Miskolc Mathematical Notes
Vol. 22 (2021), No. 1, pp. 435–441

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2021.3222

$G-$SUPPLEMENTED LATTICES

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Received 9 February, 2020

Abstract. In this work, g-supplemented lattices are defined and some properties of these lattices are investigated. g-small submodules and g-supplemented modules are generalized to lattices. Let L be a lattice and $1 = a_1 \vee a_2 \vee ... \vee a_n$ with $a_i \in L$ $(1 \leq i \leq n)$. If $a_i/0$ is g-supplemented for every $i = 1,2,...,n$, then L is also g-supplemented. If L is g-supplemented, then $1/a$ is also g-supplemented for every $a \in L$. It is also defined the g-radical of a lattice L and it is shown that if L is g-supplemented, then $1/r_g(L)$ is complemented.

2010 *Mathematics Subject Classification:* 06C05, 06C15

Keywords: Lattices, Essential Elements, Small Elements, Supplemented Lattices

1. INTRODUCTION

In this paper, every lattice is complete modular lattice with the smallest element 0 and the greatest element 1. Let L be a lattice, $x, y \in L$ and $x \leq y$. A sublattice $\{a \in L | x \leq a \leq y\}$ is called a *quotient sublattice* and denoted by y/x. An element y of a lattice L is called a *complement* of x in L if $x \wedge y = 0$ and $x \vee y = 1$, this case we denote $1 = x \oplus y$ (in this case we call x and y are *direct summands* of L). L is said to be *complemented* if each element has at least one complement in L. An element x of L is said to be *small* or *superfluous* and denoted by $x \ll L$ if $y = 1$ for every $y \in L$ such that $x \vee y = 1$. The meet of all maximal $(\neq 1)$ elements of a lattice L is called the *radical* of L and denoted by $r(L)$. An element a of L is called a *supplement* of b in L if it is minimal for $a \vee b = 1$. a is a supplement of b in a lattice L if and only if $a \vee b = 1$ and $a \wedge b \ll a/0$. A lattice L is called a *supplemented lattice* if every element of L has a supplement in L. We say that an element y of L *lies above* an element x of L if $x \leq y$ and $y \ll 1/x$. L is said to be *hollow* if every element distinct from 1 is superfluous in L, and L is said to be *local* if L has the greatest element $(\neq 1)$. We say an element $x \in L$ has *ample supplements* in L if for every $y \in L$ with $x \vee y = 1$, x has a supplement z in L with $z \leq y$. L is said to be *amply supplemented* if every element of L has ample supplements in L. It is clear that every amply supplemented lattice is supplemented. Let L be a lattice and $k \in L$. If $t = 0$ for very $t \in L$ with $k \wedge t = 0$, then k is called an *essential element* of L and denoted by $k \trianglelefteq L$.

More informations about (amply) supplemented lattices are in [1, 2, 5, 9]. More results about (amply) supplemented modules are in [8, 12].

Definition 1. Let L be a lattice and $a \in L$. If $b = 1$ for every $b \trianglelefteq L$ with $a \vee b = 1$, then a is called a generalized small (briefly, g-small) element of L and denoted by $a \ll_g L$.

It is clear that every small element is g-small, but the converse is not true in general (See Example 1 and Example 2).

G-small elements generalize g-small submodules. G-small submodules are studied in [6, 7, 11].

Lemma 1. *Let L be a lattice and $a, b, c, d \in L$. Then the followings are hold.*

(i) *If $a \leq b$ and $b \ll_g L$, then $a \ll_g L$.*
(ii) *If $a \ll_g b/0$, then $a \ll_g t/0$ for every $t \in L$ with $b \leq t$.*
(iii) *If $a \ll_g L$, then $a \vee b \ll_g 1/b$.*
(iv) *If $a \ll_g b/0$ and $c \ll_g d/0$, then $a \vee c \ll_g (b \vee d)/0$.*

Proof. (i) Let $a \vee k = 1$ with $k \trianglelefteq L$. Since $a \leq b$, $b \vee k = 1$ and since $b \ll_g L$, $k = 1$. Hence $a \ll_g L$ as desired.

(ii) Let $t \in L$ with $b \leq t$ and let $a \vee k = t$ with $k \trianglelefteq t/0$. Here $k \wedge b \trianglelefteq b/0$. Since $a \leq b$, by modularity, $b = b \wedge t = b \wedge (a \vee k) = a \vee (k \wedge b)$ and since $a \ll_g b/0$, $k \wedge b = b$ and $b \leq k$. Hence $a \leq k$ and $t = a \vee k = k$. Therefore, $a \ll_g t/0$.

(iii) Let $a \vee b \vee k = 1$ with $k \trianglelefteq 1/b$. Since $k \trianglelefteq 1/b$, we can easily see that $k \trianglelefteq L$. Since $1 = a \vee b \vee k = a \vee k$ and $a \ll_g L$, $k = 1$. Hence $a \vee b \ll_g 1/b$ as desired.

(iv) Let $a \vee c \vee k = b \vee d$ with $k \trianglelefteq (b \vee d)/0$. By (ii) $a \ll_g (b \vee d)/0$ and $c \ll_g (b \vee d)/0$. Since $a \ll_g (b \vee d)/0$ and $c \vee k \trianglelefteq (b \vee d)/0, c \vee k = b \vee d$ and since $c \ll_g (b \vee d)/0$, $k = b \vee d$. Hence $a \vee c \ll_g (b \vee d)/0$ as desired. □

Corollary 1. *If $a_i \ll_g b_i/0$ for $a_i, b_i \in L$ $(i = 1, 2, \dots, n)$, then $a_1 \vee a_2 \vee \dots \vee a_n \ll_g (b_1 \vee b_2 \vee \dots \vee b_n)/0$.*

Proof. Clear from Lemma 1(iv). □

Corollary 2. *Let $a, b \in L$ and $a \leq b$. If $b \ll_g L$, then $b \ll_g 1/a$.*

Proof. Clear from Lemma 1(iii). □

2. G-SUPPLEMENTED LATTICES

Definition 2. Let L be a lattice and $a, b \in L$. If $1 = a \vee b$ and $1 = a \vee t$ with $t \trianglelefteq b/0$ implies that $t = b$, then b is called a g-supplement of a in L. If every element of L has a g-supplement in L, then L is called a g-supplemented lattice.

G-supplemented lattices generalize g-supplemented modules. G-supplemented modules are studied in [7]. Every supplemented lattice is g-supplemented. Hollow and local lattices are g-supplemented.

Lemma 2. *Let L be a lattice and $a,b \in L$. Then b is a g-supplement of a in L if and only if* $1 = a \vee b$ *and* $a \wedge b \ll_g b/0$.

Proof. ($\Longrightarrow$) Let $(a \wedge b) \vee k = b$ with $k \trianglelefteq b/0$. Then $1 = a \vee b = a \vee (a \wedge b) \vee k = a \vee k$ hold. Since b is a g-supplement of a in L and $k \trianglelefteq b/0$, by definition, $k = b$. Hence $a \wedge b \ll_g b/0$ as desired.

($\Longleftarrow$) Let $1 = a \vee t$ with $t \trianglelefteq b/0$. Since $t \leq b$, by modularity, $b = b \wedge 1 = b \wedge (a \vee t) = (a \wedge b) \vee t$. Since $a \wedge b \ll_g b/0$, $t = b$. Hence b is a g-supplement of a in L as desired. □

Lemma 3. *Let L be a lattice and $a,b \in L$. If $a \vee b$ has a g-supplement x in L and $(a \vee x) \wedge b$ has a g-supplement y in $b/0$, then $x \vee y$ is a g-supplement of a in L.*

Proof. Since x is a g-supplement of $a \vee b$ in L, by Lemma 2,

$$1 = a \vee b \vee x \text{ and } (a \vee b) \wedge x \ll_g x/0.$$

Since y is a g-supplement of $(a \vee x) \wedge b$ in $b/0$, by Lemma 2,

$$b = ((a \vee x) \wedge b) \vee y$$

and

$$(a \vee x) \wedge y = (a \vee x) \wedge b \wedge y \ll_g y/0.$$

Then

$$1 = a \vee b \vee x = a \vee x \vee ((a \vee x) \wedge b) \vee y = a \vee x \vee y$$

and by Lemma 1,

$$\begin{aligned} a \wedge (x \vee y) &\leq ((a \vee x) \wedge y) \vee ((a \vee y) \wedge x) \\ &\leq ((a \vee x) \wedge y) \vee ((a \vee b) \wedge x) \ll_g (x \vee y)/0. \end{aligned}$$

Hence $x \vee y$ is a g-supplement of a in L. □

Corollary 3. *Let L be a lattice and $a,b \in L$. If $a \vee b$ has a g-supplement in L and $b/0$ is g-supplemented, then a has a g-supplement in L.*

Proof. Clear from Lemma 3. □

Lemma 4. *Let $1 = a \vee b$ with $a,b \in L$. If $a/0$ and $b/0$ are g-supplemented, then L is also g-supplemented.*

Proof. Let x be any element of L. Then 0 is a g-supplement of $x \vee a \vee b$ in L and since $b/0$ is g-supplemented, by Corollary 3, $x \vee a$ has a g-supplement in L. Since $a/0$ is g-supplemented, again by Corollary 3, x has a g-supplement in L. Hence L is g-supplemented. □

Corollary 4. *Let $1 = a_1 \vee a_2 \vee \ldots \vee a_n$ with $a_i \in L$ $(1 \leq i \leq n)$. If $a_i/0$ is g-supplemented for every $i = 1,2,\ldots,n$, then L is also g-supplemented.*

Proof. Clear from Lemma 4. □

Lemma 5. *Let L be a lattice and $a,b,c \in L$ with $c \leq a$. If b is a g-supplement of a in L, then $b \vee c$ is a g-supplement of a in $1/c$.*

Proof. Since b is a g-supplement of a in L, $1 = a \vee b$ and $a \wedge b \ll_g b/0$. Since $a \wedge b \ll_g b/0$, by Lemma 1 (ii),

$$a \wedge b \ll_g (b \vee c)/0$$

and by Lemma 1 (iii),

$$(a \wedge b) \vee c \ll_g (b \vee c)/c.$$

Hence $1 = a \vee b = a \vee b \vee c$ and $a \wedge (b \vee c) = (a \wedge b) \vee c \ll_g (b \vee c)/c$ and $b \vee c$ is a g-supplement of a in $1/c$. □

Corollary 5. *Let L be a g-supplemented lattice. Then $1/a$ is g-supplemented for every $a \in L$.*

Proof. Clear from Lemma 5. □

Definition 3. Let L be a lattice and t be a maximal $(\neq 1)$ element of L. If $t \trianglelefteq L$, then t is called a g-maximal element of L. The meet of all g-maximal elements of L is called the g-radical of L and denoted by $r_g(L)$. If L have not any g-maximal elements, then we call $r_g(L) = 1$.

Corollary 6. *Let L be a lattice. Then $r(L) \leq r_g(L)$.*

Proof. Clear from definitions. □

Lemma 6. *Let L be a lattice and $a \in L$. If $a \ll_g L$, then $a \leq r_g(L)$.*

Proof. Assume $a \nleq r_g(L)$. Then there exists a g-maximal element t of L with $a \nleq t$. Since t is maximal $(\neq 1)$ and $a \nleq t$, $a \vee t = 1$ and since $a \ll_g L$ and $t \trianglelefteq L$, $t = 1$. This is contradiction. Hence $a \leq r_g(L)$ as desired. □

Lemma 7. *Let L be a lattice and $a \in L$. Then $r_g(a/0) \leq r_g(L)$.*

Proof. Let t be any g-maximal element of L. If $a \leq t$, then $r_g(a/0) \leq t$. If $a \nleq t$, we can easily see that $a \wedge t$ is a g-maximal element of $a/0$ and hence $r_g(a/0) \leq t$. Therefore, $r_g(a/0) \leq r_g(L)$. □

Lemma 8. *Let L be a g-supplemented lattice. Then $1/r_g(L)$ is complemented.*

Proof. Let x be any element of $1/r_g(L)$. Since L is g-supplemented, x has a g-supplement y in L. Here $1 = x \vee y$ and $x \wedge y \ll_g y/0$. Since $x \wedge y \ll_g y/0$, by Lemma 6 and Lemma 7, $x \wedge y \leq r_g(y/0) \leq r_g(L)$. Hence $1 = x \vee y \vee r_g(L)$ and

$$x \wedge (y \vee r_g(L)) = (x \wedge y) \vee r_g(L) = r_g(L).$$

Therefore, $y \vee r_g(L)$ is a complement of x in $1/r_g(L)$ and $1/r_g(L)$ is complemented. □

Let $x, y \in L$. It is defined a relation β_* on the elements of L by $x\beta_* y$ if and only if for every $t \in L$ with $1 = x \vee t$ then $1 = y \vee t$ and for every $k \in L$ with $1 = y \vee k$ then $1 = x \vee k$. (See [10, Definition 1]. More informations about β_* relation are in [10]. More informations about β^* relation on modules are in [4].

Corollary 7. *Let L be a g-supplemented lattice. Then $1/r_g(L)$ is $\oplus-$supplemented.*

Proof. Clear from [3, Definition 1] and Lemma 8. □

Lemma 9. *Let L be a lattice and $a\beta_* b$ in L. If a and b have g-supplements in L, then they have the same g-supplements in L.*

Proof. Let x be a g-supplement of a in L. Then $1 = a \vee x$ and since $a\beta_* b$, we have $1 = b \vee x$. Let $1 = b \vee t$ with $t \trianglelefteq x/0$. Since $a\beta_* b$, we have $1 = a \vee t$ and since x is a g-supplement of a in L, we have $t = x$. Hence x is a g-supplement of b in L. Similarly, interchanging the roles of a and b we can prove that each g-supplement of b in L is also a g-supplement of a in L. □

Corollary 8. *Let L be a lattice and a lies above b in L. If a and b have g-supplements in L, then they have the same g-supplements in L.*

Proof. By [10, Theorem 3], $a\beta_* b$ and by Lemma 9, the desired is obtained. □

Lemma 10. *Let L be a lattice and $t \ll_g x/0$ for every g-supplement element x in L and for every $t \ll_g L$ with $t \leq x$. If every element of L is β_* equivalent to a g-supplement element in L, then L is g-supplemented.*

Proof. Let $a \in L$. By hypothesis, there exists a g-supplement element x in L such that $a\beta_* x$. Let x be a g-supplement of b in L. By hypothesis, there exists a g-supplement element y in L with $b\beta_* y$. By Lemma 9, x is a g-supplement of y in L. Here $1 = x \vee y$ and $x \wedge y \ll_g x/0$. Since $x \wedge y \ll_g L$ and y is a g-supplement element in L, by hypothesis, $x \wedge y \ll_g y/0$. Then by Lemma 2, y is a g-supplement of x in L. Since $a\beta_* x$, by Lemma 9, y is a g-supplement of a in L. Hence L is g-supplemented. □

Corollary 9. *Let L be a lattice and $t \ll_g x/0$ for every g-supplement element x in L and for every $t \ll_g L$ with $t \leq x$. If every element of L lies above a g-supplement element in L, then L is g-supplemented.*

Proof. Clear from [10, Theorem 3] and Lemma 10. □

Definition 4. Let L be a lattice. If every element of L with distinct from 1 is g-small in L, then L is called a g-hollow lattice.

Clearly we can see that every hollow lattice is g-hollow. But the converse is not true in general (See Example 2).

Proposition 1. *Every g-hollow lattice is g-supplemented.*

Proof. Let L be a g-hollow lattice. Then 1 is a g-supplement of every element of L with distinct from 1 and 0 is a g-supplement of 1 in L. Hence L is g-supplemented. □

Proposition 2. *Let L be a lattice with $r_g(L) \neq 1$. The following conditions are equivalent.*

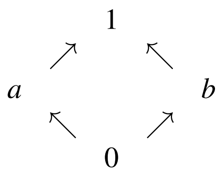
(i) *L is g-hollow.*
(ii) *L is local.*
(iii) *L is hollow.*

Proof. $(i) \Longrightarrow (ii)$ Let $x \in L$ and $x \neq 1$. Since L is g-hollow, $x \ll_g L$ and by Lemma 6, $x \leq r_g(L)$. By hypothesis, $r_g(L) \neq 1$. Hence $r_g(L)$ is the greatest element $(\neq 1)$ of L and L is local.

$(ii) \Longrightarrow (iii)$ and $(iii) \Longrightarrow (i)$ are clear. □

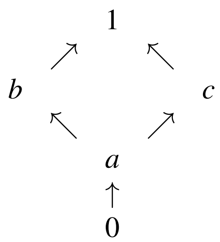
Example 1. Let L be a nonzero complemented lattice. Here $1 \ll_g L$, but not $1 \ll L$. 1 is a g-supplement of 1 in L, but 1 is not a supplement of 1 in L.

Example 2. Consider the lattice $L = \{0, a, b, 1\}$ given by the following diagram.



Then L is g-hollow but not hollow. Here $1 \ll_g L$, but not $1 \ll L$. 1 is a g-supplement of 1 in L, but 1 is not a supplement of 1 in L. Here also $r(L) = 0 \neq 1 = r_g(L)$ hold.

Example 3. Consider the lattice $L = \{0, a, b, c, 1\}$ given by the following diagram.



Then L is g-supplemented but not g-hollow.

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