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DIFFERENTIAL SUBORDINATION FOR A CERTAIN GENERALIZED OPERATOR

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Abstract. The authors have recently introduced a new generalized derivative operator $\mu_{\lambda_1,\lambda_2}^{n,m}$, that generalized many well-known operators. The trend of finding new differential or integral operators has attracted widespread interest. The aim of this paper is to use the relation

$$(1+n)\,\mu_{\lambda_1,\lambda_2}^{n+1,m} f(z) = \left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' + n\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)$$

to discuss some interesting results by using the technique of differential subordination. The results include both subordination and inclusion. In the case of $n = 0, \lambda_2 = 0$, we obtain the results of Oros [11].

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1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let S, $S^*(\alpha)$, $C(\alpha)$ $(0 \le \alpha < 1)$ denote the subclasses of $\mathcal{A}$ consisting of functions that are univalent, starlike of order α and convex of order α in U, respectively. In particular, the classes $S^*(0) = S^*$ and $C(0) = C$ are the familiar classes of starlike and convex functions in U, respectively. And a function $f \in C(\alpha)$ if $\mathrm{Re}(1 + \frac{zf''}{f'}) > \alpha$. Furthermore a function f analytic in U is said to be convex if it is univalent and $f(U)$ is convex.

Let $\mathcal{H}(U)$ be the class of holomorphic functions in unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Consider

$$\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots, \ (z \in U)\}, \text{ with } \mathcal{A}_1 = \mathcal{A}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1,2,3,\dots,\}$ we let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \dots, \quad (z \in U)\}.$$

Given two functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, the Hadamard product (or convolution) $f * g$ is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k .$$

Next, we state the basic ideas on subordination. If f and g are analytic in U, then the function f is said to be subordinate to g, written as

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in U),$$

if and only if there exists the Schwarz function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in U)$.

Furthermore if g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$. [see [14], p.36].

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \quad (z \in U), \tag{1.2}$$

then p is called a solution of the differential subordination.

The univalent function q is called a dominant of the solutions of the differential subordination, or simply a dominant, if $p \prec q$ for all p satisfying (1.2).

A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.2) is said to be the best dominant of (1.2). (Note that the best dominant is unique up to a rotation of U).

Now, $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(x)_k = \begin{cases} 1 & \text{for } k = 0,\ x \in \mathbb{C}\backslash\{0\}, \\ x(x+1)(x+2)\dots(x+k-1) & \text{for } k \in \mathbb{N} = \{1,2,3,\dots\} \text{and } x \in \mathbb{C}. \end{cases}$$

In [1], the authors introduced and studied the generalized derivative operator $\mu^{n,m}_{\lambda_1,\lambda_2} f(z)$ given by the following definition.

Definition 1. For $f \in \mathcal{A}$ the generalized derivative operator $\mu^{n,m}_{\lambda_1,\lambda_2}$ is defined by $\mu^{n,m}_{\lambda_1,\lambda_2} : \mathcal{A} \to \mathcal{A}$,

$$\mu^{n,m}_{\lambda_1,\lambda_2} f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^m}{(1+\lambda_2(k-1))^{m-1}} c(n,k) a_k z^k, \quad (z \in U),$$

where $n, m \in \mathbb{N}_0 = \{0,1,2\dots\}$, $\lambda_2 \geq \lambda_1 \geq 0$ and $c(n,k) = \binom{n+k-1}{n} = \frac{(n+1)_{k-1}}{(1)_{k-1}}$.

Special cases of this operator includes the Ruscheweyh derivative operator in two cases when $\mu^{n,1}_{0,\lambda_2} \equiv R^n$ and $\mu^{n,0}_{\lambda_1,0} \equiv R^n$ [16], the Salagean derivative operator for $\mu^{0,m}_{1,0} \equiv S^n$ [17], the generalized Ruscheweyh derivative operator in the cases $\mu^{n,1}_{\lambda_1,\lambda_2} \equiv R^n_\lambda$ and $\mu^{n,0}_{\lambda_1,\lambda_2} \equiv R^n_\lambda$ [3], the generalized Salagean derivative operator introduced by Al-Oboudi $\mu^{0,m}_{\lambda_1,0} \equiv S^n_\beta$ [2], and the generalized Al-Shaqsi and Darus derivative operator $\mu^{n,m}_{\lambda_1,0} \equiv D^n_{\lambda,\beta}$ [5]. Now, let us recall the well known Carlson-Shaffer operator $L(a,c)$ [4] associated to the incomplete beta function $\phi(a,c;z)$, defined by

$$L(a,c): \mathcal{A} \to \mathcal{A},$$
$$L(a,c)f(z) := \phi(a,c;z) * f(z) \qquad (z \in U),$$
$$\text{where } \phi(a,c;z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k.$$

It can be easily seen that

$$\mu^{0,0}_{\lambda_1,0} f(z) = \mu^{1,0}_{0,\lambda_2} f(z) = f(z)$$

and

$$\mu^{1,0}_{\lambda_1,0} f(z) = \mu^{1,1}_{0,\lambda_2} f(z) = z f'(z).$$

Also $\mu^{a-1,0}_{\lambda_1,0} f(z) = \mu^{a-1,1}_{0,\lambda_2} f(z)$ where $a = 1,2,3,\dots$.

To prove our results, we need the following equality:

$$(1+n)\,\mu^{n+1,m}_{\lambda_1,\lambda_2} f(z) = z\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' + n\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right), \quad (z \in U) \tag{1.3}$$

where $n,m \in \mathbb{N}_0 = \{0,1,2\dots\}$ and $\lambda_2 \geq \lambda_1 \geq 0$.

In addition, we need the following lemmas to prove our main results:

Lemma 1 ([9],p.71)**.** *Let h be analytic, univalent, convex in U, with $h(0) = a$, $\gamma \neq 0$ and $Re\,\gamma \geq 0$. If $p \in \mathcal{H}[a,n]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z), \quad (z \in U),$$

then

$$p(z) \prec q(z) \prec h(z), \quad (z \in U),$$

where $q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t)\, t^{(\frac{\gamma}{n})-1} dt, \quad (z \in U)$.

The function q is convex and is the best (a,n)-dominant.

Lemma 2 ([8])**.** *Let g be a convex function in U and let*

$$h(z) = g(z) + n\alpha z g'(z),$$

where $\alpha > 0$ and n is a positive integer.

If

$$p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \dots, \quad (z \in U),$$

is holomorphic in U and

$$p(z)+\alpha z p'(z) \prec h(z), \quad (z \in U),$$

then

$$p(z) \prec g(z)$$

and this result is sharp.

Lemma 3 ([10])**.** *Let $f \in \mathcal{A}$, if*

$$\mathrm{Re}\left(1+\frac{z f''(z)}{f'(z)}\right) > -\frac{1}{2},$$

then

$$\frac{2}{z}\int_0^z f(t)dt, \quad (z \in U \text{ and } z \neq 0),$$

is a convex function.

In the present paper, we shall use the method of differential subordination to derive certain properties of the generalized derivative operator $\mu_{\lambda_1,\lambda_2}^{n,m} f(z)$. Note that, differential subordination has been studied by various authors, and we follow the similar work of Oros [12] and Oros and Oros [13].

2. Main results

Before we state our first theorem, we give another definition.

Definition 2. For $n,m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$ and $0 \leq \alpha < 1$, we let $R_{\lambda_1,\lambda_2}^{n,m}(\alpha)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$\mathrm{Re}\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' > \alpha, \quad (z \in U). \tag{2.1}$$

It is clear that the class $R_{\lambda_1,0}^{0,1}(\alpha) \equiv R(\lambda_1,\alpha)$ consists of functions $f \in \mathcal{A}$ satisfying

$$\mathrm{Re}(\lambda_1 z f''(z) + f'(z)) > \alpha, \quad (z \in U),$$

studied by Ponnusamy [15] and many others.

Now we begin with our first result.

Theorem 1. *Let*

$$h(z) = \frac{1+(2\alpha-1)z}{1+z}, \quad (z \in U),$$

be convex in U, with h(0)=1 and $0 \leq \alpha < 1$. If $n,m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, and the differential subordination.

$$(\mu_{\lambda_1,\lambda_2}^{n+1,m} f(z))' \prec h(z), \quad (z \in U), \tag{2.2}$$

then

$$\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' \prec q(z) = 2\alpha - 1 + \frac{2(n+1)(1-\alpha)\sigma(n)}{z^{n+1}},$$

where σ is given by

$$\sigma(x) = \int_0^z \frac{t^x}{1+t} dt, \quad (z \in U). \tag{2.3}$$

The function q is convex and is the best dominant.

Proof. By differentiating (1.3), with respect to z, we obtain

$$\left(\mu^{n+1,m}_{\lambda_1,\lambda_2} f(z)\right)' = \frac{(1+n)\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' + z\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)''}{1+n}. \tag{2.4}$$

Using (2.4) in (2.2), differential subordination (2.2) becomes

$$\begin{aligned} \frac{(1+n)\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' + z\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)''}{1+n} &\prec h(z) \\ &= \frac{1+(2\alpha-1)z}{1+z}. \end{aligned} \tag{2.5}$$

Let

$$\begin{aligned} p(z) = \left[\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right]' &= \left[z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^m}{(1+\lambda_2(k-1))^{m-1}} c(n,k) a_k z^k\right]' \\ &= 1 + p_1 z + p_2 z^2 + \ldots, \qquad (p \in \mathcal{H}[1,1],\ z \in U). \end{aligned} \tag{2.6}$$

Using (2.6) in (2.5), the differential subordination becomes:

$$p(z) + \frac{z p'(z)}{1+n} \prec h(z) = \frac{1+(2\alpha-1)z}{1+z}.$$

By using Lemma 1, we have

$$\begin{aligned} p(z) \prec q(z) =& \frac{(n+1)\int_0^z h(t)t^n dt}{z^{n+1}}, \\ =& \frac{(n+1)\int_0^z \left(\frac{1+(2\alpha-1)t}{1+t}\right) t^n dt}{z^{n+1}}, \\ =& \frac{(n+1)}{z^{n+1}}\left[\sigma(n)+(2\alpha-1)\int_0^z \frac{t^{n+1}}{1+t}dt\right], \\ =& 2\alpha-1+\frac{2(n+1)(1-\alpha)\sigma(n)}{z^{n+1}}, \end{aligned}$$

where σ is given by (2.3), so we get

$$\left[\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right]' \prec q(z) = 2\alpha-1+\frac{2(n+1)(1-\alpha)\sigma(n)}{z^{n+1}}.$$

The functions q is convex and is the best dominant. The proof is complete. □

Theorem 2. *If $n,m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$ and $0 \leq \alpha < 1$, then we have*

$$R_{\lambda_1,\lambda_2}^{n+1,m}(\alpha) \subset R_{\lambda_1,\lambda_2}^{n,m}(\delta)$$

where

$$\delta = 2\alpha-1+2(n+1)(1-\alpha)\sigma(n),$$

where σ is given by (2.3).

Proof. Let $f \in R_{\lambda_1,\lambda_2}^{n+1,m}(\alpha)$, then from (2.1) we have

$$\mathrm{Re}(\mu_{\lambda_1,\lambda_2}^{n+1,m} f(z))' > \alpha, \quad (z \in U),$$

which is equivalent to

$$(\mu_{\lambda_1,\lambda_2}^{n+1,m} f(z))' \prec h(z) = \frac{1+(2\alpha-1)z}{1+z}.$$

Using Theorem 1, we have

$$\left[\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right]' \prec q(z) = 2\alpha-1+\frac{2(n+1)(1-\alpha)\sigma(n)}{z^{n+1}}.$$

Since q is convex and $q(U)$ is symmetric with respect to the real axis, we deduce

$$\begin{aligned} \mathrm{Re}\left[\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right]' > \mathrm{Re}\, q(1) =& \delta = \delta(\alpha,\lambda_1) \\ =& 2\alpha-1+2(n+1)(1-\alpha)\sigma(n). \end{aligned}$$

From that we deduce $R^{n+1,m}_{\lambda_1,\lambda_2}(\alpha) \subset R^{n,m}_{\lambda_1,\lambda_2}(\delta)$. This completes the proof of Theorem 2. □

Theorem 3. *Let q be a convex function in U, with $q(0) = 1$ and let*

$$h(z) = q(z) + \lambda_1 z q'(z), \quad (z \in U).$$

If $n, m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, $f \in \mathcal{A}$ and it satisfies the differential subordination

$$(\mu^{n+1,m}_{\lambda_1,\lambda_2} f(z))' \prec h(z), \quad (z \in U), \tag{2.7}$$

then

$$\left[\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right]' \prec q(z), \quad (z \in U),$$

and this result is sharp.

Proof. Let

$$p(z) = \left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)'.$$

Using (2.4), the differential subordination (2.7) becomes

$$p(z) + \frac{zp'(z)}{1+n} \prec h(z) = q(z) + \lambda_1 z q'(z), \quad (z \in U).$$

Using Lemma 2, we obtain

$$p(z) \prec q(z), \quad (z \in U).$$

Hence

$$\left[\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right]' \prec q(z), \quad (z \in U).$$

The result is sharp. This completes the proof of the theorem. □

We give a simple application for Theorem 3.

Example 1. For $n = 1$, $m = 0$, $\lambda_2 \geq \lambda_1 \geq 0$, $q(z) = \frac{1+z}{1-z}$, $f \in \mathcal{A}$ and $z \in U$ and applying Theorem 3, we have

$$h(z) = \frac{1+z}{1-z} + \lambda_1 z \left(\frac{1+z}{1-z}\right)' = \frac{1+2\lambda_1 z - z^2}{(1-z)^2}.$$

By using (2.4) we find

$$\begin{aligned}\left(\mu^{1,0}_{\lambda_1,\lambda_2}f(z)\right)' &= \left(\mu^{0,0}_{\lambda_1,\lambda_2}f(z)\right)' + z\left(\mu^{0,0}_{\lambda_1,\lambda_2}f(z)\right)'',\\ &=1+\sum_{k=2}^{\infty}(1+\lambda_2(k-1))ka_kz^{k-1}\\ &\quad+\sum_{k=2}^{\infty}(1+\lambda_2(k-1))k(k-1)a_kz^{k-1},\\ &=1+\sum_{k=2}^{\infty}(1+\lambda_2(k-1))k^2a_kz^{k-1},\\ &=\frac{f(z)*\left[z+\sum_{k=2}^{\infty}(1+\lambda_2(k-1))k^2z^k\right]}{z}.\end{aligned} \tag{2.8}$$

Similarly we compute $\left(\mu^{2,0}_{\lambda_1,\lambda_2}f(z)\right)'$. By using (2.4), we find

$$\left(\mu^{2,0}_{\lambda_1,\lambda_2}f(z)\right)' = \left(\mu^{1,0}_{\lambda_1,\lambda_2}f(z)\right)' + \frac{z}{2}\left(\mu^{1,0}_{\lambda_1,\lambda_2}f(z)\right)''. \tag{2.9}$$

Then, by using (2.8) we have

$$\left(\mu^{1,0}_{\lambda_1,\lambda_2}f(z)\right)'' = \sum_{k=2}^{\infty}(1+\lambda_2(k-1))k^2(k-1)a_k z^{k-2}. \tag{2.10}$$

After that, by (2.8) and (2.10), (2.9) becomes

$$\begin{aligned}\left(\mu^{2,0}_{\lambda_1,\lambda_2}f(z)\right)' &=1+\sum_{k=2}^{\infty}(1+\lambda_2(k-1))k^2a_kz^{k-1}\\ &\quad+\frac{1}{2}\sum_{k=2}^{\infty}(1+\lambda_2(k-1))k^2(k-1)a_k z^{k-1},\\ &=1+\sum_{k=2}^{\infty}(1+\lambda_2(k-1))k^2\frac{(k+1)}{2}a_k z^{k-1},\\ &=\frac{f(z)*\left[z+\sum_{k=2}^{\infty}\frac{1}{2}(1+\lambda_2(k-1))(1+k)k^2z^k\right]}{z}.\end{aligned}$$

From Theorem 3 we deduce that

$$\frac{f(z)*\left[z+\sum_{k=2}^{\infty}\frac{1}{2}\left(1+\lambda_2\left(k-1\right)\right)\left(1+k\right)k^2z^k\right]}{z}\prec\frac{1+2\lambda_1z-z^2}{(1-z)^2}$$

implies

$$\frac{f(z)*\left[z+\sum_{k=2}^{\infty}k^2\left(1+\lambda_2(k-1)\right)z^k\right]}{z}\prec\frac{1+z}{1-z},\quad ,(z\in U).$$

Theorem 4. *Let q be a convex function in U, with $q(0)=1$ and let*

$$h(z)=q(z)+zq'(z),\quad (z\in U).$$

If $n,m\in\mathbb{N}_0$, $\lambda_2\geq\lambda_1\geq 0$, $f\in\mathcal{A}$ and satisfies the differential subordination

$$(\mu^{n,m}_{\lambda_1,\lambda_2}f(z))'\prec h(z), \tag{2.11}$$

then

$$\frac{\mu^{n,m}_{\lambda_1,\lambda_2}f(z)}{z}\prec q(z),\quad ,(z\in U).$$

The result is sharp.

Proof.

$$\begin{aligned} p(z)=&\frac{\mu^{n,m}_{\lambda_1,\lambda_2}f(z)}{z}, \\ =&\frac{z+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1))^m}{(1+\lambda_2(k-1))^{m-1}}c(n,k)a_kz^k}{z}, \\ =&1+p_1z+p_2z^2+\dots.,\qquad (p\in\mathcal{H}[1,1],\ z\in U). \end{aligned} \tag{2.12}$$

Differentiating (2.12), with respect to z, we obtain

$$\left(\mu^{n,m}_{\lambda_1,\lambda_2}f(z)\right)'=p(z)+zp'(z),\quad (z\in U). \tag{2.13}$$

Using (2.13), the differential subordination (2.11) becomes

$$p(z)+zp'(z)\prec h(z)=q(z)+zq'(z),$$

and by using Lemma 2, we deduce

$$p(z)\prec q(z),\quad (z\in U).$$

Next using (2.12), we have

$$\frac{\mu^{n,m}_{\lambda_1,\lambda_2}f(z)}{z}\prec q(z),\quad ,(z\in U).$$

This proves Theorem 4. □

We give a simple application of Theorem 4.

Example 2. For $n=1, m=0$, $\lambda_2 \geq \lambda_1 \geq 0$, $q(z)=\frac{1}{1-z}$, $f \in \mathcal{A}$ and $z \in U$, by using Theorem 4, we obtain

$$h(z)=\frac{1}{1-z}+z\left(\frac{1}{1-z}\right)'=\frac{1}{(1-z)^2}.$$

From (1.3), we have

$$\begin{aligned}\left(\mu^{1,0}_{\lambda_1,\lambda_2}f(z)\right) &= z\left(\mu^{0,0}_{\lambda_1,\lambda_2}f(z)\right)',\\ &= z+\sum_{k=2}^{\infty}(1+\lambda_2(k-1))\,k a_k z^k,\\ &= f(z)*\left[z+\sum_{k=2}^{\infty}(1+\lambda_2(k-1))\,k z^k\right].\end{aligned}$$

From example 1, we have

$$\left(\mu^{1,0}_{\lambda_1,\lambda_2}f(z)\right)'=\frac{f(z)*\left[z+\sum_{k=2}^{\infty}(1+\lambda_2(k-1))\,k^2 z^k\right]}{z}.$$

Now, applying Theorem 4, we deduce that

$$\frac{f(z)*\left[z+\sum_{k=2}^{\infty}(1+\lambda_2(k-1))\,k^2 z^k\right]}{z}\prec\frac{1}{(1-z)^2}$$

implies

$$\frac{f(z)*\left[z+\sum_{k=2}^{\infty}(1+\lambda_2(k-1))\,k z^k\right]}{z}\prec\frac{1}{1-z}.$$

Theorem 5. *Let*

$$h(z)=\frac{1+(2\alpha-1)z}{1+z},\quad (z\in U)$$

be convex in U, with $h(0)=1$ and $0\leq\alpha<1$. If $n,m\in\mathbb{N}_0$, $\lambda_2\geq\lambda_1\geq 0$, $f\in\mathcal{A}$ and the differential subordination

$$(\mu^{n,m}_{\lambda_1,\lambda_2}f(z))'\prec h(z) \tag{2.14}$$

is satisfied, then

$$\frac{\mu^{n,m}_{\lambda_1,\lambda_2}f(z)}{z} \prec q(z) = 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}.$$

The function q is convex and is the best dominant.

Proof. Let

$$\begin{aligned} p(z) =& \frac{\mu^{n,m}_{\lambda_1,\lambda_2}f(z)}{z}, \qquad (2.15)\\ =& \frac{z + \sum\limits_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^m}{(1+\lambda_2(k-1))^{m-1}} c(n,k) a_k z^k}{z},\\ =& 1 + p_1 z + p_2 z^2 + \dots, \qquad (p \in \mathcal{H}[1,1],\ z \in U). \end{aligned}$$

Differentiating (2.15), with respect to z, we obtain

$$\left(\mu^{n,m}_{\lambda_1,\lambda_2}f(z)\right)' = p(z) + z p'(z), \quad (z \in U). \qquad (2.16)$$

Using (2.16), the differential subordination (2.14) becomes

$$p(z) + z p'(z) \prec h(z) = \frac{1+(2\alpha-1)z}{1+z}, \quad (z \in U).$$

From Lemma 1, we deduce

$$\begin{aligned} p(z) \prec q(z) =& \frac{1}{z}\int_0^z h(t)\,dt,\\ =& \frac{1}{z}\int_0^z \left(\frac{1+(2\alpha-1)t}{1+t}\right) dt,\\ =& \frac{1}{z}\left[\int_0^z \frac{1}{1+t}dt + (2\alpha-1)\int_0^z \frac{t}{1+t}dt\right],\\ =& 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}. \end{aligned}$$

Using (2.15), we have

$$\frac{\mu^{n,m}_{\lambda_1,\lambda_2}f(z)}{z} \prec q(z) = 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}.$$

The proof is complete. □

From Theorem 5, we deduce the following Corollary:

Corollary 1. *If $f \in R^{n,m}_{\lambda_1,\lambda_2}(\alpha)$, then*

$$\operatorname{Re}\left(\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z}\right) > (2\alpha-1)+2(1-\alpha)\ln 2, \quad (z \in U).$$

Proof. Since $f \in R^{n,m}_{\lambda_1,\lambda_2}(\alpha)$, and from Definition 2 we have

$$\operatorname{Re}\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' > \alpha, \quad (z \in U),$$

which is equivalent to

$$(\mu^{n,m}_{\lambda_1,\lambda_2} f(z))' \prec h(z) = \frac{1+(2\alpha-1)z}{1+z}.$$

Using Theorem 5, we have

$$\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \prec q(z) = (2\alpha-1)+2(1-\alpha)\frac{\ln(1+z)}{z}.$$

Since q is convex and $q(U)$ is symmetric with respect to the real axis, we deduce

$$\operatorname{Re}\left(\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z}\right) > \operatorname{Re} q(1) = (2\alpha-1)+2(1-\alpha)\ln 2, \quad (z \in U).$$

□

Theorem 6. *Let $h \in \mathcal{H}(U)$, with $h(0)=1$, $h'(0) \neq 0$ and assume that it satisfies the inequality*

$$\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}, \quad (z \in U).$$

If $n,m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, $f \in \mathcal{A}$ and it satisfies the differential subordination

$$\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' \prec h(z), , \quad (z \in U), \tag{2.17}$$

then

$$\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \prec q(z) = \frac{1}{z}\int_0^z h(t)dt.$$

Proof. Let

$$\begin{aligned} p(z) =& \frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z}, \\ =& \frac{z+\sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^m}{(1+\lambda_2(k-1))^{m-1}} c(n,k) a_k z^k}{z}, \\ =& 1+p_1 z+p_2 z^2+\ldots, \quad (p \in \mathcal{H}[1,1],\ z \in U). \end{aligned} \tag{2.18}$$

Differentiating (2.18), with respect to z, we have

$$\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' = p(z) + zp'(z), \quad (z \in U). \tag{2.19}$$

Using (2.19), the differential subordination (2.17) becomes

$$p(z) + zp'(z) \prec h(z), \quad (z \in U).$$

From Lemma 1, we deduce

$$p(z) \prec q(z) = \frac{1}{z}\int_0^z h(t)\,dt.$$

With (2.18), we obtain

$$\frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z} \prec q(z) = \frac{1}{z}\int_0^z h(t)dt.$$

From Lemma 3, we obtain that the function q is convex, and from Lemma 1, q is the best dominant for the subordination (2.17). This completes the proof of Theorem 6. □

3. Conclusion

We remark that several subclasses of analytic univalent functions can be derived using the operator $\mu_{\lambda_1,\lambda_2}^{n,m}$. Several of their properties can be studied with this method, for example properties related to the ones that were studied in [7] and [6].

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References

[1] M. H. Al-Abbadi and M. Darus, “Differential subordination defined by new generalised derivative operator for analytic functions,” *Int. J. Math. Math. Sci.*, vol. 2010, p. 15, 2010.

[2] F. M. Al-Oboudi, “On univalent functions defined by a generalized Sălăgean operator,” *Int. J. Math. Math. Sci.*, vol. 2004, no. 25-28, pp. 1429–1436, 2004.

[3] K. Al-Shaqsi and M. Darus, “On univalent functions with respect to k-symmetric points defined by a generalized Ruscheweyh derivatives operator,” *J. Anal. Appl.*, vol. 7, no. 1, pp. 53–61, 2009.

[4] B. Carlson and D. B. Shaffer, “Starlike and prestarlike hypergeometric functions,” *SIAM J. Math. Anal.*, vol. 15, pp. 737–745, 1984.

[5] M. Darus and K. Al-Shaqsi, “Differential sandwich theorems with generalised derivative operator,” *Int. J. Comput. Math. Sci.*, vol. 2, no. 2, pp. 75–78, 2008.

[6] M. Darus and I. Faisal, “A study of Pescar’s univalence criteria for space of analytic functions,” *Journal of Inequalities and Applications*, vol. 2011, no. 109, p. 7, 2011.

[7] M. Darus and I. Faisal, "A study on Becker's univalence criteria," *Abstr. Appl. Anal.*, vol. 2011, p. 13, 2011.
[8] S. S. Miller and P. T. Mocanu, "On some classes of first-order differential subordinations," *Mich. Math. J.*, vol. 32, pp. 185–195, 1985.
[9] S. S. Miller and P. T. Mocanu, *Differential subordinations: theory and applications*, ser. Series on monographs and textbooks in pure and applied mathematics. New York: Marcel Dekker, 2000, vol. 225.
[10] P. T. Mocanu, T. Bulboaca, and G. S. Salagean, *Teoria geometrica a functiilor univalente*. Cluj-Napoca: Casa Cartii de Stiinta, 1999.
[11] G. I. Oros, "On a class of holomorphic functions defined by the Ruscheweyh derivative," *Int. J. Math. Math. Sci.*, vol. 2003, no. 65, pp. 4139–4144, 2003.
[12] G. I. Oros, "A class of holomorphic functions defined using a differential operator," *Gen. Math.*, vol. 13, no. 4, pp. 13–18, 2005.
[13] G. Oros and O. G. Irina, "Differential superordination defined by Sălăgean operator," *Gen. Math.*, vol. 12, no. 4, pp. 3–10, 2004.
[14] K. S. Padmanabhan and R. Manjini, "Certain applications of differential subordination," *Publ. Inst. Math., Nouv. Sér.*, vol. 39, no. 53, pp. 107–118, 1986.
[15] S. Ponnusamy, "Differential subordination and starlike functions," *Complex Variables, Theory Appl.*, vol. 19, no. 3, pp. 185–194, 1992.
[16] S. Ruscheweyh, "New criteria for univalent functions," *Proc. Am. Math. Soc.*, vol. 49, pp. 109–115, 1975.
[17] G. S. Sălăgean, "Subclasses of univalent functions," in *Complex analysis - Proc. 5th Rom.-Finn. Semin., Bucharest 1981*, ser. Lect. Notes Math., vol. 1013. Springer-Verlag, 1983, pp. 362–372.

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