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ON THIRD-ORDER JACOBSTHAL POLYNOMIALS AND THEIR PROPERTIES

GAMALIEL CERDA-MORALES

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Abstract. Third-order Jacobsthal polynomial sequence is defined in this study. Some properties involving this polynomial, including the Binet-style formula and the generating function are also presented. Furthermore, we present the modified third-order Jacobsthal polynomials, and derive adaptations for some well-known identities of third-order Jacobsthal and modified third-order Jacobsthal numbers.

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1. Introduction

The Jacobsthal numbers have many interesting properties and applications in many fields of science (see, [1]). The Jacobsthal numbers $(J_n)_{n\geq 0}$ are defined by the recurrence relation

$$J_0 = 0,\ J_1 = 1,\quad J_{n+2} = J_{n+1} + 2J_n,\ n \geq 0. \tag{1.1}$$

Another important sequence is the Jacobsthal–Lucas sequence. This sequence is defined by the recurrence relation $j_{n+2} = j_{n+1} + 2j_n$, where $j_0 = 2$ and $j_1 = 1$.

In Cook and Bacon's work [5] the Jacobsthal recurrence relation is extended to higher order recurrence relations and the basic list of identities provided by A. F. Horadam [9] is expanded and extended to several identities for some of the higher order cases. In fact, the third-order Jacobsthal numbers, $\{J_n^{(3)}\}_{n\geq 0}$, and third-order Jacobsthal–Lucas numbers, $\{j_n^{(3)}\}_{n\geq 0}$, are defined by

$$J_{n+3}^{(3)} = J_{n+2}^{(3)} + J_{n+1}^{(3)} + 2J_n^{(3)},\ J_0^{(3)} = 0,\ J_1^{(3)} = J_2^{(3)} = 1,\ n \geq 0, \tag{1.2}$$

and

$$j_{n+3}^{(3)} = j_{n+2}^{(3)} + j_{n+1}^{(3)} + 2j_n^{(3)},\ j_0^{(3)} = 2,\ j_1^{(3)} = 1,\ j_2^{(3)} = 5,\ n \geq 0, \tag{1.3}$$

respectively.

Some of the following properties given for third-order Jacobsthal numbers and third-order Jacobsthal–Lucas numbers are used in this paper (for more details, see

[2–5]). Note that Eqs. (1.7) and (1.11) have been corrected in [3], since they have been wrongly described in [5]. Then, we have

$$3J_n^{(3)} + j_n^{(3)} = 2^{n+1}, \tag{1.4}$$

$$j_n^{(3)} - 3J_n^{(3)} = 2j_{n-3}^{(3)},\ n \geq 3, \tag{1.5}$$

$$J_{n+2}^{(3)} - 4J_n^{(3)} = \left\{ \begin{array}{ccc} -2 & \text{if} & n \equiv 1 \pmod 3 \\ 1 & \text{if} & n \not\equiv 1 \pmod 3 \end{array} \right. , \tag{1.6}$$

$$j_{n+1}^{(3)} + j_n^{(3)} = 3J_{n+2}^{(3)}, \tag{1.7}$$

$$j_n^{(3)} - J_{n+2}^{(3)} = \left\{ \begin{array}{ccc} 1 & \text{if} & n \equiv 0 \pmod 3 \\ -1 & \text{if} & n \equiv 1 \pmod 3 \\ 0 & \text{if} & n \equiv 2 \pmod 3 \end{array} \right. , \tag{1.8}$$

$$\left(j_{n-3}^{(3)}\right)^2 + 3J_n^{(3)} j_n^{(3)} = 4^n, \tag{1.9}$$

$$\sum_{k=0}^{n} J_k^{(3)} = \left\{ \begin{array}{ccc} J_{n+1}^{(3)} & \text{if} & n \not\equiv 0 \pmod 3 \\ J_{n+1}^{(3)} - 1 & \text{if} & n \equiv 0 \pmod 3 \end{array} \right. \tag{1.10}$$

and

$$\left(j_n^{(3)}\right)^2 - 9\left(J_n^{(3)}\right)^2 = 2^{n+2} j_{n-3}^{(3)},\ n \geq 3. \tag{1.11}$$

Using standard techniques for solving recurrence relations, the auxiliary equation, and its roots are given by

$$x^3 - x^2 - x - 2 = 0;\ x = 2,\ \text{and } x = \frac{-1 \pm i\sqrt{3}}{2}.$$

Note that the latter two are the complex conjugate cube roots of unity. Call them ω_1 and ω_2, respectively. Thus the Binet formulas can be written as

$$J_n^{(3)} = \frac{2}{7}2^n - \left(\frac{3+2i\sqrt{3}}{21}\right)\omega_1^n - \left(\frac{3-2i\sqrt{3}}{21}\right)\omega_2^n \tag{1.12}$$

and

$$j_n^{(3)} = \frac{8}{7}2^n + \left(\frac{3+2i\sqrt{3}}{7}\right)\omega_1^n + \left(\frac{3-2i\sqrt{3}}{7}\right)\omega_2^n, \tag{1.13}$$

respectively. Now, we use the notation

$$Z_n = \frac{A\omega_1^n - B\omega_2^n}{\omega_1 - \omega_2} = \left\{ \begin{array}{ccc} 2 & \text{if} & n \equiv 0 \pmod 3 \\ -3 & \text{if} & n \equiv 1 \pmod 3 \\ 1 & \text{if} & n \equiv 2 \pmod 3 \end{array} \right. , \tag{1.14}$$

where $A = -3 - 2\omega_2$ and $B = -3 - 2\omega_1$. Furthermore, note that for all $n \geq 0$ we have

$$Z_{n+2} = -Z_{n+1} - Z_n, \quad Z_0 = 2,\ Z_1 = -3. \tag{1.15}$$

From the Binet formulas (1.12), (1.13) and Eq. (1.14), we have

$$J_n^{(3)} = \frac{1}{7}\left(2^{n+1} - Z_n\right) \text{ and } j_n^{(3)} = \frac{1}{7}\left(2^{n+3} + 3Z_n\right). \tag{1.16}$$

A systematic investigation of the incomplete generalized Jacobsthal numbers and the incomplete generalized Jacobsthal–Lucas numbers was featured in [6]. In [7], Djordjević and Srivastava introduced the generalized incomplete Fibonacci polynomials and the generalized incomplete Lucas polynomials. In [8], the authors investigated some properties and relations involving generalizations of the Fibonacci numbers. In [10], Raina and Srivastava investigated the a new class of numbers associated with the Lucas numbers. Moreover they gave several interesting properties of these numbers.

In this paper, we introduce the third-order Jacobsthal polynomials and we give some properties, including the Binet-style formula and the generating functions for these sequences. Some identities involving these polynomials are also provided.

2. The Third-order Jacobsthal polynomial, Binet's formula and the generating function

The principal goals of this section will be to define the third-order Jacobsthal polynomial and to present some elementary results involving it.

For any variable quantity x such that $x^3 \neq 1$. We define the third-order Jacobsthal polynomial, denoted by $\{J_n^{(3)}(x)\}_{n\geq 0}$. This sequence is defined recursively by

$$J_{n+3}^{(3)}(x) = (x-1)J_{n+2}^{(3)}(x) + (x-1)J_{n+1}^{(3)}(x) + xJ_n^{(3)}(x), \quad n \geq 0, \tag{2.1}$$

with initial conditions $J_0^{(3)}(x) = 0$, $J_1^{(3)}(x) = 1$ and $J_2^{(3)}(x) = x-1$.

In order to find the generating function for the third-order Jacobsthal polynomial, we shall write the sequence as a power series where each term of the sequence correspond to coefficients of the series. As a consequence of the definition of generating function of a sequence, the generating function associated to $\{J_n^{(3)}(x)\}_{n\geq 0}$, denoted by $\{j(t)\}$, is defined by

$$j(t) = \sum_{n\geq 0} J_n^{(3)}(x)t^n.$$

Consequently, we obtain the following result:

Theorem 1. *The generating function for the third-order Jacobsthal polynomials* $\{J_n^{(3)}(x)\}_{n\geq 0}$ *is* $j(t) = \frac{t}{1-(x-1)t-(x-1)t^2-xt^3}$.

Proof. Using the definition of generating function, we have

$$j(t) = J_0^{(3)}(x) + J_1^{(3)}(x)t + J_2^{(3)}(x)t^2 + \cdots + J_n^{(3)}(x)t^n + \cdots.$$

Multiplying both sides of this identity by $-(x-1)t$, $-(x-1)t^2$ and by $-xt^3$, and then from Eq. (2.1), we have

$$
\begin{aligned}
&(1-(x-1)t-(x-1)t^2-xt^3)j(t)\\
&=J_0^{(3)}(x)+(J_1^{(3)}(x)-(x-1)J_0^{(3)}(x))t+(J_2^{(3)}(x)-(x-1)J_1^{(3)}-(x-1)J_0^{(3)}(x))t^2
\end{aligned}
\tag{2.2}
$$

and the result follows. □

The following result gives the Binet-style formula for $J_n^{(3)}(x)$.

Theorem 2. *For $n \geq 0$, we have*

$$
J_n^{(3)}(x)=\frac{x^{n+1}}{x^2+x+1}-\frac{\omega_1^{n+1}}{(x-\omega_1)(\omega_1-\omega_2)}+\frac{\omega_2^{n+1}}{(x-\omega_2)(\omega_1-\omega_2)},
$$

where ω_1, ω_2 are the roots of the characteristic equation associated with the respective recurrence relations $\lambda^2+\lambda+1=0$.

Proof. Since the characteristic equation has three distinct roots, the sequence $J_n^{(3)}(x)=a(x)x^n+b(x)\omega_1^n+c(x)\omega_2^n$ is the solution of the Eq. (2.1). Considering $n=0,1,2$ in this identity and solving this system of linear equations, we obtain a unique value for $a(x)$, $b(x)$ and $c(x)$, which are, in this case, $(x^2+x+1)a(x)=x$, $(x-\omega_1)(\omega_1-\omega_2)b(x)=-\omega_1$ and $(x-\omega_2)(\omega_1-\omega_2)c(x)=\omega_2$. So, using these values in the expression of $J_n^{(3)}(x)$ stated before, we get the required result. □

We define the modified third-order Jacobsthal polynomial sequence, denoted by $\{K_n^{(3)}(x)\}_{n\geq 0}$. This sequence is defined recursively by

$$
K_{n+3}^{(3)}(x)=(x-1)K_{n+2}^{(3)}(x)+(x-1)K_{n+1}^{(3)}(x)+xK_n^{(3)}(x), \tag{2.3}
$$

with initial conditions $K_0^{(3)}(x)=3$, $K_1^{(3)}(x)=x-1$ and $K_2^{(3)}(x)=x^2-1$.

We give their versions for the third-order Jacobsthal and modified third-order Jacobsthal polynomials.

For simplicity of notation, let

$$
\begin{aligned}
Z_n(x)&=\frac{1}{\omega_1-\omega_2}\left((x-\omega_2)\omega_1^{n+1}-(x-\omega_1)\omega_2^{n+1}\right),\\
Y_n&=\omega_1^n+\omega_2^n.
\end{aligned}
\tag{2.4}
$$

Then, we can write

$$
J_n^{(3)}(x)=\frac{1}{x^2+x+1}\left(x^{n+1}-Z_n(x)\right)
$$

and

$$
K_n^{(3)}(x)=x^n+Y_n.
$$

Then, $Z_n(x)=-Z_{n-1}(x)-Z_{n-2}(x)$, $Z_0(x)=x$ and $Z_1(x)=-(x+1)$.

Furthermore, we easily obtain the identities stated in the following result:

Proposition 1. *For a natural number n and m, if $J_n^{(3)}(x)$ and $K_n^{(3)}(x)$ are, respectively, the n-th third-order Jacobsthal and modified third-order Jacobsthal polynomials, then the following identities are true:*

$$K_n^{(3)}(x)=(x-1)J_n^{(3)}(x)+2(x-1)J_{n-1}^{(3)}(x)+3xJ_{n-2}^{(3)}(x),\ n\geq 2, \tag{2.5}$$

$$\begin{aligned} &J_n^{(3)}(x)J_m^{(3)}(x)+J_{n+1}^{(3)}(x)J_{m+1}^{(3)}(x)+J_{n+2}^{(3)}(x)J_{m+2}^{(3)}(x)\\ &=\frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c} (1+x^2+x^4)\cdot x^{n+m+2}\\ -x^{n+1}\left((1-x^2)Z_m(x)+x(1-x)Z_{m+1}(x)\right)\\ -x^{m+1}\left((1-x^2)Z_n(x)+x(1-x)Z_{n+1}(x)\right)\\ +(x^2+x+1)(\omega_1^n\omega_2^m+\omega_1^m\omega_2^n) \end{array}\right\}, \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\left(J_n^{(3)}(x)\right)^2+\left(J_{n+1}^{(3)}(x)\right)^2+\left(J_{n+2}^{(3)}(x)\right)^2\\ &=\frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c} (1+x^2+x^4)\cdot x^{2n+2}\\ -2x^{n+1}\left((1-x^2)Z_n(x)+x(1-x)Z_{n+1}(x)\right)\\ +2(x^2+x+1) \end{array}\right\}, \end{aligned} \tag{2.7}$$

and $Z_n(x)$ as in Eq. (2.4).

Proof. (2.5): To prove Eq. (2.5), we use induction on n. Let $n=2$, we get

$$\begin{aligned} (x-1)J_2^{(3)}(x)+2(x-1)J_1^{(3)}(x)+3xJ_0^{(3)}(x)&=(x-1)(x-1)+2(x-1)\\ &=x^2-1\\ &=K_2^{(3)}(x). \end{aligned}$$

Let us assume that $K_m^{(3)}(x)=(x-1)J_m^{(3)}(x)+2(x-1)J_{m-1}^{(3)}(x)+3xJ_{m-2}^{(3)}(x)$ is true for all values m less than or equal $n\geq 2$. Then,

$$\begin{aligned} K_{m+1}^{(3)}(x)&=(x-1)K_m^{(3)}(x)+(x-1)K_{m-1}^{(3)}(x)+xK_{m-2}^{(3)}(x)\\ &=(x-1)\left((x-1)J_m^{(3)}(x)+2(x-1)J_{m-1}^{(3)}(x)+3xJ_{m-2}^{(3)}(x)\right)\\ &\quad+(x-1)\left((x-1)J_{m-1}^{(3)}(x)+2(x-1)J_{m-2}^{(3)}(x)+3xJ_{m-3}^{(3)}(x)\right)\\ &\quad+x\left((x-1)J_{m-2}^{(3)}(x)+2(x-1)J_{m-3}^{(3)}(x)+3xJ_{m-4}^{(3)}(x)\right)\\ &=(x-1)J_{m+1}^{(3)}(x)+2(x-1)J_m^{(3)}(x)+3xJ_{m-1}^{(3)}(x). \end{aligned}$$

(2.6): Using the Binet formula of $J_n^{(3)}(x)$ in Theorem 2, we have

$$\begin{aligned} &J_n^{(3)}(x)J_m^{(3)}(x)+J_{n+1}^{(3)}(x)J_{m+1}^{(3)}(x)+J_{n+2}^{(3)}(x)J_{m+2}^{(3)}(x)\\ &\qquad=\frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c} \left(x^{n+1}-Z_n(x)\right)\left(x^{m+1}-Z_m(x)\right)\\ +\left(x^{n+2}-Z_{n+1}(x)\right)\left(x^{m+2}-Z_{m+1}(x)\right)\\ +\left(x^{n+3}-Z_{n+2}(x)\right)\left(x^{m+3}-Z_{m+2}(x)\right) \end{array}\right\}. \end{aligned}$$

Then, we obtain

$$\begin{aligned}
&J_n^{(3)}(x)J_m^{(3)}(x)+J_{n+1}^{(3)}(x)J_{m+1}^{(3)}(x)+J_{n+2}^{(3)}(x)J_{m+2}^{(3)}(x)\\
&=\frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c}(1+x^2+x^4)\cdot x^{n+m+2}\\ -x^{n+1}\left(Z_m(x)+xZ_{m+1}(x)+x^2Z_{m+2}(x)\right)\\ -x^{m+1}\left(Z_n(x)+xZ_{n+1}(x)+x^2Z_{n+2}(x)\right)\\ +Z_n(x)Z_m(x)+Z_{n+1}(x)Z_{m+1}(x)+Z_{n+2}(x)Z_{m+2}(x)\end{array}\right\}\\
&=\frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c}(1+x^2+x^4)\cdot x^{n+m+2}\\ -x^{n+1}\left((1-x^2)Z_m(x)+x(1-x)Z_{m+1}(x)\right)\\ -x^{m+1}\left((1-x^2)Z_n(x)+x(1-x)Z_{n+1}(x)\right)\\ +(x^2+x+1)(\omega_1^n\omega_2^m+\omega_1^m\omega_2^n)\end{array}\right\}.
\end{aligned}$$

Then, we obtain the Eq. (2.7) if $m=n$ in Eq. (2.6). □

3. Some identities involving this type of polynomials

In this section, we state some identities related with these type of third-order polynomials. As a consequence of the Binet formula of Theorem 2, we get for this sequence the following interesting identities.

Proposition 2 (Catalan-like identity)**.** *For a natural numbers n, s, with $n\geq s$, if $J_n^{(3)}(x)$ is the n-th third-order Jacobsthal polynomials, then the following identity*

$$\begin{aligned}
&J_{n+s}^{(3)}(x)J_{n-s}^{(3)}(x)-\left(J_n^{(3)}(x)\right)^2\\
&=\frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c}x^{n+1}\left(x^s-x^{-s}\right)X_sZ_{n+1}(x)\\ -x^{n+1}\left(2+x^sX_{s+1}-x^{-s}X_{s-1}\right)Z_n(x)\\ -(x^2+x+1)X_s^2\end{array}\right\}
\end{aligned}$$

is true, where $Z_n(x)$ as in Eq. (2.4), $X_n=\frac{\omega_1^n-\omega_2^n}{\omega_1-\omega_2}$ and ω_1, ω_2 are the roots of the characteristic equation associated with the recurrence relation $x^2+x+1=0$.

Proof. Using the Eq. (2.4) and the Binet formula of $J_n^{(3)}(x)$ in Theorem 2, we have

$$\begin{aligned}
&J_{n+s}^{(3)}(x)J_{n-s}^{(3)}(x)-\left(J_n^{(3)}(x)\right)^2\\
&=\frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c}\left(x^{n+s+1}-Z_{n+s}(x)\right)\left(x^{n-s+1}-Z_{n-s}(x)\right)\\ -\left(x^{n+1}-Z_n(x)\right)^2\end{array}\right\}\\
&=\frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c}-x^{n+1}\left(x^sZ_{n-s}(x)+x^{-s}Z_{n+s}(x)-2Z_n(x)\right)\\ +Z_{n+s}(x)Z_{n-s}(x)-(Z_n(x))^2\end{array}\right\}.
\end{aligned}$$

Using the following identity for the sequence $Z_n(x)$:

$$Z_{n+s}(x)=X_sZ_{n+1}(x)-X_{s-1}Z_n(x),$$

where $X_s = \frac{\omega_1^s - \omega_2^s}{\omega_1 - \omega_2}$ and $X_{-s} = -X_s$. Then, we obtain

$$J_{n+s}^{(3)}(x)J_{n-s}^{(3)}(x) - \left(J_n^{(3)}(x)\right)^2$$
$$= \frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c} x^{n+1}\left(x^s - x^{-s}\right)X_s Z_{n+1}(x) \\ -x^{n+1}\left(x^s X_{s+1} - x^{-s}X_{s-1} - 2\right)Z_n(x) \\ -(x^2+x+1)X_s^2 \end{array}\right\}.$$

Hence the result holds. □

Note that for $s = 1$ in the Catalan-like identity obtained, we get the Cassini-like identity for the third-order Jacobsthal polynomial. Furthermore, for $s = 1$, the identity stated in Proposition 2, yields

$$J_{n+1}^{(3)}(x)J_{n-1}^{(3)}(x) - \left(J_n^{(3)}(x)\right)^2$$
$$= \frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c} x^{n+1}\left(x^1 - x^{-1}\right)X_1 Z_{n+1}(x) \\ -x^{n+1}\left(x^1 X_{1+1} - x^{-1}X_{1-1} - 2\right)Z_n(x) \\ -(x^2+x+1) \end{array}\right\}.$$

and using $X_0 = 0$ and $X_1 = 1$ in Proposition 2, we obtain the following result.

Proposition 3 (Cassini-like identity)**.** *For a natural numbers n, if $K_n^{(3)}$ is the n-th third-order Jacobsthal numbers, then the identity*

$$J_{n+1}^{(3)}(x)J_{n-1}^{(3)}(x) - \left(J_n^{(3)}(x)\right)^2$$
$$= \frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c} x^n\left((x^2-1)Z_{n+1}(x) + x(x+2)Z_n(x)\right) \\ -(x^2+x+1) \end{array}\right\}.$$

is true.

The d'Ocagne-like identity can also be obtained using the Binet formula and in this case we obtain

Proposition 4 (d'Ocagne-like identity)**.** *For a natural numbers m, n, with $m \geq n$ and $J_n^{(3)}(x)$ is the n-th third-order Jacobsthal polynomial, then the following identity*

$$J_{m+1}^{(3)}(x)J_n^{(3)}(x) - J_m^{(3)}(x)J_{n+1}^{(3)}(x)$$
$$= \frac{1}{(x^2+x+1)^2}\left\{\begin{array}{c} x^{m+1}\left(Z_{n+1}(x) - xZ_n(x)\right) \\ -x^{n+1}\left(Z_{m+1}(x) - xZ_m(x)\right) + (x^2+x+1)X_{m-n} \end{array}\right\}$$

is true.

Proof. Using the Eq. (2.4) and the Theorem 2, we get the required result. □

In addition, some formulae involving sums of terms of the third-order Jacobsthal polynomial sequence will be provided in the following proposition.

Proposition 5. *For a natural numbers m, n, with $n \geq m$, if $J_n^{(3)}(x)$ and $K_n^{(3)}(x)$ are, respectively, the n-th third-order Jacobsthal and modified third-order Jacobsthal polynomials, then the following identities are true:*

$$\sum_{s=m}^{n} J_s^{(3)}(x) = \frac{1}{3(x-1)} \left\{ \begin{array}{c} (3x-2)J_n^{(3)}(x) + (2x-1)J_{n-1}^{(3)}(x) \\ +xJ_{n-2}^{(3)}(x) - J_{m+2}^{(3)}(x) \\ +(x-2)J_{m+1}^{(3)}(x) + (2x-3)J_m^{(3)}(x) \end{array} \right\}, \tag{3.1}$$

$$\sum_{s=0}^{n} K_s^{(3)}(x) = \frac{1}{x-1} \left\{ \begin{array}{ccc} x^{n+1}+2x-3 & \text{if} & n \equiv 0 \pmod 3 \\ x^{n+1}+x-2 & \text{if} & n \equiv 1 \pmod 3 \\ x^{n+1}-1 & \text{if} & n \equiv 2 \pmod 3 \end{array} \right. . \tag{3.2}$$

Proof. (3.1): Using Eq. (2.1), we obtain

$$\begin{aligned} \sum_{s=m}^{n} J_s^{(3)}(x) &= J_m^{(3)}(x) + J_{m+1}^{(3)}(x) + J_{m+2}^{(3)}(x) + \sum_{s=m+3}^{n} J_s^{(3)}(x) \\ &= J_m^{(3)}(x) + J_{m+1}^{(3)}(x) + J_{m+2}^{(3)}(x) + (x-1)\sum_{s=m+2}^{n-1} J_s^{(3)}(x) \\ &\quad + (x-1)\sum_{s=m+1}^{n-2} J_s^{(3)}(x) + x\sum_{s=m}^{n-3} J_s^{(3)}(x) \end{aligned}$$

Then,

$$\begin{aligned} \sum_{s=m}^{n} J_s^{(3)}(x) &= (3x-2)\sum_{s=m}^{n} J_s^{(3)}(x) + J_{m+2}^{(3)}(x) - (x-2)J_{m+1}^{(3)}(x) - (2x-3)J_m^{(3)}(x) \\ &\quad - (3x-2)J_n^{(3)}(x) - (2x-1)J_{n-1}^{(3)}(x) - xJ_{n-2}^{(3)}(x). \end{aligned}$$

Finally, the result in Eq. (3.1) is completed.

(3.2): As a consequence of the Eq. (2.4) of Theorem 2 and

$$\begin{aligned} \sum_{s=0}^{n} Y_s &= \sum_{s=0}^{n} (\omega_1^s + \omega_2^s) \\ &= \frac{\omega_1^{n+1}-1}{\omega_1 - 1} + \frac{\omega_2^{n+1}-1}{\omega_2 - 1} \\ &= \frac{1}{3}(Y_n - Y_{n+1}) + 1, \end{aligned}$$

we have

$$\begin{aligned} \sum_{s=0}^{n} K_s^{(3)}(x) &= \sum_{s=0}^{n} x^s + \sum_{s=0}^{n} Y_s \\ &= \frac{x^{n+1}-1}{x-1} + \frac{1}{3}(Y_n - Y_{n+1}) + 1 \end{aligned}$$

$$= \frac{1}{x-1} \begin{cases} x^{n+1} + 2x - 3 & \text{if} \quad n \equiv 0 \pmod 3 \\ x^{n+1} + x - 2 & \text{if} \quad n \equiv 1 \pmod 3 \\ x^{n+1} - 1 & \text{if} \quad n \equiv 2 \pmod 3 \end{cases} .$$

Hence, we obtain the result. □

For example, if $n \equiv 0 (\bmod 3)$ we have that $x^{n+1} + 2x - 3$ is divisible by $x - 1$.

For negative subscripts terms of the sequence of modified third-order Jacobsthal polynomial we can establish the following result:

Proposition 6. *For a natural number n and $x^3 \neq 0$ the following identities are true:*

$$K_{-n}^{(3)}(x) = K_n^{(3)}(x) + x^{-n} - x^n, \tag{3.3}$$

$$\sum_{s=0}^{3n} K_{-s}^{(3)}(x) = \frac{1}{x-1}\left(3x - 2 - x^{-3n}\right). \tag{3.4}$$

Proof. (3.3): Since $Y_{-n} = Y_n$, using the Binet formula stated in Theorem 2 and the fact that $\omega_1\omega_2 = 1$, all the results of this Proposition follow. In fact,

$$\begin{aligned} K_{-n}^{(3)}(x) &= x^{-n} + Y_{-n} \\ &= x^{-n} + x^n + Y_n - x^n \\ &= K_n^{(3)}(x) + x^{-n} - x^n. \end{aligned}$$

So, the proof is completed.

(3.4): The proof is similar to the proof of Eq. (3.1) using Eq. (3.3). □

4. Conclusion

Sequences of polynomials have been studied over several years, including the well-known Tribonacci polynomial and, consequently, on the Tribonacci-Lucas polynomial. In this paper, we have also contributed for the study of third-order Jacobsthal and modified third-order Jacobsthal polynomials, deducing some formulae for the sums of such polynomials, presenting the generating functions and their Binet-style formula. It is our intention to continue the study of this type of sequences, exploring some their applications in the science domain. For example, a new type of sequences in the quaternion algebra with the use of these polynomials and their combinatorial properties.

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Author's address

Gamaliel Cerda-Morales

Instituto de Matemáticas, Pontificia Universidad Católica de Valparaíso, Blanco Viel 596, Valparaíso, Chile

E-mail address: gamaliel.cerda.m@mail.pucv .cl