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ON GOLDIE*-SUPPLEMENTED MODULES

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Abstract. Let M be a left R-module. X,Y of M are β^* equivalent, $X\beta^*Y$, if and only if $\frac{X+Y}{X}$ is small in $\frac{M}{X}$ and $\frac{X+Y}{Y}$ is small in $\frac{M}{Y}$. A module M is called G^*-supplemented if for every submodule X of M there is a supplement submodule S of M such that $X\beta^*S$. In this work some new properties of β^* are given and G^*-supplemented modules are studied. Also completely G^*-supplemented modules and G^*-radical supplemented modules are defined.

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1. INTRODUCTION

Throughout this paper R denotes an associative ring with unity and all R-modules are unital left R-modules. Let M be an R-module. A submodule S is called a *small* submodule of M if for every proper submodule A of M, $M \neq A+S$. We will use the notation $S \ll M$ to indicate that a submodule S is small in M.

Let M be an R-module. Let N be a submodule of M. A *supplement* of N in M is a submodule K of M minimal with respect to the property $M=N+K$, equivalently, $M=N+K$ and $N\cap K \ll K$. An R-module M is called a *supplemented* module if every submodule of M has a supplement in M. A submodule N of M has *ample supplements* in M if every submodule L such that $M=N+L$ contains a supplement of N in M. The module M is called *amply supplemented* if every submodule of M has ample supplements in M. More generally, a submodule N of M has a *weak supplement* L in M if $M=N+L$ and $N\cap L \ll M$ and M is called *weakly supplemented* if every submodule of M has a weak supplement in M. The R-module M is called $\oplus$-*supplemented* if every submodule of M has a supplement that is a direct summand of M. M is called *completely* $\oplus$-*supplemented* if every direct summand of M is $\oplus$-supplemented.

Let M be an R-module and $K \leq U \leq M$. If $\frac{U}{K} \ll \frac{M}{K}$ then we say U *lies above* K. It is well known that, U lies above a submodule K of M if and only if $K \leq U$ and for every submodule T of M with $U+T=M$, then $K+T=M$. Let M be an R-module. M satisfies $(D1)$ if for every submodule N of M there exist submodules K_1 and K_2 of

M such that $M = K_1 \oplus K_2$, $K_1 \leq N$ and $N \cap K_2 \ll K_2$. Furthermore M satisfies $(D1)$ iff every submodule of M lies above a direct summand of M.

Other terminologies and notations can be found in [2,4,8].

2. The β^* Relation

Let M be an R-module. The relation "β^*" on the set of submodules of M is defined by $X\beta^*Y$ if and only if $\frac{X+Y}{X}$ is small in $\frac{M}{X}$ and $\frac{X+Y}{Y}$ is small in $\frac{M}{Y}$. Moreover, β^* is an equivalence relation [1].

Lemma 1. *Let M be an R-module and $X, Y \leq M$. The following are equivalent:*

(i) *$X\ \beta^*Y$.*
(ii) *For each $A \leq M$ such that $X+Y+A = M$ then $X+A = M$ and $Y+A = M$.*
(iii) *If $K \leq M$ with $X+K = M$ then $Y+K = M$ and if $H \leq M$ with $Y+H = M$ then $X+H = M$.*

Proof. See [1, Theorem 2.3]. □

Lemma 2. *Let V be a supplement of U in M. If $X, Y \leq V$ such that $X\beta^*Y$ on the set of submodules of M, then $X\beta^*Y$ on the set of submodules of V.*

Proof. Let $X+K = V$, for some $K \leq V$. Then $X+K+U = M$ and since $X\beta^*Y$ on the set of submodules of M, $Y+K+U = M$. Then by V being a supplement of U in M, $Y+K = V$. Similarly, $Y+H = V$ with $H \leq V$ then $X+H = V$. Thus $X\ \beta^*Y$ on the set of submodules of V. □

Corollary 1. *Let $M = A \oplus B$ be an R-module. If $X, Y \leq A$ such that $X\beta^*Y$ on the set of submodules of M, then $X\beta^*Y$ on the set of submodules of A.*

Proof. Clear from Lemma 2. □

Lemma 3. *Let M be an R-module, $X, Y \leq M$ and $Rad\,(M/X) = 0$. If $X\beta^*Y$, then $Y \leq X$.*

Proof. Since $X\ \beta^*Y$, $\frac{X+Y}{X} \ll M/X$. Then $\frac{X+Y}{X} \leq Rad\,(M/X) = 0$. Hence $X+Y = X$ and $Y \leq X$. □

Corollary 2. *Let M be an R-module, $X, Y \leq M$, $Rad\,(M/X) = 0$ and $Rad\,(M/Y) = 0$. Then $X\ \beta^*Y$ if and only if $X = Y$.*

Proof. Clear from Lemma 3. □

Theorem 1. *Let M be a semisimple R-module and $X, Y \leq M$. Then $X\ \beta^*Y$ if and only if $X = Y$.*

Proof. Since M is semisimple, M/X and M/Y are semisimple. Then $Rad\,(M/X) = 0$ and $Rad\,(M/Y) = 0$. The rest is obvious by Corollary 2. □

Theorem 2. *Let M be an R-module. M is hollow if and only if all proper submodules of M are equivalent to each other with β^*.*

Proof. Clear. □

Let M be an R-module. M is called *distributive* if for arbitrary submodules K, L, N of M, $N + (K \cap L) = (N + K) \cap (N + L)$ this equivalent to $N \cap (K + L) = (N \cap K) + (N \cap L)$.

Theorem 3. *Let M be a distributive module and $X \leq M$. If $M = M_1 \oplus M_2$ and $M_1 \beta^* X$ then $M_1 \leq X$ and $M_2 \cap X \ll M$.*

Proof. Since $M = M_1 \oplus M_2$ and $M_1 \beta^* X$, $M = X + M_2$. Thus $M_1 = M_1 \cap M = M_1 \cap (X + M_2) = M_1 \cap X + M_1 \cap M_2 = M_1 \cap X$ and $M_1 \leq X$. Since $M_1 \leq X$ and $M_1 \beta^* X$, X lies above M_1 and $M_2 \cap X \ll M$. □

Corollary 3. *Let M be a distributive module. Assume that, for a submodule X of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq X$ and $M_2 \cap X \ll M$. If $X \beta^* Y$, then the decomposition $M = M_1 \oplus M_2$ exists for Y such that $M_1 \leq Y$ and $M_2 \cap Y \ll M$.*

Proof. By hypothesis, $M_1 \beta^* X$. Since $X \beta^* Y$, $M_1 \beta^* Y$. From Theorem 3, $M_1 \leq Y$ and $M_2 \cap Y \ll M$. □

Theorem 4. *Let M be an R-module. M is weakly supplemented if and only if for each $X \leq M$, there exists a weak supplement W in M such that $X \beta^* W$.*

Proof.

($\Rightarrow$) Assume that, M is weakly supplemented. Then every submodule of M is a weak supplement. Since $X \beta^* X$ for each $X \leq M$, every submodule of M is β^* equivalent to a weak supplement.

($\Leftarrow$) Let $X \leq M$. By hypothesis there exists a weak supplement W in M such that $X \beta^* W$. Since W is a weak supplement in M, there exists $A \leq M$ such that $W + A = M$ and $W \cap A \ll M$. Hence, A is a weak supplement of X by [1, Theorem 2.6]. So M is weakly supplemented.

□

Corollary 4. *Let M be an R-module. M is weakly supplemented if and only if for each $X \leq M$ there exists a weak supplement W and a small submodule H of M such that $X + H = W + H = X + W$.*

Proof.

($\Rightarrow$) Let $X \leq M$. Since M is weakly supplemented, by Theorem 4, there exists a weak supplement W in M such that $X \beta^* W$. Hence there exists $A \leq M$ such that $W + A = M$ and $W \cap A \ll M$. From [1, Proposition 2.11], $X \beta^* (X + W)$ and $W \beta^* (X + W)$. By [1, Theorem 2.6], A is a weak supplement of X and

$X+W$. By the modular law, $X+H=W+H=X+W$, where $H=(X+W)\cap A\ll M$.

($\Leftarrow$) It can be seen easily.

□

3. G^*-SUPPLEMENTED MODULES

Let M be an R-module. M is called *G^*-supplemented* (*G^*-lifting = H-supplemented*) if for every submodule X of M, there is a supplement submodule S (direct summand D) of M such that $X\ \beta^* S$ ($X\ \beta^* D$). [1]

Theorem 5. *Let M be a G^*-supplemented module and $X\le M$. If for every supplement submodule S of M, $\frac{(X+S)}{X}$ is a supplement submodule of $\frac{M}{X}$, then $\frac{M}{X}$ is G^*-supplemented.*

Proof. Let $\frac{N}{X}\le\frac{M}{X}$. Since M is G^*-supplemented, there exists a supplement submodule S of M such that $N\beta^* S$. Then by [1, Proposition 2.9(i)], $\frac{N}{X}\beta^*\frac{S+X}{X}$. Since S is a supplement submodule of M, then by hypothesis, $\frac{S+X}{X}$ is a supplement submodule of $\frac{M}{X}$. Hence $\frac{M}{X}$ is G^*-supplemented. □

Corollary 5. *Let M be a G^*-supplemented module. If M is a distributive module, then $\frac{M}{X}$ is G^*-supplemented for every submodule X of M.*

Proof. Let S be a supplement submodule of M. There exists a submodule S' of M such that $M=S+S'$and $S\cap S'\ll S$. Then $\frac{(S+X)}{X}+\frac{(S'+X)}{X}=\frac{M}{X}$. Let $\left[\frac{(S+X)}{X}\cap\frac{(S'+X)}{X}\right]+\frac{K}{X}=\frac{(S+X)}{X}$ for some $\frac{K}{X}\le\frac{(S+X)}{X}$. Then $[X+(S\cap S')]+K=S+X$. Since $S\cap S'\ll S$, $K=S+X$. So $\frac{(S+X)}{X}$ is a supplement submodule of $\frac{M}{X}$. Thus $\frac{M}{X}$ is G^*-supplemented by Theorem 5. □

Definition 1. Let M be an R-module and $K\le M$. We say that a submodule $\overline{T}$ of $\frac{M}{K}$ lifts to a submodule T of M, if under the natural morphism $\pi: M\to\frac{M}{K}$, $\pi(T)=\overline{T}$.

Theorem 6. *Let M be an R-module. If $Rad(M)\ll M$, then M is G^*-supplemented if and only if $\overline{M}=\frac{M}{Rad(M)}$ is semisimple and each submodule of $\overline{M}$ lifts to a supplement submodule of M.*

Proof.

($\Rightarrow$) Suppose that M is G^*-supplemented and $\overline{A}\le\overline{M}$. Then M is a supplemented module by [1, Theorem 3.6]. So the full inverse image A of $\overline{A}$ has a supplement B in M. Then $A\cap B$ is small in B hence in M. Therefore $A\cap B\le Rad(M)$ and consequently $\overline{A}\oplus\overline{B}=\overline{M}$. We conclude that $\overline{M}$ is semisimple.

Returning to $\overline{A}\le\overline{M}$, we have a supplement submodule $S\le M$ such that $A\beta^* S$. Then $\overline{A}\ \beta^*\frac{(S+Rad(M))}{Rad(M)}$. Since $\overline{M}$ is semisimple, $\overline{A}=\frac{(S+Rad(M))}{Rad(M)}$ by Theorem 1. Consequently $\overline{A}$ lifts to the supplement submodule S.

($\Leftarrow$) If $N \leq M$ is given, there exists a supplement submodule S of M such that $\overline{S} = \overline{N}$. Since $Rad(M) \ll M$, $N\beta^*S$. Thus M is G^*-supplemented.

□

Theorem 7. *Let R be a complete discrete valuation ring. Then every R-module is G^*-supplemented if and only if every R-module is amply supplemented.*

Proof.

($\Rightarrow$) Let M be an R-module. Then M is G^*-supplemented. By [1, Theorem 3.6], M is supplemented and hence by [9, Theorem 2.2(c)], M is amply supplemented.

($\Leftarrow$) Let M be an R-module. Then M is amply supplemented. From [1, Proposition 3.11], M is G^*-supplemented.

□

Theorem 8. *Let R be any ring. Then R is left perfect if and only if every projective R-module is G^*-supplemented.*

Proof.

($\Rightarrow$) Let R be a left perfect ring and M be a projective R-module. By [4, Theorem 4.41], M is supplemented. Since M is projective, M is G^*-supplemented by [1, Proposition 3.12].

($\Leftarrow$) Let M be a left R-module and $f : P \to M$ be an epimorphism with a projective R-module P. By assumption, P is G^*-supplemented. Then P is supplemented by [1, Theorem 3.6]. Since P is projective, P is π-projective. By [8, Theorem 41.16], every supplement submodule of P is a direct summand of P. Since P is G^*-supplemented, there exists a supplement submodule S in P such that $\text{Ker}(f)\,\beta^*S$ and also S is a direct summand. Let $P = S \oplus S'$ for some submodules S' of P. Then $P = \text{Ker}(f) + S'$ and $\text{Ker}(f) \cap S' \ll S'$. Let $g = f\,|_{S'}$. Then $g : S' \to M$ is a projective cover of M. Hence R is left perfect.

□

It is unknown whether every direct summand of a G^*-supplemented (G^*-lifting) module is G^*-supplemented (G^*-lifting).

Definition 2. Let M be an R-module. M is called completely G^*-supplemented (G^*-lifting) if every direct summand of M is G^*-supplemented (G^*-lifting).

It is clear that lifting and completely G^*-lifting modules are completely G^*-supplemented.

Theorem 9. *Suppose that M is G^*-supplemented and distributive R-module. If the intersection of any two supplement submodules of M is again a supplement submodule in M, then M is completely G^*-supplemented.*

Proof. Let D be a direct summand of M. There exists a direct summand D' of M such that $M = D \oplus D'$. Let A be a submodule of D. Since M is G^*-supplemented and $A \leq M$, there exists a supplement submodule S of M such that $A\ \beta^* S$. Since M is distributive, $A\ \beta^* (D \cap S)$. Also $D \cap S$ is a supplement submodule in D. Consequently D is G^*-supplemented and hence M is completely G^*-supplemented. □

The SSP Property. A module M is said to have the *summand sum property*, if the sum of any two direct summands of M is again a direct summand of M.

Theorem 10. *Let M be a projective module with SSP. The followings are equivalent.*

(1) M is supplemented.
(2) M is quasi-discrete.
(3) M is discrete.
(4) M is lifting.
(5) M is G^*-lifting($=H$-supplemented).
(6) M is completely G^*-lifting module.
(7) M is amply supplemented.
(8) M is $\oplus$-supplemented.
(9) M is completely $\oplus$-supplemented.
(10) M is G^*-supplemented.
(11) M is completely G^*-supplemented.
(12) M is semiperfect.

Proof. Result of [3, Theorem 2.11] and [1, Proposition 3.12]. □

4. G^*-RADICAL SUPPLEMENTED MODULES

For submodules U and V of a module M, the submodule V is said to be a *radical supplement* (or briefly *Rad-supplement*)of U in M or U is said to have a *radical supplement* V in M if $U + V = M$ and $U \cap V \leq Rad(V)$. A module M is called a *radical supplemented* (or briefly *Rad-supplemented*) module if every submodule of M has a Rad-supplement in M (according to [5], a *generalized supplemented* module) and it is called amply Rad-supplemented in case $M = A + B$ implies that A has a Rad-supplement $B' \leq B$.

Lemma 4. *Let M be an R-module and $U, V \leq M$. V is a radical supplement of U in M if and only if $U + V = M$ and for every $m \in U \cap V$, $Rm \ll V$.*

Proof. See [6, Proposition 4]. □

Lemma 5. *Let M be an R-module. If T is a Rad-supplement in M and $a \in T$ then $Ra \ll T$ iff $Ra \ll M$.*

Proof.
($\Rightarrow$) Clear.

($\Leftarrow$) Assume $Ra \ll M$. Let $X \leq T$ such that $Ra + X = T$. Since T is a Rad-supplement in M, there exists a submodule A of M such that $A + T = M$ and $A \cap T \leq Rad(T)$. Hence $Ra + X + A = M$. Since $Ra \ll M$, $X + A = M$. By the modular law, $X + (T \cap A) = T$. Since $a \in T$, there exist $a_1 \in X$ and $a_2 \in T \cap A$ such that $a = a_1 + a_2$. Thus $Ra \leq Ra_1 + Ra_2$. Since $Ra + X = T$ and $a_1 \in X$, $T = Ra_2 + X$. By Lemma 4, $Ra_2 \ll T$ and so $X = T$. Finally $Ra \ll T$.

□

Theorem 11. *Let M be an R-module and $S, T \leq M$. If S is a radical supplement of T in M and T is a radical supplement in M, then T is a radical supplement of S in M.*

Proof. By hypothesis, there exist $A \leq M$ such that $A + T = M$ and $A \cap T \leq Rad(T)$ and also $S + T = M$ and $S \cap T \leq Rad(S)$. Let $X \leq T$ and a is an arbitrary element of $S \cap T$ such that $Ra + X = T$. Then $Ra + X + A = M$. Since $a \in S \cap T \leq Rad(S), Ra \ll S$. Thus $Ra \ll M$. Since $a \in T$ and $Ra \ll M$, $Ra \ll T$ by Lemma 5. Hence T is a Rad-supplement of S in M by Lemma 4. □

Theorem 12. *Let $X, Y \leq M$ such that $X\ \beta^* Y$. Then, for $U \leq M$, U is a Rad-supplement of X in M if and only if U is a Rad-supplement of Y in M.*

Proof. Assume U is a Rad-supplement of X in M. Then $X + U = M$ and $X \cap U \leq Rad(U)$. Since $X\ \beta^* Y$, $Y + U = M$ by Lemma 1. Let $a \in U \cap Y$ and $T \leq U$ such that $Ra + T = U$. Then $Y + Ra + T = M$ and by $Ra \leq Y$, $Y + T = M$. Since $X\ \beta^* Y$, $X + T = M$ by Lemma 1. Hence $U = T + (U \cap X)$. So we can write $a = a_1 + a_2$ such that $a_1 \in T$ and $a_2 \in U \cap X$. Since $Ra \leq Ra_1 + Ra_2 \leq U$, $Ra_1 + Ra_2 + T = U$ and since $a_2 \in U \cap X$, $Ra_2 \ll U$ by Lemma 4. Thus $Ra_1 + T = U$ and since $Ra_1 \leq T$, $U = T$. So $Ra \ll U$. By Lemma 4, U is a Rad-supplement of Y. □

Theorem 13. *Let M be an R-module. If M is amply Rad-supplemented then its submodules which have the same Rad-supplements are equivalent with β^*.*

Proof. Let U and V be submodules of M. Assume that they have the same Rad-supplements and $U + T = M$ with $T \leq M$. Since M is amply Rad-supplemented, there exists a submodule $T' \leq T$ such that T' is a Rad-supplement of U in M. Since U and V have the same Rad-supplements, T' is a Rad-supplement of V too. Thus $V + T' = M$ and since $T' \leq T$, $V + T = M$. Similarly, if $V + K = M$ with $K \leq M$, then $U + K = M$. Therefore, from Lemma 1, $U\ \beta^* V$. □

Corollary 6. *Let M be an R-module. If M is amply supplemented then its submodules which have the same Rad-supplements are equivalent with β^*.*

Proof. Since M is amply supplemented, M is amply Rad-supplemented. So the proof is clear by Theorem 13. □

Definition 3. Let M be an R-module. M is called G^*-Radical supplemented (or briefly $G^* - Rad$-supplemented) if for every submodule X of M, there is a Rad-supplement submodule U in M such that $X\ \beta^* U$.

Theorem 14. *Let M be an R-module. Consider the following conditions:*

(1) M is G^*-supplemented.
(2) M is $G^* - Rad$-supplemented.
(3) M is Rad-supplemented.

Then $(1) \Rightarrow (2) \Rightarrow (3)$.

Proof.

$(1) \Rightarrow (2)$ Let M be G^*-supplemented. Since every supplement submodule is a Rad-supplement submodule, M *is* $G^* - Rad$-supplemented.

$(2) \Rightarrow (3)$ Let M be $G^* - Rad$-supplemented. For every submodule X of M, there exists $V \leq M$ such that V is a Rad-supplement submodule in M and $X\ \beta^* V$. Hence, there exists $T \leq M$ such that $V + T = M$ and $V \cap T \leq Rad(V)$. Since M is $G^* - Rad$-supplemented, there exists a submodule K of M such that K is a Rad-supplement submodule in M and $K\ \beta^* T$. Hence, V is Rad-supplement of K in M by Theorem 12. Since K is Rad-supplement submodule in M, K is a Rad-supplement of V in M by Theorem 11. Since $X \beta^* V$, K is Rad-supplement of X in M by Theorem 12. So M *is* Rad-supplemented.

□

Example 1. Let K be the quotient field of a Dedekind domain R which is not local. Let $M = K \oplus K$. Since $Rad(M) = M$, M is Rad-supplemented but not G^*-supplemented. To see this, assume that M is G^*-supplemented. Then M is supplemented by [1, Theorem 3.6]. Hence K is supplemented. Since K is supplemented, R is local by [10, Remark 2 in the proof of Theorem 3.1], a contradiction.

Theorem 15. *Let M be an R-module. Consider the following conditions:*

(1) M is lifting.
(2) M is G^*-lifting$(= H$-supplemented$)$.
(3) M is G^*-supplemented.
(4) M is $G^* - Rad$-supplemented.
(5) M is Rad-supplemented.

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ hold. If every Rad-supplement submodule of M is projective then $(5) \Rightarrow (1)$ also holds.

Proof. $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ is clear by [1, Theorem 3.6] and Theorem 14.

$(5) \Rightarrow (1)$ Let M be Rad-supplemented and every Rad-supplement submodule of M be projective. Let $U < M$. Since M is Rad-supplemented, there is a Rad-supplement V of U in M. So $M = U + V$ and $U \cap V \leq Rad(V)$ and also $V \neq 0$. By hypothesis and $V \neq 0$, there is a maximal submodule T of V. Since $M =$

$U+V$ and $U \cap V \leq Rad(V) \leq T$, $\frac{M}{(U+T)} = \frac{(U+V)}{(U+T)} \simeq \frac{V}{[V \cap (U+T)]} = \frac{V}{U \cap V+T} = \frac{V}{T}$. So $U+T$ is a maximal submodule of M and every proper submodule of M is contained in a maximal submodule in M. Hence $Rad(M) \ll M$. Let $X \leq M$. Since M is Rad-supplemented, there is a Rad-supplement Y of X in M. Then $M = X+Y$ and $X \cap Y \leq Rad(Y)$. So Y is a direct summand in M by [5, Lemma II.1]. Finally Y is a supplement of X in M and so M is supplemented. Since M is supplemented and projective, M is also lifting.

□

Let R be a ring. R is called a *Bass* ring, if every R-module has a maximal submodule. Also, a ring R is a Bass ring if and only if for every R-module M, $\mathrm{Rad}(M) \ll M$ [2].

Theorem 16. *Let R be a Bass ring. Then an R-module M is G^*-Rad-supplemented if and only if every R-module M is G^*-supplemented.*

Proof. Since R is a Bass ring, every Rad-supplement submodule is a supplement submodule. Therefore, proof is clear. □

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