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SOLUTION OF COMPLEX DIFFERENTIAL EQUATIONS BY USING REDUCED DIFFERENTIAL TRANSFORM METHOD

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Abstract. In this paper, complex differential equations from first and second order with constant coefficients are solved using reduced differential transform method (RDTM). Such equations were previously solved by other methods. While obtaining a solution with these methods, the solution was reached by dividing the equation into real and imaginary parts. With the RDTM used in this study, we found that the solution can be reached without dividing the real and imaginary parts of the equation.An iteration relation is given for the solution of such equations . In addition, a variable coefficient equation was solved with RDTM.

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1. INTRODUCTION

The emergence of complex partial differential equations dates back to the beginning of 1900. One of the important mathematicians who made serious studies and left a name in this field was D. Pompeiu, and he described the Pompeiu integral operator, which is called by his name and which forms the basis of the theory of complex differential equations even today.

Some applications of complex partial differential equations emerged at a mechanical congress in Canada. The difficulties of some problems in real space have been overcome by the solution methods of complex equations. For instance, the $\Delta u = 0$ Laplace equation, an elliptic differential equation (see e.g. [11]), does not have a general solution in real space, but there is a general solution of this equation in complex space. One can find more information about on complex equation in [5, 9, 12].

In this study, solutions of complex differential equations from first and second order with constant coefficients are obtained. Equations which are studied are form that:

$$A\frac{\partial w}{\partial z} + B\frac{\partial w}{\partial \bar{z}} + Cw = F(z, \bar{z}) \tag{1.1}$$

and

$$A\frac{\partial^2 w}{\partial z^2}+B\frac{\partial^2 w}{\partial z\partial \bar{z}}+C\frac{\partial^2 w}{\partial \bar{z}^2}+D\frac{\partial w}{\partial z}+E\frac{\partial w}{\partial \bar{z}}+Fw=G(z,\bar{z}) \tag{1.2}$$

where coefficients of equations are constant.

Complex differential equations were solved by using integral transforms, Adomian decomposition method [2–4]. While making the solution with these methods, the equation was divided into real and imaginary parts and the process was performed. In this article, the reduced differential transformation method can be used to solve the equation without separating it into real and imaginary parts. Therefore, the solution can be obtained by performing fewer operations. RDTM was proposed firstly by Keskin in [7]. Linear and nonlinear equations and fractional equations are solved with RDTM [1, 6, 8, 10].

This paper has been organized as follows. Basic definitions and theorems associated with RDTM and complex derivative are given in Section 2. In Section 3, for solve of 1.1 and 1.2 equations have been obtained a recursive relation and have been given some examples for validity of the method.

2. Basic Definitions and Theorems

Suppose that the two-variable $u(x,y)$ function can be written as $u(x,y)=f(x)g(y)$. With the help of one dimensional differential transformation, the function $u(x,y)$ can be written as follows.

$$u(x,y)=\left(\sum_{i=0}^{\infty}F(i)x^i\right)\left(\sum_{j=0}^{\infty}G(j)y^j\right)=\left(\sum_{k=0}^{\infty}U_k(x)y^k\right) \tag{2.1}$$

where $U_k(x)$ is called y dimensional spectrum function $u(x,y)$, and $F(i)$, $G(j)$ are differential transform of $f(x)$ and $g(y)$, respectively.

Definition 1. If a function $u(x,y)$ is analytic and differentiated continuously with respect to y and space x in the domain of interest then let

$$U_k(x)=\frac{1}{k!}\left(\frac{\partial^k}{\partial y^k}u(x,y)\right)_{y=0} \tag{2.2}$$

where the y dimensional spectrum function $U_k(x)$ is the transformed function and $u(x,y)$ is the original function.

Definition 2. The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,y)=\sum_{k=0}^{\infty}U_k(x)y^k \tag{2.3}$$

From (2.2) and (2.3), we get

$$u(x,y) = \sum_{k=0}^{\infty} \frac{y^k}{k!} \left(\frac{\partial^k}{\partial y^k} u(x,y) \right)_{y=0} \tag{2.4}$$

Theorem 1 ([7,8])**.** *If* $f(x,y) = ag(x,y) + bh(x,y)$ *then* $F_k(x) = aG_k(x) + bH_k(x)$, *where a and b are constants.*

Theorem 2 ([7,8])**.** *If* $f(x,y) = x^m y^n$, *then* $F_k(x) = x^m \delta(k-n)$

Theorem 3 ([7, 8])**.** *If* $f(x,y) = \dfrac{\partial^n g(x,y)}{\partial y^n}$, *then* $F_k(x) = (k+1)(k+2)...(k+n)G_{k+n}(x)$

Theorem 4 ([7,8])**.** *If* $f(x,y) = \dfrac{\partial^n g(x,y)}{\partial x^n}$, *then* $F_k(x) = \dfrac{\partial^n G_k(x)}{\partial x^n}$.

Theorem 5 ([7,8])**.** *If* $f(x,y) = g(x,y).h(x,y)$, *then* $F_k(x) = \sum_{r=0}^{k} G_r(x).H_{k-r}(x)$.

Now, let's give the equals of the first and second order derivatives of a complex function from kind of real derivatives.

Definition 3. First and second order derivatives of $w = w(z,\bar{z})$ as $z = x+iy$ are as follows.

$$\frac{\partial w}{\partial z} = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) \tag{2.5}$$

$$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right) \tag{2.6}$$

$$\frac{\partial^2 w}{\partial z^2} = \frac{1}{4}\left[\frac{\partial^2 w}{\partial x^2} - 2i\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2}\right] \tag{2.7}$$

$$\frac{\partial^2 w}{\partial \bar{z}^2} = \frac{1}{4}\left[\frac{\partial^2 w}{\partial x^2} + 2i\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2}\right] \tag{2.8}$$

$$\frac{\partial^2 w}{\partial z \partial \bar{z}} = \frac{1}{4}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] \tag{2.9}$$

3. SOLUTION OF COMPLEX DIFFERENTIAL EQUATIONS FROM FIRST AND SECOND ORDER WHICH IS CONSTANT COEFFIENTS

In this section, two theorems for which solution of complex equations from first and second order with constant coefficients have been given and then the samples have been solved according to these theorems.

Theorem 6. *Let A, B, C are real constants and $F(z,\bar{z})$ is a function of $z,\bar{z}$. Then a special solution of equation*

$$A\frac{\partial w}{\partial z}+B\frac{\partial w}{\partial \bar{z}}+Cw=F(z,\bar{z}) \tag{3.1}$$

with condition

$$w(x,0)=f(x)$$

is

$$w(z,\bar{z})=\sum_{k=0}^{\infty}W_k(x)\,y^k \tag{3.2}$$

where

$$(B-A)i(k+1)W_{k+1}(x)=2F_k^*(x)-2CW_k(x)-(A+B)\frac{\partial W_k(x)}{\partial x}, k\geq 0$$

$$x=\frac{z+\bar{z}}{2}, y=\frac{z-\bar{z}}{2i}, \text{ and } W_0(x)=f(x).$$

Proof. If (2.5), (2.6) equalities are used in equality (3.1), which is given in theorem, following equality is obtained .

$$\left(\frac{A+B}{2}\right)\frac{\partial w}{\partial x}+\left(\frac{B-A}{2}\right)i\frac{\partial w}{\partial y}+Cw=F^*(x,y) \tag{3.3}$$

where $F^*(x,y)$ is obtained by writing $x+iy$ in place of z and $x-iy$ in place of $\bar{z}$ in $F(z,\bar{z})$. If equality(3.3) is regulated then following equality is obtained.

$$(A+B)\frac{\partial w}{\partial x}+(B-A)i\frac{\partial w}{\partial y}+2Cw=2F^*(x,y) \tag{3.4}$$

If reduced differential transform is applied in equality(3.4) then following equalities is obtained.

$$(A+B)\frac{\partial W_k(x)}{\partial x}+(B-A)i(k+1)W_{k+1}(x)+2CW_k(x)=2F_k^*(x)$$

$$(B-A)i(k+1)W_{k+1}(x)=2F_k^*(x)-2CW_k(x)-(A+B)\frac{\partial W_k(x)}{\partial x} \tag{3.5}$$

In equality (3.5) $k\geq 0$, $W_0(x)=w(x,0)=f(x)$. □

Theorem 7. *Let A,B,C,D,E,F are real constants and $G(z,\bar{z})$ is a polynomial of $z,\bar{z}$. Then a special solution of equation*

$$A\frac{\partial^2 w}{\partial z^2}+B\frac{\partial^2 w}{\partial z\partial \bar{z}}+C\frac{\partial^2 w}{\partial \bar{z}^2}+D\frac{\partial w}{\partial z}+E\frac{\partial w}{\partial \bar{z}}+Fw=G(z,\bar{z}) \tag{3.6}$$

with conditions

$$w(x,0)=f(x)$$

$$\frac{\partial w}{\partial y}(x,0) = g(x)$$

is

$$w(z,\bar{z}) = \sum_{k=0}^{\infty} W_k(x)\, y^k$$

where

$$W_{k+2}(x) = \frac{4\left[G_k^*(x) - FW_k(x) - \frac{(E-D)}{2} i(k+1)W_{k+1}(x)\right]}{(B-A-C)(k+1)(k+2)}$$
$$-\frac{4\left[\frac{(D+E)}{2}\frac{\partial W_k}{\partial x} + 2i\frac{(C-A)}{4}(k+1)\frac{\partial W_{k+1}}{\partial x} + \frac{(A+B+C)}{4}\frac{\partial^2 W_k}{\partial x^2}\right]}{(B-A-C)(k+1)(k+2)}, k \geq 0$$
$$W_0(x) = f(x), W_1(x) = g(x),\ x = \frac{z+\bar{z}}{2}, y = \frac{z-\bar{z}}{2i}$$

Proof. If it is used equalities (2.5)-(2.9) in equation (3.6), which is given in theorem, following equality is obtained .

$$\left(\frac{A+B+C}{4}\right)\frac{\partial^2 w}{\partial x^2} + 2i\left(\frac{C-A}{4}\right)\frac{\partial^2 w}{\partial x \partial y} + \left(\frac{B-A-C}{4}\right)\frac{\partial^2 w}{\partial y^2}$$
$$+\left(\frac{D+E}{2}\right)\frac{\partial w}{\partial x} + \left(\frac{E-D}{2}\right) i\frac{\partial w}{\partial y} + Fw = G^*(x,y) \tag{3.7}$$

where $G^*(x,y)$ is obtained by writing $x+iy$ in place of z and $x-iy$ in place of $\bar{z}$ in $G(z,\bar{z})$.

If reduced differential transform is applied in equality (3.7) then following iterative relation is obtained.

$$\left(\frac{A+B+C}{4}\right)\frac{\partial^2 W_k}{\partial x^2} + 2i\left(\frac{C-A}{4}\right)(k+1)\frac{\partial W_{k+1}}{\partial x}$$
$$+\left(\frac{B-A-C}{4}\right)(k+1)(k+2)W_{k+2} + \left(\frac{D+E}{2}\right)\frac{\partial W_k}{\partial x}$$
$$+\left(\frac{E-D}{2}\right) i(k+1)W_{k+1} + FW_k = G_k^*(x) \tag{3.8}$$

Using (3.8) iterative relation, following equality can be written.

$$W_{k+2}(x) = \frac{4\left[G_k^*(x) - FW_k(x) - \frac{(E-D)}{2} i(k+1)W_{k+1}(x)\right]}{(B-A-C)(k+1)(k+2)} \tag{3.9}$$

$$-\frac{4\left[\frac{(D+E)}{2}\frac{\partial W_k}{\partial x}+2i\frac{(C-A)}{4}(k+1)\frac{\partial W_{k+1}}{\partial x}+\frac{(A+B+C)}{4}\frac{\partial^2 W_k}{\partial x^2}\right]}{(B-A-C)(k+1)(k+2)}$$

where we set $W_0(x)=f(x), W_1(x)=g(x)$ due to the conditions. □

Example 1. Solve the following problem

$$\frac{\partial w}{\partial z}+2\frac{\partial w}{\partial \bar{z}}=3z^2+2$$

$$w(x,0)=x^3+x.$$

Solution. From Theorem 6, we get $A=1, B=2, C=0, F(z,\bar{z})=3z^2+2$, $F^*(x,y)=3x^2-3y^2+2+6ixy$.

From equality (3.5)

$$i(k+1)W_{k+1}(x)=2\left(3x^2+2\right)\delta(k)-6\delta(k-2)+12ix\delta(k-1)-3\frac{\partial W_k(x)}{\partial x}$$

$$W_0(x)=x^3+x$$

$$\begin{aligned} iW_1(x) &= 6x^2+4-3\left(3x^2+1\right)=-3x^2+1, W_1(x)=i\left(3x^2-1\right)\\ 2iW_2(x) &= 12ix-3.6ix, W_2(x)=-3x\\ 3iW_3(x) &= -6-3(-3), W_3(x)=-i\\ W_k(x) &= 0, k>3. \end{aligned}$$

$$\begin{aligned} w(x,y) &= \sum_{k=0}^{\infty} W_k(x)y^k = W_0(x)+W_1(x).y+W_2(x)y^2+W_3(x)y^3\\ &= x^3+x+yi\left(3x^2-1\right)-3xy^2-iy^3\\ &= x^3+3ix^2y-3xy^2-iy^3+x-iy. \end{aligned}$$

Therefore, solution is

$$w(z,\bar{z})=z^3+\bar{z}.$$

Example 2. Solve the following problem

$$2\frac{\partial w}{\partial z}-\frac{\partial w}{\partial \bar{z}}=4z+1$$

$$w(x,0)=x^2+5x$$

Solution. Coeffients of equation are $A = 2, B = -1, C = 0, F(z,\bar{z}) = 4z + 1$, $F^*(x,y) = 4x + 1 + 4iy$

From equality (3.5)

$$-3i(k+1)W_{k+1}(x) = 2(4x+1)\delta(k) + 8i\delta(k-1) - \frac{\partial W_k(x)}{\partial x}$$

$$W_0(x) = x^2 + 5x$$

$$\begin{aligned}
&-3iW_1(x) = 8x + 2 - (2x+5) = 6x - 3, W_1(x) = i(2x-1) \\
&-6iW_2(x) = 8i - 2i = 6i, W_2(x) = -1, \\
&W_k(x) = 0, \quad k > 2
\end{aligned}$$

$$\begin{aligned}
w(x,y) &= \sum_{k=0}^{\infty} W_k(x) y^k = W_0(x) + W_1(x) y + W_2(x) y^2 \\
&= x^2 + 5x + i(2x-1) y - y^2 = x^2 + 2ixy + 2(x+iy) + 3(x-iy).
\end{aligned}$$

Therefore, solution of the example is

$$w(z,\bar{z}) = z^2 + 2z + 3\bar{z}.$$

Example 3. Solve the following problem

$$z\frac{\partial w}{\partial z} - \bar{z}\frac{\partial w}{\partial \bar{z}} = 2z^2 + 5\bar{z}$$

with the condition

$$w(x,0) = 2x^2 - 5x$$

Solution. This equation hasn't constant coefficients. Coefficients of the equation are variable. Writing $x+iy$ in place of z, $x-iy$ in place of $\bar{z}$ in the example, from (2.5) and (2.6), the following equalities is obtained as

$$(x+iy)\frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) - (x-iy)\frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right) = 2(x+iy)^2 + 5(x-iy) \quad (3.10)$$

$$-ix\frac{\partial w}{\partial y} + iy\frac{\partial w}{\partial x} = 2x^2 - 2y^2 + 5x + i(4xy - 5y) \quad (3.11)$$

$$\begin{aligned}
-ix(k+1)W_{k+1}(x) &= \left(2x^2 + 5x\right)\delta(k) + i(4x-5)\delta(k-1) - 2\delta(k-2) \\
&\quad - i\sum_{r=0}^{k}\delta(r-1)\frac{\partial W_{k-r}(x)}{\partial x} \\
W_0(x) &= 2x^2 - 5x \\
-ixW_1(x) &= 2x^2 + 5x, W_1(x) = i(2x+5)
\end{aligned}$$

$$-2ixW_2(x) = i(4x-5) - i(4x-5) = 0, W_2(x) = 0$$
$$-3ixW_3(x) = -2 - i2i = 0, W_k(x) = 0 (k > 3)$$
$$w(x,y) = \sum_{k=0}^{\infty} W_k(x) y^k = W_0(x) + W_1(x) y + W_2(x) y^2$$
$$= 2x^2 - 5x + i(2x+5)y.$$

Then,

$$w(z,\bar{z}) = z^2 + z.\bar{z} - 5\bar{z}.$$

Example 4. Solve the following problem

$$\frac{\partial w}{\partial z} - \frac{\partial w}{\partial \bar{z}} - w = 0$$

with the condition

$$w(x,0) = e^{3x}$$

Solution. This equation is homogeneous. Coefficients are $A = 1, B = -1$, $C = -1$.

We can write from (3.5) following equality

$$W_{k+1}(x) = i\frac{W_k(x)}{k+1}, k \geq 0$$

$$W_0(x) = e^{3x}, W_1(x) = ie^{3x}, W_2(x) = -\frac{e^{3x}}{2}, W_3(x) = -i\frac{e^{3x}}{6},$$
$$W_4(x) = \frac{e^{3x}}{24}, W_n(x) = i^n \frac{e^{3x}}{n!}$$

Solution of problem is

$$W_0(x) + W_1(x) y + W_2(x) y^2 + W_3(x) y^3 + ... = e^{3x}(1 + iy - \frac{y^2}{2} - \frac{iy^3}{6} + \frac{y^4}{24} + ...)$$
$$= e^{3x} e^{iy}.$$

So,

$$w(z,\bar{z}) = e^{2z+\bar{z}}.$$

Example 5. Solve the following problem

$$\frac{\partial^2 w}{\partial z \partial \bar{z}} = 4$$

with the conditions

$$w(x,0) = 9x^2$$
$$\frac{\partial w}{\partial y}(x,0) = 10ix$$

Solution. According to Theorem 7, coefficients of equation which in example are $B=1, A=C=D=E=F=0, G(z,\bar{z})=G^{*}(x,y)=4$.

According to (3.9)

$$W_0(x)=9x^2, W_1(x)=10ix$$

$$W_2(x)=4\left[\frac{4\delta(k)-\frac{1}{4}\frac{\partial^2 W_0(x)}{\partial x^2}}{2}\right]=-1, W_k(x)=0\ (k=3,4,5,\ldots)$$

Solution of problem is

$$\begin{aligned}&W_0(x)+W_1(x)y+W_2(x)y^2+W_3(x)y^3+\ldots.\\&=9x^2+10ixy-y^2\\&=9\left(\frac{z+\bar{z}}{2}\right)^2+10i\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right)-\left(\frac{z-\bar{z}}{2i}\right)^2.\end{aligned}$$

Therefore,

$$w(z,\bar{z})=4z\bar{z}+5z^2.$$

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