphism $f\colon P \longrightarrow M$ with small kernel. R is called *semiperfect* if every finitely generated left (or right) R-module is supplemented. Now we give a characterization of semiperfect (left perfect) rings.

Lemma 3. *Let R be an arbitrary ring. Then ${}_RR$ is ss-lifting if and only if R is semiperfect and $Rad(R) \subseteq Soc({}_RR)$.*

Proof. By Theorem 4. □

Theorem 5. *The following statements are equivalent for a ring R.*

(1) ${}_RR$ *is ss-lifting.*
(2) ${}_RR$ *is ss-supplemented.*
(3) *Every left R-module is ss-supplemented.*
(4) *R is semiperfect and $Rad(R) \subseteq Soc({}_RR)$.*

Proof. (1) $\Rightarrow$ (2) It is clear.
(2) $\Rightarrow$ (3) $\Rightarrow$ (4) It follows from [4, Theorem 41].
(4) $\Rightarrow$ (1) By Theorem 4. □

Now we characterize the rings with the property that every left module is *ss*-lifting. Firstly, we need following lemma.

Lemma 4. *Let M be a lifting module and $Rad(M) \subseteq Soc(M)$. Then M is ss-lifting.*

Proof. The proof is clear. □

Theorem 6. *The following statements are equivalent:*

(1) *R is a left and right artinian serial ring and $Rad(R) \subseteq Soc({}_RR)$.*
(2) *Every left R-module is ss-lifting.*

Proof. (1)⇒(2) By the hypothesis and Lemma 3, it is clear that $Rad(R) \subseteq Soc({}_RR)$. On the other side, if every left R-module is semisimple lifting, then every left R-module is lifting by [2, 29.10].

(2)⇒(1) Since $Rad(R) \subseteq Soc({}_RR)$, we have $Rad(R)^2 = 0$ by [7, 21.12 (4)]. Moreover, we say that every left R-module is lifting by [2, 29.10]. We can write $Rad(M) = Rad(R)M \subseteq Soc({}_RR)M \subseteq Soc(M)$ because R is an artinian ring. Therefore M is *ss*-lifting by previous Lemma. □

Example 2. Consider the local ring $R = \mathbb{Z}_4$ is left and right artinian serial ring and $Rad(R) = \{0,2\} = Soc({}_RR)$ and so every left R-module is *ss*-lifting by Theorem 6.

Theorem 7. *Let M be a ss-lifting module. If $\frac{K+X}{X}$ is a direct summand of $\frac{M}{X}$ for every direct summand K of M, then $\frac{M}{X}$ is ss-lifting.*

Proof. Let $\frac{A}{X} \leq \frac{M}{X}$. Since M is *ss*-lifting, there exists a direct summand K of M with $K \leq A$ and $\frac{A}{K} \subseteq Soc_s\left(\frac{M}{K}\right)$ by Lemma 2. It is clear that $\frac{K+X}{X} \leq \frac{A}{X}$. If we say $\frac{A}{K+X} \subseteq Soc_s\left(\frac{M}{K+X}\right)$, the proof is completed. Since $\frac{\left(\frac{M}{K}\right)}{\left(\frac{K+X}{K}\right)} \cong \frac{M}{K+X}$, we get that $\frac{A}{K+X} \subseteq Soc_s\left(\frac{M}{K+X}\right)$. Therefore, $\frac{M}{X}$ is a *ss*-lifting module by Lemma 2. □

Recall from [2] that a submodule U of M is called *fully invariant* if $f(U)$ is contained in U for every R-endomorphism f of M. Recall from [2] that a module M is called *duo* if every submodule of M is fully invariant in M.

Theorem 8. *Let M be a ss-lifting module and X be a fully invariant submodule of M. Then $\frac{M}{X}$ is ss-lifting.*

Proof. Suppose that $M = K \oplus L$. Then $e(M) = K$ and $(1-e)(M) = L$ for some $e \in End(M)$. Since X is fully invariant, $e(X) = X \cap K$ and $(1-e)(X) = X \cap L$. From here, $X = e(X) \oplus (1-e)(X) = (X \cap K) \oplus (X \cap L)$ and we can write $\frac{K+X}{X} = \frac{K+[(X\cap K)\oplus(X\cap L)]}{X} = \frac{K\oplus(X\cap L)}{X}$ and $\frac{L+X}{X} = \frac{L+[(X\cap K)\oplus(X\cap L)]}{X} = \frac{L\oplus(X\cap K)}{X}$. Hence $M = K + X + L + X = [K \oplus (X \cap L)] + L + X$ implies that $[K \oplus (X \cap L)] \cap [L + X] = [K \oplus (X \cap L)] \cap [L + (X \cap K)] = (X \cap K) \oplus (X \cap L) = X$ and $\frac{M}{X} = \left(\frac{K\oplus(X\cap L)}{X}\right) \oplus \left(\frac{L+X}{X}\right)$. Thus $\frac{M}{X}$ is *ss*-lifting by the previous theorem. □

Recall from [1] that a submodule U is called a *weak distributive* of M if $U = (U \cap X) + (U \cap Y)$ for all submodules $X, Y \leq M$ such that $M = X + Y$. A module M is said to be *weakly distributive* if every submodule of M is a weak distributive submodule of M.

Theorem 9. *Let M be a weakly distributive module and $X \leq M$. Then $\frac{M}{X}$ is ss-lifting.*

Proof. Let $M = K \oplus L$. Then we have $\frac{M}{X} = \left(\frac{K+X}{X}\right) + \left(\frac{L+X}{X}\right)$ and $X = X + K \cap L = (X + K) \cap (X + L)$. Thus $\frac{M}{X} = \left(\frac{K+X}{X}\right) \oplus \left(\frac{L+X}{X}\right)$ and so $\frac{M}{X}$ is *ss*-lifting by Theorem 7. □

Theorem 10. *Let $M = M_1 \oplus M_2$ be a duo module. If M_1 and M_2 are ss-lifting modules, then M is ss-lifting.*

Proof. Suppose that L be a submodule of M. We can write $L = \bigoplus_{i=1}^{2} (L \cap M_i)$ by Lemma 2.1 of [2]. For each $i \in \{1, 2\}$, there exists a direct summand D_i of M_i such that $M_i = D_i \oplus D_i'$ with $D_i \leq L \cap M_i$ and $L \cap D_i' \subseteq Soc_s(D_i')$. From here

$$M = M_1 \oplus M_2 = \left(D_1 \oplus D_1'\right) \oplus \left(D_2 \oplus D_2'\right) = (D_1 \oplus D_2) \oplus \left(D_1' \oplus D_2'\right).$$

It is clear that $D_1 \oplus D_2 \leq L$. Since $L \cap D_i' \subseteq Soc_s(D_i')$, $L \cap (D_1' \oplus D_2') \subseteq Soc_s(D_1' \oplus D_2')$. Therefore M is *ss*-lifting. □

Lemma 5 (see [5, Lemma 5]). *The following statements are equivalent for a module $M = M_1 \oplus M_2$.*
(i) M_2 is M_1-projective.
(ii) For each submodule N of M with $M = M_1 + N$ there exists a submodule N' of N such that $M = M_1 \oplus N'$.

Theorem 11. *Let the module $M = M_1 \oplus M_2$ with M_1 and M_2 are relatively projective modules. If M_1 is semisimple and M_2 is ss-lifting, then M is ss-lifting.*

Proof. Suppose that K be a non-zero submodule of M.
Case 1: Assume that $T = M_1 \cap (K + M_2) \neq 0$. Since M_1 is semisimple, we can write $M_1 = T \oplus T_1$ for some submodule T_1 of M_1 and so $M = T \oplus T_1 \oplus M_2 = [(M_1 \cap (K + M_2))] \oplus T_1 \oplus M_2 = K \oplus (M_2 \oplus T_1)$. Using Prop. 4.31, Prop. 4.32 and

Prop. 4.33 in [6], we can say that T is $M_2 \oplus T_1$-projective. By 41.14 in [7], there exists a submodule K_1 of K such that $M = K_1 \oplus (M_2 \oplus T_1)$. Let A be any submodule of M_2 and $K \cap (M_2 \oplus T_1) \neq 0$. Since $K \cap (A + T_1) \leq A \cap (K + T_1) + T_1 \cap (K + A)$ and $T_1 \cap (K + A) = 0$, then $K \cap (A + T_1) \leq A \cap (K + T_1)$. Similarly, $A \cap (K + T_1) \leq K \cap (A + T_1)$. Hence $A \cap (K + T_1) = K \cap (A + T_1)$. Moreover, if we consider M_2 is ss-lifting, then there exists a submodule X_1 of $M_2 \cap (K + T_1) = K \cap (M_2 + T_1)$ such that $M_2 = X_1 \oplus X_2$ and $X_2 \cap (K + T_1) \subseteq Soc_s(X_2)$ for some submodule X_2 of M_2. Therefore $M = (K_1 \oplus X_1) \oplus (X_2 \oplus T_1)$, $K_1 \oplus T_1 \leq K$ and $K \cap (X_2 \oplus T_1) = = X_2 \cap (K + T_1) \subseteq Soc_s(X_2 \oplus T_1)$.

Case 2: Assume that $T = M_1 \cap (K + M_2) = 0$. From here T is a submodule of M_2. Since M_2 is ss-lifting, there exists a submodule Y_1 of K such that $M_2 = Y_1 \oplus Y_2$, $K \cap Y_2 \subseteq Soc_s(Y_2)$ for some submodule Y_2 of M_2. Thus $M = M_1 \oplus M_2 = = M_1 \oplus (Y_1 \oplus Y_2) = Y_1 \oplus (M_1 \oplus Y_2)$ and $K \cap (M_1 \oplus Y_2) = K \cap Y_2 \subseteq Soc_s(M_1 \oplus Y_2)$. As a result M is ss-lifting. □

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