Differential subordination for certain generalized operator

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DIFFERENTIAL SUBORDINATION FOR A CERTAIN GENERALIZED OPERATOR

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Abstract. The authors have recently introduced a new generalized derivative operator $\mu_{\lambda_1,\lambda_2}^{n,m}$, that generalized many well-known operators. The trend of finding new differential or integral operators has attracted widespread interest. The aim of this paper is to use the relation

$$(1 + n)\mu_{\lambda_1,\lambda_2}^{n+1,m} f(z) = \left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' + n\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)$$

to discuss some interesting results by using the technique of differential subordination. The results include both subordination and inclusion. In the case of $n = 0, \lambda_2 = 0$, we obtain the results of Oros [11].

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1. INTRODUCTION AND DEFINITIONS

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let $S, S^*(\alpha), C(\alpha) (0 \leq \alpha < 1)$ denote the subclasses of $A$ consisting of functions that are univalent, starlike of order $\alpha$ and convex of order $\alpha$ in $U$, respectively. In particular, the classes $S^*(0) = S^*$ and $C(0) = C$ are the familiar classes of starlike and convex functions in $U$, respectively. And a function $f \in C(\alpha)$ if $\Re(1 + \frac{zf''}{f'}) > \alpha$. Furthermore a function $f$ analytic in $U$ is said to be convex if it is univalent and $f(U)$ is convex.

Let $\mathcal{H}(U)$ be the class of holomorphic functions in unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Consider

$$\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \ldots, (z \in U)\}, \text{ with } \mathcal{A}_1 = \mathcal{A}.$$
For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, 3, \ldots \}$ we let

$$H[a, n] = \{ f \in H(U) : f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \ldots, \ (z \in U) \}. $$

Given two functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ analytic in the unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$, the Hadamard product (or convolution) $f \ast g$ is defined by

$$f(z) \ast g(z) = (f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

Next, we state the basic ideas on subordination. If $f$ and $g$ are analytic in $U$, then the function $f$ is said to be subordinate to $g$, written as

$$f < g \quad \text{or} \quad f(z) < g(z) \quad (z \in U),$$

if and only if there exists the Schwarz function $w$, analytic in $U$, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z)) \ (z \in U)$.

Furthermore if $g$ is univalent in $U$, then $f < g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$, [see [14], p.36].

Let $\psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ and let $h$ be univalent in $U$. If $p$ is analytic in $U$ and satisfies the (second-order) differential subordination

$$\psi (p(z), z p'(z), z^2 p''(z); z) < h(z), \quad (z \in U), \quad (1.2)$$

then $p$ is called a solution of the differential subordination.

The univalent function $q$ is called a dominant of the solutions of the differential subordination, or simply a dominant, if $p < q$ for all $p$ satisfying (1.2).

A dominant $\tilde{q}$ that satisfies $\tilde{q} < q$ for all dominants $q$ of (1.2) is said to be the best dominant of (1.2). (Note that the best dominant is unique up to a rotation of $U$).

Now, $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(x)_k = \begin{cases} 1 & \text{for } k = 0, x \in \mathbb{C}\setminus\{0\}, \\ x(x+1)(x+2)\ldots(x+k-1) & \text{for } k \in \mathbb{N} = \{1, 2, 3, \ldots \} \text{ and } x \in \mathbb{C}. \end{cases}$$

In [1], the authors introduced and studied the generalized derivative operator $\mu^{n,m}_{\lambda_1, \lambda_2} f(z)$ given by the following definition.

**Definition 1.** For $f \in A$ the generalized derivative operator $\mu^{n,m}_{\lambda_1, \lambda_2}$ is defined by

$$\mu^{n,m}_{\lambda_1, \lambda_2} f(z) = z + \sum_{k=2}^{\infty} \frac{(1 + \lambda_1 (k-1))^m}{(1 + \lambda_2 (k-1))^{m-1}} c(n,k) a_k z^k, \quad (z \in U),$$

where $n, m \in \mathbb{N}_0 = \{0, 1, 2, \ldots \}$, $\lambda_2 \geq \lambda_1 \geq 0$ and $c(n,k) = \binom{n+k-1}{n} = \frac{(n+1)!}{k! (n-k-1)!}$. 

Special cases of this operator includes the Ruscheweyh derivative operator in two cases when $\mu_{\lambda_1,0}^{n,0} \equiv R^n_\lambda$ and $\mu_{\lambda_1,0}^{n,1} \equiv R^n_\lambda$ [16], the Salagean derivative operator for $\mu_{\lambda_1,0}^{0,m} \equiv S^n_\lambda$ [17], the generalized Ruscheweyh derivative operator in the cases $\mu_{\lambda_1,0}^{n,1} \equiv R^n_\lambda$ and $\mu_{\lambda_1,0}^{n,0} \equiv R^n_\lambda$ [3], the generalized Salagean derivative operator introduced by Al-Oboudi $\mu_{\lambda_1,0}^{m,0} \equiv S^n_\beta$ [2], and the generalized Al-Shaqsi and Darus derivative operator $\mu_{\lambda_1,0}^{n,m} \equiv D^n_{\lambda_1,\beta}$ [5]. Now, let us recall the well known Carlson-Shaffer operator $L(a,c)$ [4] associated to the incomplete beta function $I_{\alpha,\beta}$, defined by

$$L(a,c): \mathcal{A} \to \mathcal{A},$$

$$L(a,c)f(z) := \phi(a,c;z) * f(z) \quad (z \in U),$$

where $\phi(a,c;z) = z + \sum_{k=2}^{\infty} \binom{a}{k-1} \frac{z^k}{c^{k-1}}$.

It can be easily seen that

$$\mu_{\lambda_1,0}^{0,0} f(z) = \mu_{0,\lambda_1}^{1,0} f(z) = f(z)$$

and

$$\mu_{\lambda_1,0}^{1,0} f(z) = \mu_{0,\lambda_1}^{1,1} f(z) = z f'(z).$$

Also $\mu_{\lambda_1,0}^{a-1,0} f(z) = \mu_{0,\lambda_1}^{a-1,1} f(z)$ where $a = 1, 2, 3, \ldots$.

To prove our results, we need the following equality:

$$(1 + n) \mu_{\lambda_1,\lambda_2}^{n+1,m} f(z) = z \left( \mu_{\lambda_1,\lambda_2}^{n,m} f(z) \right)' + n \left( \mu_{\lambda_1,\lambda_2}^{n,m} f(z) \right), \quad (z \in U) \quad (1.3)$$

where $n, m \in \mathbb{N}_0 = \{0, 1, 2, \ldots \}$ and $\lambda_2 \geq \lambda_1 \geq 0$.

In addition, we need the following lemmas to prove our main results:

**Lemma 1** ([9], p.71). Let $h$ be analytic, univalent, convex in $U$, with $h(0) = a$, $\gamma \neq 0$ and $\Re \gamma \geq 0$. If $p \in \mathcal{H}[a,n]$ and

$$p(z) + \frac{zp'(z)}{\gamma} < h(z), \quad (z \in U),$$

then

$$p(z) < q(z) < h(z), \quad (z \in U),$$

where $q(z) = \frac{\gamma}{n \pi} \int_0^\frac{\pi}{n} h(t) t'(\frac{z}{\gamma})^{-1} dt, \quad (z \in U)$.

The function $q$ is convex and is the best $(a,n)$-dominant.

**Lemma 2** ([8]). Let $g$ be a convex function in $U$ and let

$$h(z) = g(z) + naz g'(z),$$

where $\alpha > 0$ and $n$ is a positive integer.

If

$$p(z) = g(0) + pn z^n + pn+1 z^{n+1} + \ldots, \quad (z \in U),$$

then

$$p(z) < q(z) < h(z), \quad (z \in U),$$

where $q(z) = \frac{\gamma}{n \pi} \int_0^\frac{\pi}{n} h(t) t'(\frac{z}{\gamma})^{-1} dt, \quad (z \in U)$.
is holomorphic in $U$ and
\[ p(z) + \alpha z p'(z) < h(z), \quad (z \in U), \]
then
\[ p(z) < g(z) \]
and this result is sharp.

**Lemma 3** ([10]). Let $f \in \mathcal{A}$, if
\[ \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \frac{1}{2}, \]
then
\[ \frac{2}{z} \int_0^z f(t) \, dt, \quad (z \in U \text{ and } z \neq 0), \]
is a convex function.

In the present paper, we shall use the method of differential subordination to derive certain properties of the generalized derivative operator $\mu_{\lambda_1,\lambda_2}^{n,m} f(z)$. Note that, differential subordination has been studied by various authors, and we follow the similar work of Oros [12] and Oros and Oros [13].

## 2. Main Results

Before we state our first theorem, we give another definition.

**Definition 2.** For $n, m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$ and $0 \leq \alpha < 1$, we let $R_{\lambda_1,\lambda_2}^{n,m} (\alpha)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the condition
\[ \Re \left( \mu_{\lambda_1,\lambda_2}^{n,m} f(z) \right)' > \alpha, \quad (z \in U). \]  
(2.1)

It is clear that the class $R_{\lambda_1,0}^{0,1} (\alpha) \equiv R(\lambda_1, \alpha)$ consists of functions $f \in \mathcal{A}$ satisfying
\[ \Re(\lambda_1 z f''(z) + f'(z)) > \alpha, \quad (z \in U), \]
studied by Ponnusamy [15] and many others.

Now we begin with our first result.

**Theorem 1.** Let
\[ h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}, \quad (z \in U), \]
be convex in $U$, with $h(0)=1$ and $0 \leq \alpha < 1$. If $n, m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, and the differential subordination.
\[ (\mu_{\lambda_1,\lambda_2}^{n+1,m} f(z))' < h(z), \quad (z \in U), \]  
(2.2)
then
\[
\left( \mu_{\lambda_1, \lambda_2}^{n,m} f(z) \right)' < q(z) = 2\alpha - 1 + \frac{2(n+1)(1-\alpha)\sigma(n)}{z^{n+1}},
\]
where \( \sigma \) is given by
\[
\sigma(x) = \int_0^x \frac{t^x}{1+t} \, dt, \quad (z \in U).
\] (2.3)
The function \( q \) is convex and is the best dominant.

**Proof.** By differentiating (1.3), with respect to \( t \), we obtain
\[
\left( \mu_{\lambda_1, \lambda_2}^{n+1,m} f(z) \right)' = \frac{(1+n)\left( \mu_{\lambda_1, \lambda_2}^{n,m} f(z) \right)' + z \left( \mu_{\lambda_1, \lambda_2}^{n,m} f(z) \right)''}{1+n}.
\] (2.4)
Using (2.4) in (2.2), differential subordination (2.2) becomes
\[
\frac{(1+n)\left( \mu_{\lambda_1, \lambda_2}^{n,m} f(z) \right)' + z \left( \mu_{\lambda_1, \lambda_2}^{n,m} f(z) \right)''}{1+n} < h(z)
\]
\[= 1 + (2\alpha - 1)z \]
\[= \frac{1 + (2\alpha - 1)z}{1 + z}. \] (2.5)
Let
\[
p(z) = \left[ \mu_{\lambda_1, \lambda_2}^{n,m} f(z) \right]' = \left[ z + \sum_{k=2}^\infty \frac{(1+\lambda_1(k-1))^{m}}{(1+\lambda_2(k-1))^{m-1}} c(n,k)a_k z^k \right]'
\]
\[= 1 + p_1 z + p_2 z^2 + \ldots, \quad (p \in \mathbb{H}[1,1], \ z \in U). \] (2.6)
Using (2.6) in (2.5), the differential subordination becomes:
\[
p(z) + \frac{zp'(z)}{1+n} < h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}. \]
By using Lemma 1, we have

\[
p(z) < q(z) = \frac{(n+1) \int_0^z t^n dt}{z^{n+1}},
\]

\[
= \frac{(n+1) \int_0^z \left( \frac{1+(2\alpha-1)t}{1+t} \right) t^n dt}{z^{n+1}},
\]

\[
= \frac{(n+1)}{z^{n+1}} \left[ \sigma(n) + (2\alpha-1) \int_0^z \frac{t^{n+1}}{1+t} dt \right],
\]

\[
= 2\alpha - 1 + \frac{2(n+1)(1-\alpha)\sigma(n)}{z^{n+1}},
\]

where \( \sigma \) is given by (2.3), so we get

\[
\left[ \mu_{\lambda_1,\lambda_2}^{n,m} f(z) \right]' < q(z) = 2\alpha - 1 + \frac{2(n+1)(1-\alpha)\sigma(n)}{z^{n+1}}.
\]

The functions \( q \) is convex and is the best dominant. The proof is complete. \( \square \)

**Theorem 2.** If \( n, m \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0 \) and \( 0 \leq \alpha < 1 \), then we have

\[
R_{\lambda_1,\lambda_2}^{n+1,m}(\alpha) \subset R_{\lambda_1,\lambda_2}^{n,m}(\delta)
\]

where

\[
\delta = 2\alpha - 1 + 2(n+1)(1-\alpha)\sigma(n),
\]

where \( \sigma \) is given by (2.3).

**Proof.** Let \( f \in R_{\lambda_1,\lambda_2}^{n+1,m}(\alpha) \), then from (2.1) we have

\[
\text{Re}(\mu_{\lambda_1,\lambda_2}^{n+1,m} f(z))' > \alpha, \quad (z \in U),
\]

which is equivalent to

\[
(\mu_{\lambda_1,\lambda_2}^{n+1,m} f(z))' < h(z) = \frac{1 + (2\alpha - 1)z}{1+z}.
\]

Using Theorem 1, we have

\[
\left[ \mu_{\lambda_1,\lambda_2}^{n,m} f(z) \right]' < q(z) = 2\alpha - 1 + \frac{2(n+1)(1-\alpha)\sigma(n)}{z^{n+1}}.
\]

Since \( q \) is convex and \( q(U) \) is symmetric with respect to the real axis, we deduce

\[
\text{Re} \left[ \mu_{\lambda_1,\lambda_2}^{n,m} f(z) \right]' > \text{Re} q(1) = \delta = \delta(\alpha, \lambda_1)
\]

\[
= 2\alpha - 1 + 2(n+1)(1-\alpha)\sigma(n).
\]
From that we deduce $R^{n+1,m}_{\lambda_1,\lambda_2}(\alpha) \subset R^{n,m}_{\lambda_1,\lambda_2}(\delta)$. This completes the proof of Theorem 2.

**Theorem 3.** Let $q$ be a convex function in $U$, with $q(0) = 1$ and let

$$h(z) = q(z) + \lambda_1 z q'(z), \quad (z \in U).$$

If $n, m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, $f \in \mathcal{A}$ and it satisfies the differential subordination

$$(\mu^{n+1,m}_{\lambda_1,\lambda_2} f(z))' < h(z), \quad (z \in U),$$

then

$$\left[ \mu^{n,m}_{\lambda_1,\lambda_2} f(z) \right]' < q(z), \quad (z \in U),$$

and this result is sharp.

**Proof.** Let

$$p(z) = \left( \mu^{n,m}_{\lambda_1,\lambda_2} f(z) \right)'.'$$

Using (2.4), the differential subordination (2.7) becomes

$$p(z) + \frac{z p'(z)}{1+n} < h(z) = q(z) + \lambda_1 z q'(z), \quad (z \in U).$$

Using Lemma 2, we obtain

$$p(z) < q(z), \quad (z \in U).$$

Hence

$$\left[ \mu^{n,m}_{\lambda_1,\lambda_2} f(z) \right]' < q(z), \quad (z \in U).$$

The result is sharp. This completes the proof of the theorem.

We give a simple application for Theorem 3.

**Example 1.** For $n = 1$, $m = 0$, $\lambda_2 \geq \lambda_1 \geq 0$, $q(z) = \frac{1+z}{1-z}$, $f \in \mathcal{A}$ and $z \in U$ and applying Theorem 3, we have

$$h(z) = \frac{1+z}{1-z} + \lambda_1 z \left( \frac{1+z}{1-z} \right)' = \frac{1+2\lambda_1 z - z^2}{(1-z)^2}. $$
By using (2.4) we find
\[
\left( \mu^{1,0}_{\lambda_1,\lambda_2} f(z) \right)' = \left( \mu^{0,0}_{\lambda_1,\lambda_2} f(z) \right)' + z \left( \mu^{0,0}_{\lambda_1,\lambda_2} f(z) \right)'',
\]
\[
= 1 + \sum_{k=2}^{\infty} (1 + \lambda_2 (k - 1)) k a_k z^{k-1}
\]
\[
+ \sum_{k=2}^{\infty} (1 + \lambda_2 (k - 1)) k (k - 1) a_k z^{k-1},
\]
(2.8)
\[
= 1 + \sum_{k=2}^{\infty} (1 + \lambda_2 (k - 1)) k^2 a_k z^{k-1},
\]
\[
f(z) * \left[ z + \sum_{k=2}^{\infty} (1 + \lambda_2 (k - 1)) k^2 z^{k-1} \right]_{z}.
\]

Similarly we compute \( \left( \mu^{2,0}_{\lambda_1,\lambda_2} f(z) \right)' \). By using (2.4), we find
\[
\left( \mu^{2,0}_{\lambda_1,\lambda_2} f(z) \right)' = \left( \mu^{1,0}_{\lambda_1,\lambda_2} f(z) \right)' + \frac{z}{2} \left( \mu^{1,0}_{\lambda_1,\lambda_2} f(z) \right)'',
\]
(2.9)
Then, by using (2.8) we have
\[
\left( \mu^{1,0}_{\lambda_1,\lambda_2} f(z) \right)'' = \sum_{k=2}^{\infty} (1 + \lambda_2 (k - 1)) k^2 (k - 1) a_k z^{k-2}.
\]
(2.10)

After that, by (2.8) and (2.10), (2.9) becomes
\[
\left( \mu^{2,0}_{\lambda_1,\lambda_2} f(z) \right)' = 1 + \sum_{k=2}^{\infty} (1 + \lambda_2 (k - 1)) k^2 a_k z^{k-1}
\]
\[
+ \frac{1}{2} \sum_{k=2}^{\infty} (1 + \lambda_2 (k - 1)) k^2 (k - 1) a_k z^{k-1},
\]
\[
= 1 + \sum_{k=2}^{\infty} (1 + \lambda_2 (k - 1)) k^2 \frac{(k + 1)}{2} a_k z^{k-1},
\]
\[
f(z) * \left[ z + \sum_{k=2}^{\infty} \frac{1}{2} (1 + \lambda_2 (k - 1)) (1 + k) k^2 z^{k-1} \right]_{z}.
\]
From Theorem 3 we deduce that
\[
    f(z) \left[ z + \sum_{k=2}^{\infty} \frac{1}{2} (1 + \lambda_2 (k-1)) (1 + k) k^2 z^k \right] < \frac{1 + 2\lambda_1 z - z^2}{(1-z)^2}
\]
implies
\[
    f(z) \left[ z + \sum_{k=2}^{\infty} k^2 (1 + \lambda_2 (k-1)) z^k \right] < \frac{1+z}{1-z}, \quad (z \in U).
\]

**Theorem 4.** Let \( q \) be a convex function in \( U \), with \( q(0) = 1 \) and let
\[
    h(z) = q(z) + zq'(z), \quad (z \in U).
\]
If \( n, m \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0, f \in \mathcal{A} \) and satisfies the differential subordination
\[
    \left( \mu_{n,m}^{\lambda_1,\lambda_2} f(z) \right)' < h(z), \quad (2.11)
\]
then
\[
    \frac{\mu_{n,m}^{\lambda_1,\lambda_2} f(z)}{z} < q(z), \quad (z \in U).
\]
The result is sharp.

**Proof.**
\[
p(z) = \frac{\mu_{n,m}^{\lambda_1,\lambda_2} f(z)}{z}, \quad (2.12)
\]
\[
= 1 + p_1 z + p_2 z^2 + \ldots, \quad (p \in \mathcal{H}[1,1], z \in U).
\]
Differentiating (2.12), with respect to \( z \), we obtain
\[
    \left( \mu_{n,m}^{\lambda_1,\lambda_2} f(z) \right)' = p(z) + z p'(z), \quad (z \in U). \quad (2.13)
\]
Using (2.13), the differential subordination (2.11) becomes
\[
p(z) + z p'(z) < q(z) + z q'(z),
\]
and by using Lemma 2, we deduce
\[
p(z) < q(z), \quad (z \in U).
\]
Next using (2.12), we have
\[
    \frac{\mu_{n,m}^{\lambda_1,\lambda_2} f(z)}{z} < q(z), \quad (z \in U).
\]
This proves Theorem 4. □

We give a simple application of Theorem 4.

Example 2. For \( n = 1, m = 0, \lambda_2 \geq \lambda_1 \geq 0, q(z) = \frac{1}{1-z} \), \( f \in A \) and \( z \in U \), by using Theorem 4, we obtain

\[
h(z) = \frac{1}{1-z} + z \left( \frac{1}{1-z} \right)' = \frac{1}{(1-z)^2}.
\]

From (1.3), we have

\[
\left( \mu_{\lambda_1, \lambda_2}^{1,0} f(z) \right)' = z \left( \mu_{\lambda_1, \lambda_2}^{0,0} f(z) \right)' = z + \sum_{k=2}^{\infty} (1 + \lambda_2 (k-1)) ka_k z^k,
\]

\[
= f(z) * \left[ z + \sum_{k=2}^{\infty} (1 + \lambda_2 (k-1)) k z^k \right].
\]

From example 1, we have

\[
\left( \mu_{\lambda_1, \lambda_2}^{1,0} f(z) \right)' = \frac{f(z) * \left[ z + \sum_{k=2}^{\infty} (1 + \lambda_2 (k-1)) k^2 z^k \right]}{z}.
\]

Now, applying Theorem 4, we deduce that

\[
f(z) * \frac{\left[ z + \sum_{k=2}^{\infty} (1 + \lambda_2 (k-1)) k^2 z^k \right]}{z} < \frac{1}{(1-z)^2}
\]

implies

\[
f(z) * \frac{\left[ z + \sum_{k=2}^{\infty} (1 + \lambda_2 (k-1)) k z^k \right]}{z} < \frac{1}{1-z}.
\]

**Theorem 5.** Let

\[
h(z) = \frac{1 + (2\alpha - 1) z}{1 + z}, \quad (z \in U)
\]

be convex in \( U \), with \( h(0) = 1 \) and \( 0 \leq \alpha < 1 \). If \( n, m \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0, f \in A \) and the differential subordination

\[
(\mu_{\lambda_1, \lambda_2}^{n,m} f(z))' < h(z)
\]

(2.14)
is satisfied, then
\[ \frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} < q(z) = 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}. \]

The function \( q \) is convex and is the best dominant.

**Proof.** Let
\[ p(z) = \frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z}, \]
where
\[ z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^m}{(1+\lambda_2(k-1))^m} c(n,k)a_k z^k = 1 + p_1 z + p_2 z^2 + \ldots, \quad (p \in \mathcal{K}[1,1], \ z \in U). \]

Differentiating (2.15), with respect to \( z \), we obtain
\[ \left( \frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \right)' = p(z) + z p'(z). \quad (z \in U). \]

(2.16)

Using (2.16), the differential subordination (2.14) becomes
\[ p(z) + z p'(z) < h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}, \quad (z \in U). \]

From Lemma 1, we deduce
\[ p(z) < q(z) = \frac{1}{z} \int_0^z h(t) \, dt, \]
where
\[ = \frac{1}{z} \left[ \int_0^z \frac{1 + (2\alpha - 1)t}{1 + t} \, dt \right], \]
\[ = \frac{1}{z} \left[ \left( \int_0^z \frac{1}{1 + t} \, dt \right) + (2\alpha - 1) \int_0^z \frac{t}{1 + t} \, dt \right], \]
\[ = 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}. \]

Using (2.15), we have
\[ \frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} < q(z) = 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}. \]

The proof is complete. \( \Box \)

From Theorem 5, we deduce the following Corollary:
Corollary 1. If \( f \in R_{\lambda_1, \lambda_2}^{n,m}(\alpha) \), then
\[
\text{Re} \left( \frac{\mu_{\lambda_1, \lambda_2}^{n,m} f(z)}{z} \right) > (2\alpha - 1) + 2(1 - \alpha) \ln 2, \quad (z \in U).
\]

Proof. Since \( f \in R_{\lambda_1, \lambda_2}^{n,m}(\alpha) \), and from Definition 2 we have
\[
\text{Re} \left( \frac{\mu_{\lambda_1, \lambda_2}^{n,m} f(z)}{z} \right) > \alpha, \quad (z \in U),
\]
which is equivalent to
\[
\left( \mu_{\lambda_1, \lambda_2}^{n,m} f(z) \right)' < h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}.
\]
Using Theorem 5, we have
\[
\frac{\mu_{\lambda_1, \lambda_2}^{n,m} f(z)}{z} < q(z) = (2\alpha - 1) + 2(1 - \alpha) \frac{\ln(1 + z)}{z}.
\]
Since \( q \) is convex and \( q(U) \) is symmetric with respect to the real axis, we deduce
\[
\text{Re} \left( \frac{\mu_{\lambda_1, \lambda_2}^{n,m} f(z)}{z} \right) > \text{Re} q(1) = (2\alpha - 1) + 2(1 - \alpha) \ln 2, \quad (z \in U).
\]

\[\square\]

Theorem 6. Let \( h \in H(U) \), with \( h(0) = 1 \), \( h'(0) \neq 0 \) and assume that it satisfies the inequality
\[
\text{Re} \left( 1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2}, \quad (z \in U).
\]
If \( n, m \in \mathbb{N}_0 \), \( \lambda_2 \geq \lambda_1 \geq 0 \), \( f \in \mathcal{A} \) and it satisfies the differential subordination
\[
\left( \mu_{\lambda_1, \lambda_2}^{n,m} f(z) \right)' < h(z), \quad (z \in U),
\]
then
\[
\frac{\mu_{\lambda_1, \lambda_2}^{n,m} f(z)}{z} < q(z) = \frac{1}{z} \int_0^z h(t) dt.
\]

Proof. Let \( p(z) = \frac{\mu_{\lambda_1, \lambda_2}^{n,m} f(z)}{z} \),
\[
= 1 + p_1z + p_2z^2 + \ldots, \quad (p \in H[1,1], \quad z \in U).
\]

(2.18)
Differentiating (2.18), with respect to \( z \), we have
\[
\left( \mu_{n,m}^{1,2} f(z) \right)' = p(z) + z p'(z), \quad (z \in U). \tag{2.19}
\]
Using (2.19), the differential subordination (2.17) becomes
\[
p(z) + z p'(z) < h(z), \quad (z \in U).
\]
From Lemma 1, we deduce
\[
p(z) < q(z) = \frac{1}{z} \int_{0}^{z} h(t) \, dt.
\]
With (2.18), we obtain
\[
\frac{\mu_{n,m}^{1,2} f(z)}{z} < q(z) = \frac{1}{z} \int_{0}^{z} h(t) \, dt.
\]
From Lemma 3, we obtain that the function \( q \) is convex, and from Lemma 1, \( q \) is the best dominant for the subordination (2.17). This completes the proof of Theorem 6. □

3. CONCLUSION

We remark that several subclasses of analytic univalent functions can be derived using the operator \( \mu_{n,m}^{1,2} \). Several of their properties can be studied with this method, for example properties related to the ones that were studied in [7] and [6].

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