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# ON EIGHT SOLVABLE SYSTEMS OF DIFFERENCE EQUATIONS IN TERMS OF GENERALIZED PADOVAN SEQUENCES

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*Abstract.* In this study we show that the systems of difference equations

$$x_{n+1} = f^{-1}\left(af\left(p_{n-1}\right) + bf\left(q_{n-2}\right)\right),\ \ y_{n+1} = f^{-1}\left(af\left(r_{n-1}\right) + bf\left(s_{n-2}\right)\right),$$

for $n \in \mathbb{N}_0$, where the sequences $p_n$, $q_n$, $r_n$ and $s_n$ are some of the sequences $x_n$ and $y_n$, $f : D_f \longrightarrow \mathbb{R}$ is a "$1-1$" continuous function on its domain $D_f \subseteq \mathbb{R}$, initial values $x_{-j}$, $y_{-j}$, $j \in \{0,1,2\}$, are arbitrary real numbers in $D_f$ and the parameters $a, b$ are arbitrary complex numbers, with $b \neq 0$, can be explicitly solved in terms of generalized Padovan sequences. Some analytical examples are given to demonstrate the theoretical results.

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## 1. INTRODUCTION

Firstly, recall that $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, stand for natural, non-negative integer, integer, real and complex numbers, respectively. If $m, n \in \mathbb{Z}$, $m \leq n$ the notation $i = \overline{m,n}$ stands for $\{i \in \mathbb{Z} : m \leq i \leq n\}$.

Difference equations for which the solutions can be constructed explicitly are useful due to numerous applications. As particular, difference equations related to Fibonacci, Lucas, Padovan, Tetranacci, Horadam, Pell, Jacobsthal, and Jacobsthal-Lucas sequences and their generalizations are of much interest. Many related references can be found, for example, in [5–7, 10, 11, 17–19, 21, 23].

The equation

$$x_{n+1} = \frac{ax_{n-l}x_{n-k}}{bx_{n-p} \pm cx_{n-q}},\ \ n \in \mathbb{N}_0, \tag{1.1}$$

where the initial conditions are arbitrary positive real numbers, $k$, $l$, $p$, $q$ are non-negative integers and $a$, $b$, $c$ are positive constants, is one of the difference equations whose solutions are associated with number sequences. Positive solutions of concrete special cases of equation (1.1) have been studied by several authors. For the first time, Elabbasy et al., in [6, 7], obtained positive solutions of some special cases of equation (1.1) by using induction principle. In addition, they didn't give theoretical explanation of how solutions were obtained. One of the special cases is

$$x_{n+1}=\frac{x_{n-1}x_{n-2}}{x_{n-1}+x_{n-2}},\quad n\in\mathbb{N}_0, \tag{1.2}$$

whose solutions are associated with the well known Padovan numbers in literature. Moreover, the multi-dimensional expansion of the concrete some special cases of equation (1.1) can be seen in the literature (see, for example, [2–4, 8, 9, 13, 14, 24]). Another equation

$$x_{n+1}=a+\frac{b}{x_n}+\frac{c}{x_nx_{n-1}},\quad n\in\mathbb{N}_0, \tag{1.3}$$

where the parameters $a,b,c$ and initial values $x_{-1}$ and $x_0$ are complex numbers and $c\neq 0$, which is one of these equations. The solutions of equation (1.3) are associated with number sequences, has been studied in [16]. Unlike the method used to obtain solutions of some special cases of (1.1), by using convenient transformation the equation in (1.3) reduce to the next third-order linear difference equation with constant coefficients

$$x_{n+1}=ax_n+bx_{n-1}+cx_{n-2},\quad n\in\mathbb{N}_0, \tag{1.4}$$

which has actually the general solution

$$x_n=x_0S_n+x_{-1}\left(S_{n+1}-aS_n\right)+cx_{-2}S_{n-1},\quad n\in\mathbb{N}_0, \tag{1.5}$$

where $(S_n)_{n\geq -2}$ of equation (1.4) satisfying the initial values $S_{-2}=S_{-1}=0$, $S_0=1$. Quite recently in [15], among other things, a generalization of (1.4) is treated as

$$x_n=f^{-1}\left(af\left(x_{n-1}\right)+bf\left(x_{n-2}\right)+cf\left(x_{n-3}\right)\right),\ n\in\mathbb{N}_0, \tag{1.6}$$

where $f:D_f\longrightarrow\mathbb{R}$ is a "$1-1$" continuous function on its domain $D_f\subseteq\mathbb{R}$, parameters $a,b,c$ and the initial values $x_{-3},x_{-2}$ and $x_{-1}$ are real numbers. In addition, the authors obtained the solution of the equation (1.6) in relation to the solution given in (1.5).

On the other hand, one of the popular topics for system of difference equations is also symmetric and close-to-symmetric systems such as

$$x_{n+1}=g\left(p_{n-k},q_{n-l}\right),\quad y_{n+1}=g\left(r_{n-k},s_{n-l}\right),\quad n\in\mathbb{N}_0, \tag{1.7}$$

where the sequences $p_n$, $q_n$, $r_n$, $s_n$ are some of the sequences $x_n$ and $y_n$ and $k$, $l$ are fixed natural numbers. There are studies related to some special cases of the system (1.7) (see, for example, [1, 12, 20, 22]).

Motivated by this line of investigations, here we show that the systems of difference equations

$$x_{n+1}=f^{-1}\left(af\left(p_{n-1}\right)+bf\left(q_{n-2}\right)\right),\quad y_{n+1}=f^{-1}\left(af\left(r_{n-1}\right)+bf\left(s_{n-2}\right)\right), \tag{1.8}$$

for $n\in\mathbb{N}_0$, where the sequences $p_n$, $q_n$, $r_n$ and $s_n$ are some of the sequences $x_n$ and $y_n$, $f:D_f\longrightarrow\mathbb{R}$ is a "$1-1$" continuous function on its domain $D_f\subseteq\mathbb{R}$, the initial values $x_{-j}$, $y_{-j}$, $j\in\{0,1,2\}$ are arbitrary real numbers and the parameters and $a,b$

are arbitrary complex numbers, can be solved. To do this, we will use the solutions given in (1.5) and the solutions obtained by rearranging these solutions. In this way we also give analytical examples for the general solution of special cases of system (1.8).

## 2. Main Results

In this section, we consider the eight special cases of systems (1.8), where the sequences $p_n$, $q_n$, $r_n$, $s_n$ are some of the sequences $x_n$ and $y_n$, for $n \geq -2$, and initial values $x_{-j}$, $y_{-j}$, $j \in \{0,1,2\}$, are arbitrary real numbers.

### 2.1. *Case 1:* $p_n = x_n$, $q_n = x_n$, $r_n = y_n$, $s_n = y_n$

In this case, system (1.8) is expressed as

$$x_{n+1} = f^{-1}\left(af(x_{n-1}) + bf(x_{n-2})\right), \quad y_{n+1} = f^{-1}\left(af(y_{n-1}) + bf(y_{n-2})\right), \tag{2.1}$$

for $n \in \mathbb{N}_0$. Since $f$ is "$1-1$", from (2.1)

$$f(x_{n+1}) = af(x_{n-1}) + bf(x_{n-2}), \quad f(y_{n+1}) = af(y_{n-1}) + bf(y_{n-2}), \tag{2.2}$$

for $n \in \mathbb{N}_0$. By using the change of variables

$$f(x_n) = u_n, \quad \text{and} \quad f(y_n) = v_n, \quad n \geq -2, \tag{2.3}$$

system (2.2) is transformed to the following one

$$u_{n+1} = au_{n-1} + bu_{n-2}, \quad v_{n+1} = av_{n-1} + bv_{n-2}, \tag{2.4}$$

for $n \in \mathbb{N}_0$. By taking $a = 0$, $b = a$, $c = b$ in (1.4) and $S_n = J_{n+1}$, for all $n \geq -2$, which is called generalized Padovan sequence, in (1.5), the solutions to equations in (2.4) are given by

$$u_n = u_0 J_{n+1} + u_{-1} J_{n+2} + bu_{-2} J_n, \tag{2.5}$$

$$v_n = v_0 J_{n+1} + v_{-1} J_{n+2} + bv_{-2} J_n, \tag{2.6}$$

for $n \in \mathbb{N}_0$. From (2.3), (2.5) and (2.6), it follows that the general solution to system (2.2) is given by

$$x_n = f^{-1}\left(f(x_0) J_{n+1} + f(x_{-1}) J_{n+2} + bf(x_{-2}) J_n\right), \ n \geq -2, \tag{2.7}$$

$$y_n = f^{-1}\left(f(y_0) J_{n+1} + f(y_{-1}) J_{n+2} + bf(y_{-2}) J_n\right), \ n \geq -2. \tag{2.8}$$

### 2.2. *Case 2:* $p_n = x_n$, $q_n = x_n$, $r_n = x_n$, $s_n = x_n$

In this case, system (1.8) becomes

$$x_{n+1} = f^{-1}\left(af(x_{n-1}) + bf(x_{n-2})\right), \quad y_{n+1} = f^{-1}\left(af(x_{n-1}) + bf(x_{n-2})\right), \tag{2.9}$$

for $n \in \mathbb{N}_0$. It should be first note that from the equations in (2.9) it immediately follows that $x_n = y_n$, for all $n \in \mathbb{N}$. From (2.7), the general solution to system (2.9) is

$$x_n = y_n = f^{-1}\left(f(x_0) J_{n+1} + f(x_{-1}) J_{n+2} + bf(x_{-2}) J_n\right), \ n \in \mathbb{N}. \tag{2.10}$$

2.3. *Case 3:* $p_n = y_n$, $q_n = y_n$, $r_n = y_n$, $s_n = y_n$

In this case, we obtain the system

$$x_{n+1} = f^{-1}\left(af\left(y_{n-1}\right)+bf\left(y_{n-2}\right)\right), \quad y_{n+1} = f^{-1}\left(af\left(y_{n-1}\right)+bf\left(y_{n-2}\right)\right), \tag{2.11}$$

for $n \in \mathbb{N}_0$, which is an analogue of the system (2.9). By interchanging the variables $x_n$ and $y_n$, the system in (2.9) is transformed into (2.11). So, by interchanging $x_j$ and $y_j$ for $j \in \{0,1,2\}$, the formula in (2.10) is transformed into the formula

$$x_n = y_n = f^{-1}\left(f\left(y_0\right)J_{n+1} + f\left(y_{-1}\right)J_{n+2} + bf\left(y_{-2}\right)J_n\right), \ n \in \mathbb{N}. \tag{2.12}$$

2.4. *Case 4:* $p_n = x_n$, $q_n = x_n$, $r_n = y_n$, $s_n = x_n$

In this case, system (1.8) is written as in the form

$$x_{n+1} = f^{-1}\left(af\left(x_{n-1}\right)+bf\left(x_{n-2}\right)\right), \quad y_{n+1} = f^{-1}\left[af\left(y_{n-1}\right)+bf\left(x_{n-2}\right)\right], \tag{2.13}$$

for $n \in \mathbb{N}_0$. Since $f$ is "$1-1$", from (2.13)

$$f\left(x_{n+1}\right) = af\left(x_{n-1}\right)+bf\left(x_{n-2}\right), \quad f\left(y_{n+1}\right) = af\left(y_{n-1}\right)+bf\left(x_{n-2}\right), \tag{2.14}$$

for $n \in \mathbb{N}_0$. By using the change of variables

$$f\left(x_n\right) = u_n, \ \ n \geq -2, \ \text{ and } \ f\left(y_n\right) = v_n, \ \ n \geq -1, \tag{2.15}$$

system (2.14) is transformed to the following one

$$u_{n+1} = au_{n-1} + bu_{n-2}, \quad v_{n+1} = av_{n-1} + bu_{n-2}, \tag{2.16}$$

for $n \in \mathbb{N}_0$. From (2.5), we can write the solution of the first equation in (2.16) as

$$u_n = u_0 J_{n+1} + u_{-1} J_{n+2} + bu_{-2} J_n, \ n \in \mathbb{N}_0. \tag{2.17}$$

By subtracting the second one from the first equations in (2.16), we have

$$u_{n+1} - v_{n+1} = a\left(u_{n-1} - v_{n-1}\right), \ n \in \mathbb{N}_0. \tag{2.18}$$

From (2.18) we see that the sequence $\left(u_n - v_n\right)_{n \geq -1}$ satisfies the difference equation

$$w_n = aw_{n-2}, \ n \geq 1, \tag{2.19}$$

from which it follows that

$$u_{2n+i} - v_{2n+i} = a^{n+1}\left(u_{i-2} - v_{i-2}\right), \tag{2.20}$$

for $n \in \mathbb{N}_0$, $i \in \{1,2\}$.

From (2.17) and (2.20), we get

$$\begin{aligned} v_{2n+i} &= u_{2n+i} - a^{n+1}u_{i-2} + a^{n+1}v_{i-2}, \\ &= u_0 J_{2n+i+1} + u_{-1} J_{2n+i+2} + bu_{-2} J_{2n+i} - a^{n+1}u_{i-2} + a^{n+1}v_{i-2}, \end{aligned} \tag{2.21}$$

for $n \in \mathbb{N}_0$, $i \in \{1,2\}$.

Employing (2.17) and (2.21) in (2.15) and after some calculation, we obtain

$$x_n = f^{-1}\left(f\left(x_0\right)J_{n+1} + f\left(x_{-1}\right)J_{n+2} + bf\left(x_{-2}\right)J_n\right), \ n \geq -2, \tag{2.22}$$

$$y_{2n+1} = f^{-1}\big(f(x_0)J_{2n+2} + f(x_{-1})\left(J_{2n+3} - a^{n+1}\right) + bf(x_{-2})J_{2n+1} + a^{n+1}f(y_{-1})\big),\ n \geq -1, \tag{2.23}$$

$$y_{2n+2} = f^{-1}\big(f(x_0)\left(J_{2n+3} - a^{n+1}\right) + f(x_{-1})J_{2n+4} + bf(x_{-2})J_{2n+2} + a^{n+1}f(y_0)\big),\ n \geq -1. \tag{2.24}$$

## 2.5. *Case 5:* $p_n = x_n$, $q_n = y_n$, $r_n = y_n$, $s_n = y_n$

In this case, system (1.8) is expressed as

$$x_{n+1} = f^{-1}\big(af(x_{n-1}) + bf(y_{n-2})\big),\quad y_{n+1} = f^{-1}\big(af(y_{n-1}) + bf(y_{n-2})\big), \tag{2.25}$$

for $n \in \mathbb{N}_0$. Note that system (2.25) is obtained from equations (2.13) by interchanging the letters $x$ and $y$, and hence all the statements concerning solutions to the equations follow from the corresponding statements in Case 4.

The general solution to the system (2.25) is given

$$x_{2n+1} = f^{-1}\big(f(y_0)J_{2n+2} + f(y_{-1})\left(J_{2n+3} - a^{n+1}\right) + bf(y_{-2})J_{2n+1} + a^{n+1}f(x_{-1})\big),\ n \geq -1, \tag{2.26}$$

$$x_{2n+2} = f^{-1}\big(f(y_0)\left(J_{2n+3} - a^{n+1}\right) + f(y_{-1})J_{2n+4} + bf(y_{-2})J_{2n+2} + a^{n+1}f(x_0)\big),\ n \geq -1, \tag{2.27}$$

$$y_n = f^{-1}\big(f(y_0)J_{n+1} + f(y_{-1})J_{n+2} + bf(y_{-2})J_n\big),\ n \geq -2. \tag{2.28}$$

## 2.6. *Case 6:* $p_n = y_n$, $q_n = y_n$, $r_n = x_n$, $s_n = x_n$

In this case, we obtain the system

$$x_{n+1} = f^{-1}\big(af(y_{n-1}) + bf(y_{n-2})\big),\quad y_{n+1} = f^{-1}\big(af(x_{n-1}) + bf(x_{n-2})\big), \tag{2.29}$$

for $n \in \mathbb{N}_0$. Since $f$ is "$1-1$", from (2.29)

$$f(x_{n+1}) = af(y_{n-1}) + bf(y_{n-2}),\quad f(y_{n+1}) = af(x_{n-1}) + bf(x_{n-2}), \tag{2.30}$$

for $n \in \mathbb{N}_0$. By using the change of variables

$$f(x_n) = u_n,\ \text{ and }\ f(y_n) = v_n,\ \ n \geq -2, \tag{2.31}$$

system (2.30) is transformed to the following one

$$u_{n+1} = av_{n-1} + bv_{n-2},\ \ v_{n+1} = au_{n-1} + bu_{n-2},\ n \in \mathbb{N}_0. \tag{2.32}$$

By summing the equations in (2.32) we get

$$u_{n+1} + v_{n+1} = a(u_{n-1} + v_{n-1}) + b(u_{n-2} + v_{n-2}),\ n \in \mathbb{N}_0, \tag{2.33}$$

whereas by subtracting the second one from the first, we have

$$u_{n+1} - v_{n+1} = -a(u_{n-1} - v_{n-1}) - b(u_{n-2} - v_{n-2}),\ n \in \mathbb{N}_0. \tag{2.34}$$

From (2.5), we can write the solution of equation (2.33) as

$$u_n + v_n = (u_0 + v_0)\, J_{n+1} + (u_{-1} + v_{-1})\, J_{n+2} + b\,(u_{-2} + v_{-2})\, J_n, \tag{2.35}$$

for $n \geq -2$. On the other hand, by taking $a = 0$, $b = -a$, $c = -b$ in (1.4) and $S_n = (-1)^n J'_{n+1}$, for all $n \geq -2$, which is called generalized Padovan sequence, in (1.5), from (2.34), we also have

$$u_n - v_n = (-1)^n \big( (u_0 - v_0)\, J'_{n+1} - (u_{-1} - v_{-1})\, J'_{n+2} + b\,(u_{-2} - v_{-2})\, J'_n \big), \tag{2.36}$$

for $n \geq -2$. From (2.36) we obtain

$$u_{2n} - v_{2n} = (u_0 - v_0)\, J'_{2n+1} - (u_{-1} - v_{-1})\, J'_{2n+2} + b\,(u_{-2} - v_{-2})\, J'_{2n}, \tag{2.37}$$

for $n \geq -1$. From (2.35)

$$u_{2n} + v_{2n} = (u_0 + v_0)\, J_{2n+1} + (u_{-1} + v_{-1})\, J_{2n+2} + b\,(u_{-2} + v_{-2})\, J_{2n}, \tag{2.38}$$

for $n \geq -1$. By summing the equations (2.37) and (2.38) we get

$$\begin{aligned} u_{2n} = {} & \frac{\left(J_{2n+1} + J'_{2n+1}\right) u_0 + \left(J_{2n+1} - J'_{2n+1}\right) v_0 + \left(J_{2n+2} - J'_{2n+2}\right) u_{-1}}{2} \\ & + \frac{\left(J_{2n+2} + J'_{2n+2}\right) v_{-1} + b\,(J_{2n} + J'_{2n})\, u_{-2} + b\,(J_{2n} - J'_{2n})\, v_{-2}}{2}, \end{aligned} \tag{2.39}$$

for $n \geq -1$. By subtracting equation (2.37) from equation (2.38), we have

$$\begin{aligned} v_{2n} = {} & \frac{\left(J_{2n+1} - J'_{2n+1}\right) u_0 + \left(J_{2n+1} + J'_{2n+1}\right) v_0 + \left(J_{2n+2} + J'_{2n+2}\right) u_{-1}}{2} \\ & + \frac{\left(J_{2n+2} - J'_{2n+2}\right) v_{-1} + b\,(J_{2n} - J'_{2n})\, u_{-2} + b\,(J_{2n} + J'_{2n})\, v_{-2}}{2}, \end{aligned} \tag{2.40}$$

for $n \geq -1$. From (2.36) we have

$$u_{2n+1} - v_{2n+1} = -(u_0 - v_0)\, J'_{2n+2} + (u_{-1} - v_{-1})\, J'_{2n+3} - b\,(u_{-2} - v_{-2})\, J'_{2n+1}, \tag{2.41}$$

for $n \geq -1$. From (2.35)

$$u_{2n+1} + v_{2n+1} = (u_0 + v_0)\, J_{2n+2} + (u_{-1} + v_{-1})\, J_{2n+3} + b\,(u_{-2} + v_{-2})\, J_{2n+1}, \tag{2.42}$$

for $n \geq -1$. By summing the equations (2.41) and (2.42) we get

$$\begin{aligned} u_{2n+1} = {} & \frac{\left(J_{2n+2} - J'_{2n+2}\right) u_0 + \left(J_{2n+2} + J'_{2n+2}\right) v_0 + \left(J_{2n+3} + J'_{2n+3}\right) u_{-1}}{2} \\ & + \frac{\left(J_{2n+3} - J'_{2n+3}\right) v_{-1} + b\left(J_{2n+1} - J'_{2n+1}\right) u_{-2}}{2} + \frac{b\left(J_{2n+1} + J'_{2n+1}\right) v_{-2}}{2}, \end{aligned} \tag{2.43}$$

for $n \geq -1$. By subtracting equation (2.41) from equation (2.42), we have

$$v_{2n+1} = \frac{\left(J_{2n+2} + J'_{2n+2}\right) u_0 + \left(J_{2n+2} - J'_{2n+2}\right) v_0 + \left(J_{2n+3} - J'_{2n+3}\right) u_{-1}}{2}$$

$$+\frac{\left(J_{2n+3}+J'_{2n+3}\right)v_{-1}+b\left(J_{2n+1}+J'_{2n+1}\right)u_{-2}}{2}+\frac{b\left(J_{2n+1}-J'_{2n+1}\right)v_{-2}}{2}, \tag{2.44}$$

for $n\geq -1$. Employing (2.39), (2.40), (2.43), (2.44) in (2.31) and after some calculation, we obtain

$$\begin{aligned}x_{2n}=f^{-1}\Bigg(&\frac{\left(J_{2n+1}+J'_{2n+1}\right)f(x_0)+\left(J_{2n+1}-J'_{2n+1}\right)f(y_0)}{2}\\&+\frac{\left(J_{2n+2}-J'_{2n+2}\right)f(x_{-1})+\left(J_{2n+2}+J'_{2n+2}\right)f(y_{-1})+b\left(J_{2n}+J'_{2n}\right)f(x_{-2})}{2}\\&+\frac{b\left(J_{2n}-J'_{2n}\right)f(y_{-2})}{2}\Bigg),\end{aligned} \tag{2.45}$$

$$\begin{aligned}y_{2n}=f^{-1}\Bigg(&\frac{\left(J_{2n+1}-J'_{2n+1}\right)f(x_0)+\left(J_{2n+1}+J'_{2n+1}\right)f(y_0)}{2}\\&+\frac{\left(J_{2n+2}+J'_{2n+2}\right)f(x_{-1})+\left(J_{2n+2}-J'_{2n+2}\right)f(y_{-1})+b\left(J_{2n}-J'_{2n}\right)f(x_{-2})}{2}\\&+\frac{b\left(J_{2n}+J'_{2n}\right)f(y_{-2})}{2}\Bigg),\end{aligned} \tag{2.46}$$

$$\begin{aligned}x_{2n+1}=f^{-1}\Bigg(&\frac{\left(J_{2n+2}-J'_{2n+2}\right)f(x_0)+\left(J_{2n+2}+J'_{2n+2}\right)f(y_0)}{2}\\&+\frac{\left(J_{2n+3}+J'_{2n+3}\right)f(x_{-1})+\left(J_{2n+3}-J'_{2n+3}\right)f(y_{-1})}{2}\\&+\frac{b\left(J_{2n+1}-J'_{2n+1}\right)f(x_{-2})+b\left(J_{2n+1}+J'_{2n+1}\right)f(y_{-2})}{2}\Bigg),\end{aligned} \tag{2.47}$$

and

$$\begin{aligned}y_{2n+1}=f^{-1}\Bigg(&\frac{\left(J_{2n+2}+J'_{2n+2}\right)f(x_0)+\left(J_{2n+2}-J'_{2n+2}\right)f(y_0)}{2}\\&+\frac{\left(J_{2n+3}-J'_{2n+3}\right)f(x_{-1})+\left(J_{2n+3}+J'_{2n+3}\right)f(y_{-1})}{2}\\&+\frac{b\left(J_{2n+1}+J'_{2n+1}\right)f(x_{-2})+b\left(J_{2n+1}-J'_{2n+1}\right)f(y_{-2})}{2}\Bigg),\end{aligned} \tag{2.48}$$

for $n\geq -1$.

2.7. *Case 7:* $p_n=y_n$, $q_n=x_n$, $r_n=x_n$, $s_n=y_n$

In this case, system (1.8) is expressed as

$$x_{n+1}=f^{-1}\left(af(y_{n-1})+bf(x_{n-2})\right),\quad y_{n+1}=f^{-1}\left(af(x_{n-1})+bf(y_{n-2})\right), \tag{2.49}$$

for $n \in \mathbb{N}_0$. Since $f$ is "$1-1$", from (2.49)

$$f(x_{n+1}) = af(y_{n-1}) + bf(x_{n-2}), \quad f(y_{n+1}) = af(x_{n-1}) + bf(y_{n-2}), \tag{2.50}$$

for $n \in \mathbb{N}_0$. By using the change of variables

$$f(x_n) = u_n, \text{ and } f(y_n) = v_n, \ n \geq -2, \tag{2.51}$$

system (2.50) is transformed to the following one

$$u_{n+1} = av_{n-1} + bu_{n-2}, \ v_{n+1} = au_{n-1} + bv_{n-2}, \ n \in \mathbb{N}_0. \tag{2.52}$$

By summing the equations in (2.52) we get

$$u_{n+1} + v_{n+1} = a(u_{n-1} + v_{n-1}) + b(u_{n-2} + v_{n-2}), \ n \in \mathbb{N}_0, \tag{2.53}$$

whereas by subtracting the second one from the first, we have

$$u_{n+1} - v_{n+1} = -a(u_{n-1} - v_{n-1}) + b(u_{n-2} - v_{n-2}), \ n \in \mathbb{N}_0. \tag{2.54}$$

From (2.5), we can write the solution of equation (2.53) as

$$u_n + v_n = (u_0 + v_0) J_{n+1} + (u_{-1} + v_{-1}) J_{n+2} + b(u_{-2} + v_{-2}) J_n, \tag{2.55}$$

for $n \geq -2$. On the other hand, by taking $a = 0, \ b = -a, \ c = b$ in (1.4) and $S_n = J'_{n+1}$, for all $n \geq -2$, which is called generalized Padovan sequence, in (1.5), from (2.54), we also have that

$$u_n - v_n = (u_0 - v_0) J'_{n+1} + (u_{-1} - v_{-1}) J'_{n+2} + b(u_{-2} - v_{-2}) J'_n, \tag{2.56}$$

for $n \geq -2$. By summing the equations (2.55) and (2.56) we get

$$\begin{aligned} u_n = & \frac{J_{n+1} + J'_{n+1}}{2} u_0 + \frac{J_{n+1} - J'_{n+1}}{2} v_0 + \frac{J_{n+2} + J'_{n+2}}{2} u_{-1} \\ & + \frac{J_{n+2} - J'_{n+2}}{2} v_{-1} + b\frac{J_n + J'_n}{2} u_{-2} + b\frac{J_n - J'_n}{2} v_{-2}, \ n \geq -2. \end{aligned} \tag{2.57}$$

By subtracting equation (2.56) from equation (2.55), we have

$$\begin{aligned} v_n = & \frac{J_{n+1} - J'_{n+1}}{2} u_0 + \frac{J_{n+1} + J'_{n+1}}{2} v_0 + \frac{J_{n+2} - J'_{n+2}}{2} u_{-1} \\ & + \frac{J_{n+2} + J'_{n+2}}{2} v_{-1} + b\frac{J_n - J'_n}{2} u_{-2} + b\frac{J_n + J'_n}{2} v_{-2}, \ n \geq -2. \end{aligned} \tag{2.58}$$

From (2.51), (2.57) and (2.58) and after some calculation, we obtain

$$\begin{aligned} x_n = f^{-1} \Bigg( & \frac{J_{n+1} + J'_{n+1}}{2} f(x_0) + \frac{J_{n+1} - J'_{n+1}}{2} f(y_0) \\ & + \frac{J_{n+2} + J'_{n+2}}{2} f(x_{-1}) + \frac{J_{n+2} - J'_{n+2}}{2} f(y_{-1}) \\ & + b\frac{J_n + J'_n}{2} f(x_{-2}) + b\frac{J_n - J'_n}{2} f(y_{-2}) \Bigg), \ n \geq -2, \end{aligned} \tag{2.59}$$

and

$$y_n = f^{-1}\Bigg(\frac{J_{n+1}-J'_{n+1}}{2}f(x_0)+\frac{J_{n+1}+J'_{n+1}}{2}f(y_0) + \frac{J_{n+2}-J'_{n+2}}{2}f(x_{-1})+\frac{J_{n+2}+J'_{n+2}}{2}f(y_{-1}) + b\frac{J_n-J'_n}{2}f(x_{-2})+b\frac{J_n+J'_n}{2}f(y_{-2})\Bigg),\ n\geq -2. \tag{2.60}$$

2.8. *Case 8:* $p_n = x_n$, $q_n = y_n$, $r_n = y_n$, $s_n = x_n$

In this case, system (1.8) is written as in the form

$$x_{n+1} = f^{-1}\big(af(x_{n-1})+bf(y_{n-2})\big),\ \ y_{n+1} = f^{-1}\big(af(y_{n-1})+bf(x_{n-2})\big), \tag{2.61}$$

for $n\in\mathbb{N}_0$. Since $f$ is "$1-1$", from (2.61)

$$f(x_{n+1}) = af(x_{n-1})+bf(y_{n-2}),\ \ f(y_{n+1}) = af(y_{n-1})+bf(x_{n-2}), \tag{2.62}$$

for $n\in\mathbb{N}_0$. By using the change of variables

$$f(x_n) = u_n,\ \text{ and }\ f(y_n) = v_n,\ \ n\geq -2, \tag{2.63}$$

system (2.62) is transformed to the following one

$$u_{n+1} = au_{n-1}+bv_{n-2},\ \ v_{n+1} = av_{n-1}+bu_{n-2},\ n\in\mathbb{N}_0. \tag{2.64}$$

By summing the equations in (2.64) we get

$$u_{n+1}+v_{n+1} = a(u_{n-1}+v_{n-1})+b(u_{n-2}+v_{n-2}),\ n\in\mathbb{N}_0, \tag{2.65}$$

whereas by subtracting the second one from the first, we have

$$u_{n+1}-v_{n+1} = a(u_{n-1}-v_{n-1})-b(u_{n-2}-v_{n-2}),\ n\in\mathbb{N}_0. \tag{2.66}$$

From (2.5), we can write the solution of equation (2.65) as

$$u_n+v_n = (u_0+v_0)J_{n+1}+(u_{-1}+v_{-1})J_{n+2}+b(u_{-2}+v_{-2})J_n, \tag{2.67}$$

for $n\geq -2$. On the other hand, by taking $a=0$, $b=a$, $c=-b$ in (1.4) and $S_n = (-1)^n J_{n+1}$, for all $n\geq -2$, which is called generalized Padovan sequence, in (1.5), from (2.66), we also have that

$$u_n-v_n = (-1)^n\big((u_0-v_0)J_{n+1}-(u_{-1}-v_{-1})J_{n+2}+b(u_{-2}-v_{-2})J_n\big), \tag{2.68}$$

for $n\geq -2$. From (2.68) we have

$$u_{2n}-v_{2n} = (u_0-v_0)J_{2n+1}-(u_{-1}-v_{-1})J_{2n+2}+b(u_{-2}-v_{-2})J_{2n}, \tag{2.69}$$

for $n\geq -1$ and

$$u_{2n+1}-v_{2n+1} = -(u_0-v_0)J_{2n+2}+(u_{-1}-v_{-1})J_{2n+3}-b(u_{-2}-v_{-2})J_{2n+1}, \tag{2.70}$$

for $n\geq -1$. From (2.67)

$$u_{2n}+v_{2n} = (u_0+v_0)J_{2n+1}+(u_{-1}+v_{-1})J_{2n+2}+b(u_{-2}+v_{-2})J_{2n}, \tag{2.71}$$

for $n \geq -1$. By summing the equations (2.69) and (2.71)

$$u_{2n} = u_0 J_{2n+1} + v_{-1} J_{2n+2} + b u_{-2} J_{2n},\ n \geq -1. \tag{2.72}$$

By subtracting equation (2.69) from equation (2.71), we have

$$v_{2n} = v_0 J_{2n+1} + u_{-1} J_{2n+2} + b v_{-2} J_{2n},\ n \geq -1. \tag{2.73}$$

From (2.67)

$$u_{2n+1} + v_{2n+1} = (u_0 + v_0) J_{2n+2} + (u_{-1} + v_{-1}) J_{2n+3} + b(u_{-2} + v_{-2}) J_{2n+1}, \tag{2.74}$$

for $n \geq -1$. By summing the equations (2.70) and (2.74) we get

$$u_{2n+1} = v_0 J_{2n+2} + u_{-1} J_{2n+3} + b v_{-2} J_{2n+1},\ n \geq -1. \tag{2.75}$$

By subtracting equation (2.70) from equation (2.74), we have

$$v_{2n+1} = u_0 J_{2n+2} + v_{-1} J_{2n+3} + b u_{-2} J_{2n+1},\ n \geq -1. \tag{2.76}$$

From (2.63), (2.72), (2.73), (2.75), (2.76) and after some calculation, we obtain

$$x_{2n} = f^{-1}\left(f(x_0) J_{2n+1} + f(y_{-1}) J_{2n+2} + b f(x_{-2}) J_{2n}\right),\ n \geq -1, \tag{2.77}$$

$$y_{2n} = f^{-1}\left(f(y_0) J_{2n+1} + f(x_{-1}) J_{2n+2} + b f(y_{-2}) J_{2n}\right),\ n \geq -1, \tag{2.78}$$

$$x_{2n+1} = f^{-1}\left(f(y_0) J_{2n+2} + f(x_{-1}) J_{2n+3} + b f(y_{-2}) J_{2n+1}\right),\ n \geq -1, \tag{2.79}$$

and

$$y_{2n+1} = f^{-1}\left(f(x_0) J_{2n+2} + f(y_{-1}) J_{2n+3} + b f(x_{-2}) J_{2n+1}\right), n \geq -1. \tag{2.80}$$

## 3. Analytical Examples

In this section, we give examples for Case 1, 4 and 7. Examples for the other cases can be constructed similarly.

*Example* 1. Let

$$f(t) = t. \tag{3.1}$$

Then, $D_f = \mathbb{R}$ and system (2.1) becomes

$$x_{n+1} = a x_{n-1} + b x_{n-2},\ \ y_{n+1} = a y_{n-1} + b y_{n-2},\ \ n \in \mathbb{N}_0. \tag{3.2}$$

Here we can also assume that parameters $a$, $b$ and initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}$, and $y_0$ are complex numbers, since function (3.1) is "$1-1$" on $D_f = \mathbb{C}$. Function (3.1) is obviously an involution:

$$f^{-1}(t) = f(t),\quad t \in D_f.$$

We see that formulas (2.7) and (2.8) hold. Using (3.1) in (2.7) and (2.8), we obtain that the general solution to system (3.2) is

$$\begin{aligned} x_n &= f^{-1}\left(f(x_0) J_{n+1} + f(x_{-1}) J_{n+2} + b f(x_{-2}) J_n\right) \\ &= x_0 J_{n+1} + x_{-1} J_{n+2} + b x_{-2} J_n,\ n \geq -2, \end{aligned} \tag{3.3}$$

$$\begin{aligned}y_n &= f^{-1}\big(f(y_0)J_{n+1}+f(y_{-1})J_{n+2}+bf(y_{-2})J_n\big)\\ &= y_0J_{n+1}+y_{-1}J_{n+2}+by_{-2}J_n,\ n\ge -2.\end{aligned}\tag{3.4}$$

*Example* 2. Let

$$f(t)=\frac{1}{t}.\tag{3.5}$$

Then $D_f=\mathbb{R}\setminus\{0\}$ and system (2.13) becomes

$$x_{n+1}=\left(\frac{a}{x_{n-1}}+\frac{b}{x_{n-2}}\right)^{-1},\quad y_{n+1}=\left(\frac{a}{y_{n-1}}+\frac{b}{x_{n-2}}\right)^{-1},\quad n\in\mathbb{N}_0.\tag{3.6}$$

Here we can also assume that parameters $a$, $b$ and initial values $x_{-2}$, $x_{-1}$, $x_0$, $y_{-1}$ and $y_0$ are complex numbers, since function (3.5) is "$1-1$" on $D_f=\mathbb{C}\setminus\{0\}$.

Clearly, function (3.5) is an involution. We see that (2.22)–(2.24) hold. Using (3.5) in (2.22)–(2.24), we obtain the general solution to system (3.6):

$$\begin{aligned}x_n &= f^{-1}\big(f(x_0)J_{n+1}+f(x_{-1})J_{n+2}+bf(x_{-2})J_n\big)\\ &= \left(\frac{1}{x_0}J_{n+1}+\frac{1}{x_{-1}}J_{n+2}+\frac{b}{x_{-2}}J_n\right)^{-1}\\ &= \frac{x_0x_{-1}x_{-2}}{x_{-1}x_{-2}J_{n+1}+x_0x_{-2}J_{n+2}+bx_0x_{-1}J_n},\ n\ge -2,\end{aligned}\tag{3.7}$$

$$\begin{aligned}y_{2n+1} &= f^{-1}\Big(f(x_0)J_{2n+2}+f(x_{-1})\left(J_{2n+3}-a^{n+1}\right)+bf(x_{-2})J_{2n+1}+a^{n+1}f(y_{-1})\Big)\\ &= \left(\frac{1}{x_0}J_{2n+2}+\frac{1}{x_{-1}}\left(J_{2n+3}-a^{n+1}\right)+\frac{b}{x_{-2}}J_{2n+1}+\frac{a^{n+1}}{y_{-1}}\right)^{-1}\\ &= \left(\frac{x_{-1}x_{-2}y_{-1}J_{2n+2}+x_0x_{-2}y_{-1}\left(J_{2n+3}-a^{n+1}\right)}{x_0x_{-1}x_{-2}y_{-1}}\right.\\ &\qquad\left.+\frac{bx_0x_{-1}y_{-1}J_{2n+1}+a^{n+1}x_0x_{-1}x_{-2}}{x_0x_{-1}x_{-2}y_{-1}}\right)^{-1},\ n\ge -1,\end{aligned}\tag{3.8}$$

$$\begin{aligned}y_{2n+2} &= f^{-1}\Big(f(x_0)\left(J_{2n+3}-a^{n+1}\right)+f(x_{-1})J_{2n+4}+bf(x_{-2})J_{2n+2}+a^{n+1}f(y_0)\Big)\\ &= \left(\frac{1}{x_0}\left(J_{2n+3}-a^{n+1}\right)+\frac{1}{x_{-1}}J_{2n+4}+\frac{b}{x_{-2}}J_{2n+2}+\frac{a^{n+1}}{y_0}\right)^{-1}\\ &= \left(\frac{x_{-1}x_{-2}y_0\left(J_{2n+3}-a^{n+1}\right)+x_0x_{-2}y_0J_{2n+4}}{x_0x_{-1}x_{-2}y_0}\right.\\ &\qquad\left.+\frac{bx_0x_{-1}y_0J_{2n+2}+a^{n+1}x_0x_{-1}x_{-2}}{x_0x_{-1}x_{-2}y_0}\right)^{-1},\ n\ge -1.\end{aligned}\tag{3.9}$$

*Example* 3. Let

$$f_k(t)=t^{2k+1},\quad k\in\mathbb{N}_0.\tag{3.10}$$

Then $D_{f_k} = \mathbb{R}$ and system (2.49) becomes

$$x_{n+1} = \left(ay_{n-1}^{2k+1} + bx_{n-2}^{2k+1}\right)^{\frac{1}{2k+1}}, \quad y_{n+1} = \left(ax_{n-1}^{2k+1} + by_{n-2}^{2k+1}\right)^{\frac{1}{2k+1}}, \quad n \in \mathbb{N}_0. \tag{3.11}$$

Here we can also assume that parameters $a$, $b$ and initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ are complex numbers, since function (3.10) is "$1-1$" on $D_{f_k} = \mathbb{C}$.

Function (3.10) is an involution:

$$f_k^{-1}(t) = t^{\frac{1}{2k+1}}, \quad t \in D_{f_k}.$$

We see that (2.59) and (2.60) hold. Using (3.10) in (2.59) and (2.60), we obtain the general solution to system (3.11):

$$\begin{aligned} x_n &= f_k^{-1}\Bigg(\frac{J_{n+1}+J'_{n+1}}{2} f_k(x_0) + \frac{J_{n+1}-J'_{n+1}}{2} f_k(y_0) + \frac{J_{n+2}+J'_{n+2}}{2} f_k(x_{-1}) \\ &\quad + \frac{J_{n+2}-J'_{n+2}}{2} f_k(y_{-1}) + b\frac{J_n+J'_n}{2} f_k(x_{-2}) + b\frac{J_n-J'_n}{2} f_k(y_{-2})\Bigg) \\ &= \Bigg(\frac{J_{n+1}+J'_{n+1}}{2} x_0^{2k+1} + \frac{J_{n+1}-J'_{n+1}}{2} y_0^{2k+1} + \frac{J_{n+2}+J'_{n+2}}{2} x_{-1}^{2k+1} \\ &\quad + \frac{J_{n+2}-J'_{n+2}}{2} y_{-1}^{2k+1} + b\frac{J_n+J'_n}{2} x_{-2}^{2k+1} + b\frac{J_n-J'_n}{2} y_{-2}^{2k+1}\Bigg)^{\frac{1}{2k+1}}, \end{aligned} \tag{3.12}$$

for $n \geq -2$,

$$\begin{aligned} y_n &= f_k^{-1}\Bigg(\frac{J_{n+1}-J'_{n+1}}{2} f_k(x_0) + \frac{J_{n+1}+J'_{n+1}}{2} f_k(y_0) + \frac{J_{n+2}-J'_{n+2}}{2} f_k(x_{-1}) \\ &\quad + \frac{J_{n+2}+J'_{n+2}}{2} f_k(y_{-1}) + b\frac{J_n-J'_n}{2} f_k(x_{-2}) + b\frac{J_n+J'_n}{2} f_k(y_{-2})\Bigg) \\ &= \Bigg(\frac{J_{n+1}-J'_{n+1}}{2} x_0^{2k+1} + \frac{J_{n+1}+J'_{n+1}}{2} y_0^{2k+1} + \frac{J_{n+2}-J'_{n+2}}{2} x_{-1}^{2k+1} \\ &\quad + \frac{J_{n+2}+J'_{n+2}}{2} y_{-1}^{2k+1} + b\frac{J_n-J'_n}{2} x_{-2}^{2k+1} + b\frac{J_n+J'_n}{2} y_{-2}^{2k+1}\Bigg)^{\frac{1}{2k+1}}, \end{aligned} \tag{3.13}$$

for $n \geq -2$.

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