



FIXED POINT THEOREMS UNDER LOCALLY T -TRANSITIVE BINARY RELATIONS EMPLOYING MATKOWSKI CONTRACTIONS

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Abstract. In this paper, we extend the relation-theoretic contraction principle due to Alam and Imdad (J. Fixed Point Theory Appl. 17 (2015) 693-702) for Matkowski contractions employing a locally T -transitive binary relation. Our results improve and enrich several fixed point theorems of the existing literature.

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1. INTRODUCTION

Banach contraction principle [8] is one of the most fruitful results in nonlinear analysis. Various noted generalizations of this core result are available in the existing literature. Role of non-negative constant $\alpha < 1$ in $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$ is that many authors generalized the Banach contraction principle by replacing the involved constant α with an appropriate mapping, say ϕ depending on the contractivity condition. A mapping $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfying $\phi(t) < t$ for each $t > 0$ is said to be control function. A self-mapping T defined on a metric space (X, d) is said to be a nonlinear contraction with respect to control function ϕ (or, in short, ϕ -contraction) if

$$d(Tx, Ty) \leq \phi(d(x, y)) \quad \forall x, y \in X.$$

In fact, the idea of ϕ -contraction initiated by Browder [10] in 1968, wherein author assumed ϕ to be right continuous and increasing control function and utilized the same to generalize the Banach contraction principle. Thereafter, many authors generalized Browder fixed point theorem by varying the properties of control function ϕ (see Boyd-Wong and Matkowski contractions [9, 16]. In 2004, Ran and Reurings [22] established a variant of Banach contraction principle in the setting of ordered metric spaces, which was further refined by Nieto and Rodríguez-López [19]. In this continuation, Agarwal et al. [1] extended the results of Nieto and Rodríguez-López for Matkowski type nonlinear contractions, which was later refined by O'Regan and

Petruşel [20]. In [20], authors enlarges the class of spaces by replacing class of metric spaces with the class of L -spaces. In [20], authors also proved the order-theoretic fixed point theorems in the context of L -spaces. Petruşel and Rus [21] further proved some generalized fixed point results in L -spaces as well as metric spaces endowed with partial ordered relations.

In 2015, Alam and Imdad [4] obtained a noted generalization of Banach contraction principle employing an amorphous (arbitrary) binary relation, which was further improved by the (same) authors to Boyd-Wong type nonlinear contractions patterned after [6]. For further generalizations of these lines, we refer [2, 3, 5, 6].

In this paper, we establish a variant of the Banach contraction principle for Matkowski type nonlinear contractions under the relaxed transitivity condition by employing locally T -transitive relation. In order to establish our results, we will utilize the relation-theoretic analogues of certain involved metrical notions such as: contraction, completeness, continuity, *etc.* Indeed, under the universal relation these newly defined notions reduce to their corresponding natural analogues. Our newly established results generalize and unify a multitude of corresponding results of the existing literature. Also, we furnish some examples to demonstrate the utility of our results over corresponding existing ones.

2. PRELIMINARIES

Given a nonempty set X , a subset \mathcal{R} of X^2 is called a binary relation on X . For simplicity, we sometimes write $x\mathcal{R}y$ instead of $(x, y) \in \mathcal{R}$. Given subset $E \subseteq X$ and a binary relation \mathcal{R} on X , the restriction of \mathcal{R} to E , denoted by $\mathcal{R}|_E$, are in fact $\mathcal{R} \cap E^2$. Indeed, $\mathcal{R}|_E$ is a relation on E induced by \mathcal{R} .

Out of various kind of binary relations, the following are relevant to our present discussion:

A binary relation \mathcal{R} defined on a nonempty set X is called

- “amorphous” if it has no specific property at all,
- “universal” if $\mathcal{R} = X^2$,
- “empty” if $\mathcal{R} = \emptyset$,
- “reflexive” if $(x, x) \in \mathcal{R} \ \forall x \in X$,
- “symmetric” if whenever $(x, y) \in \mathcal{R}$ implies $(y, x) \in \mathcal{R}$,
- “antisymmetric” if whenever $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$ imply $x = y$,
- “transitive” if whenever $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$ imply $(x, z) \in \mathcal{R}$,
- “complete” if $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R} \ \forall x, y \in X$,
- “partial order” if \mathcal{R} is reflexive, antisymmetric and transitive.

Throughout this paper, \mathcal{R} stands for a nonempty binary relation but for the sake of simplicity, we often write ‘binary relation’ instead of ‘nonempty binary relation’. Also, \mathbb{N} stands for the set of natural numbers, \mathbb{N}_0 for the set of whole numbers (*i.e.*, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$), \mathbb{Q} for the set of rational numbers and \mathbb{Q}^+ for set of positive rational numbers.

Definition 1. [12, 18, 25] Let X be a nonempty set equipped with partial order \preceq . A self-mapping T defined on X is called increasing (or isotone or order-preserving) if for any $x, y \in X$,

$$x \preceq y \Rightarrow T(x) \preceq T(y).$$

The following notion is formulated by using a suitable property with a view to relax the continuity requirement of the underlying mapping especially in the hypotheses of a fixed point theorem due to Nieto and Rodríguez-López [19].

Definition 2. [7] Let (X, d) be a metric space equipped with a partial order \preceq . We say that the triplet (X, d, \preceq) has “*ICU* (increasing-convergence-upper bound) property” if every increasing convergent sequence in X is bounded above by its limit (as an upper bound).

Definition 3. [16] A function $\phi : [0, \infty) \rightarrow [0, \infty)$ is called “comparison function” if

- (i) ϕ is increasing,
- (ii) $\lim_{n \rightarrow \infty} \phi^n(t) = 0 \forall t > 0$.

The class of control functions of Boyd and Wong [9] is given by:

$$\Omega = \left\{ \phi : [0, \infty) \rightarrow [0, \infty) : \phi(t) < t \text{ for each } t > 0 \text{ and } \limsup_{r \rightarrow t^+} \phi(r) < t \text{ for each } t > 0 \right\}.$$

Recall that ϕ -contractions via comparison functions are known as Matkowski contractions, later it will be shown that the comparison function is control function.

Notice that, the classes of Boyd-Wong and Matkowski contractions are independent. To differentiate this fact, consider the following two functions from $[0, \infty)$ to $[0, \infty)$ defined by:

$$\phi_1(t) = \begin{cases} 0, & \text{if } t = 0, \\ \frac{1}{n+1}, & \text{if } t \in \left(\frac{1}{n+1}, \frac{1}{n}\right], n = 1, 2, 3, \dots, \\ 1, & \text{if } t > 1, \end{cases}$$

$$\phi_2(t) = \begin{cases} \frac{t}{5}, & \text{if } t < 2, \\ \frac{1}{t}, & \text{if } t \geq 2. \end{cases}$$

Then ϕ_1 is a comparison function but not lie in Ω as it is not upper semi continuous from the right (see [16]). On the other hand, the decreasing function ϕ_2 is a member of Boyd-Wong class, but it is not comparison function.

It is worth mentioning here that Ćirić [11] proved that for a member $\phi \in \Omega$,

$$\lim_{n \rightarrow \infty} \phi^n(t) = 0, \forall t > 0.$$

Unfortunately, this fact is not true in general as mentioned in Jachymski [13].

Employing comparison functions, Agarwal et al. [1] proved the following:

Theorem 1. [1] Let (X, d) be a metric space endowed with a partial order \preceq and T a self-mapping on X . Suppose that the following conditions hold:

- (a) (X, d) is complete,
- (b) T is increasing,
- (c) either T is continuous or (X, d, \preceq) has ICU property,
- (d) there exists $x_0 \in X$ such that $x_0 \preceq T(x_0)$,
- (e) there exists a comparison function ϕ such that

$$d(Tx, Ty) \leq \phi(d(x, y)) \quad \forall x, y \in X \text{ with } x \preceq y.$$

Then T has a fixed point.

Inspired by Roldán-López-de-Hierro et al. [23], Alam and Imdad introduced the following: (i.e., a notion originated from T -transitive subset of X^2 is essentially due to [23]).

Definition 4. [6] Let X be a nonempty set and T a self-mapping on X . A binary relation \mathcal{R} defined on X is called “ T -transitive” if for any $x, y, z \in X$,

$$(Tx, Ty), (Ty, Tz) \in \mathcal{R} \Rightarrow (Tx, Tz) \in \mathcal{R}.$$

Inspired by Turinici [28, 29], Alam and Imdad [6] introduced the following notions by localizing the transitivity condition.

Definition 5. [6] Let X be a nonempty set. A binary relation \mathcal{R} defined on X is called “locally transitive” if for each (effectively) \mathcal{R} -preserving sequence $\{x_n\} \subset X$ (with range $E = \{x_n : n \in \mathbb{N}\}$), such that $\mathcal{R}|_E$ is transitive.

Clearly, the notions of “ T -transitivity” and “locally transitivity” both are relatively weaker than the notions of transitivity, but they are independent of each other. In order to make them compatible, Alam and Imdad [6] introduced the following notion of transitivity.

Definition 6. [6] Let X be a nonempty set and T a self-mapping on X . A binary relation \mathcal{R} defined on X is called locally T -transitive if for each (effectively) \mathcal{R} -preserving sequence $\{x_n\} \subset T(X)$ (with range $E = \{x_n : n \in \mathbb{N}\}$), such that $\mathcal{R}|_E$ is transitive.

The following result accomplishes the dominant idea of ‘locally T -transitivity’ over other variants of ‘transitivity’:

Proposition 1. [6] Let X be a nonempty set, \mathcal{R} a binary relation on X and T a self-mapping on X . Then

- (i) \mathcal{R} is T -transitive $\Leftrightarrow \mathcal{R}|_{T(X)}$ is transitive,
- (ii) \mathcal{R} is locally T -transitive $\Leftrightarrow \mathcal{R}|_{T(X)}$ is locally transitive,
- (iii) \mathcal{R} is transitive $\Rightarrow \mathcal{R}$ is locally transitive $\Rightarrow \mathcal{R}$ is locally T -transitive,
- (iv) \mathcal{R} is transitive $\Rightarrow \mathcal{R}$ is T -transitive $\Rightarrow \mathcal{R}$ is locally T -transitive.

3. RELEVANT NOTIONS AND AUXILIARY RESULTS

In this section, for the sake of completeness, we summarize some relevant definitions and basic results for our subsequent discussions:

Definition 7. [4] Let \mathcal{R} be a binary relation on a nonempty set X and $x, y \in X$. We say that x and y are \mathcal{R} -comparative if either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. We denote it by $[x, y] \in \mathcal{R}$

Definition 8. [15] Let X be a nonempty set and \mathcal{R} a binary relation on X .

- (1) The inverse or transpose or dual relation of \mathcal{R} , denoted by \mathcal{R}^{-1} , is defined by $\mathcal{R}^{-1} = \{(x, y) \in X^2 : (y, x) \in \mathcal{R}\}$.
- (2) The symmetric closure of \mathcal{R} (denoted by \mathcal{R}^s) is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$ (i.e., $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$). Indeed, \mathcal{R}^s is the smallest symmetric relation on X containing \mathcal{R} .

Proposition 2. [4] For a binary relation \mathcal{R} defined on a nonempty set X ,

$$(x, y) \in \mathcal{R}^s \iff [x, y] \in \mathcal{R}.$$

Definition 9. [4] Let \mathcal{R} be a binary relation defined on a nonempty set X . A sequence $\{x_n\} \subset X$ is called “ \mathcal{R} -preserving” if

$$(x_n, x_{n+1}) \in \mathcal{R} \quad \forall n \in \mathbb{N}_0.$$

Definition 10. [4] Let X be a nonempty set and T a self-mapping on X . A binary relation \mathcal{R} defined on X is called T -closed if for any $x, y \in X$,

$$(x, y) \in \mathcal{R} \implies (Tx, Ty) \in \mathcal{R}.$$

Proposition 3. [5] Let X be a nonempty set endowed with a binary relation \mathcal{R} and T a self-mapping on X such that \mathcal{R} is T -closed, then \mathcal{R}^s is also T -closed.

Proposition 4. [6] Let \mathcal{R} be a binary relation defined on a nonempty set X and T a self-mapping on X . If \mathcal{R} is T -closed, then for all $n \in \mathbb{N}_0$, \mathcal{R} is also T^n -closed, where T^n denotes n^{th} iterate of T .

Definition 11. [5] Let \mathcal{R} be a binary relation defined on a nonempty set X . We say that (X, d) is \mathcal{R} -complete if every \mathcal{R} -preserving Cauchy sequence in X converges.

Notice that every complete metric space is \mathcal{R} -complete. Particularly, under the universal relation the notion of \mathcal{R} -completeness coincides with usual completeness.

Definition 12. [5] Let \mathcal{R} be a binary relation defined on a nonempty set X with $x \in X$. A mapping $T : X \rightarrow X$ is called \mathcal{R} -continuous at x if for any \mathcal{R} -preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, we have $T(x_n) \xrightarrow{d} T(x)$. Moreover, T is called \mathcal{R} -continuous if it is \mathcal{R} -continuous at each point of X .

Clearly, every continuous mapping is \mathcal{R} -continuous, for any binary relation \mathcal{R} . Particularly, under the universal relation the notion of \mathcal{R} -continuity coincides with usual continuity.

The following notion is a generalization of d -self-closedness of a partial order relation (\preceq) contained in Turinici [26, 27]:

Definition 13. [4] A binary relation \mathcal{R} defined on a metric space (X, d) is called d -self-closed if for any \mathcal{R} -preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in \mathcal{R} \quad \forall k \in \mathbb{N}_0$.

Definition 14. [24] Let \mathcal{R} be a binary relation defined on a nonempty set X . A subset E of X is called \mathcal{R} -directed if for each $x, y \in E$, there exists $z \in X$ such that $(x, z) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$.

Definition 15. [14] Let \mathcal{R} be a binary relation defined on a nonempty set X . For $x, y \in X$, a path of length k (where k is a natural number) in \mathcal{R} from x to y is a finite sequence $\{z_0, z_1, z_2, \dots, z_k\} \subset X$ satisfying the following:

- (i) $z_0 = x$ and $z_k = y$,
- (ii) $(z_i, z_{i+1}) \in \mathcal{R}$ for each i ($0 \leq i \leq k-1$).

Notice that a path of length k involves $k+1$ elements of X , although they may or may not be distinct.

Definition 16. [5] Let \mathcal{R} be a binary relation defined on a nonempty set X . A subset E of X is called \mathcal{R} -connected if for each pair $x, y \in E$, there exists a path in \mathcal{R} from x to y .

Given a binary relation \mathcal{R} and a self-mapping T on a nonempty set X , we use the following notations:

- (i) $F(T)$:= the set of all fixed points of T ,
- (ii) $X(T, \mathcal{R}) := \{x \in X : (x, Tx) \in \mathcal{R}\}$.

The following result is a relation-theoretic extension of Banach contraction principle:

Theorem 2. [4] Let (X, d) be a metric space, \mathcal{R} a binary relation on X and T a self-mapping on X . Suppose that the following conditions hold:

- (a) (X, d) is \mathcal{R} -complete,
- (b) \mathcal{R} is T -closed,
- (c) either T is \mathcal{R} -continuous or \mathcal{R} is d -self-closed,
- (d) $X(T, \mathcal{R})$ is nonempty,
- (e) there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) \quad \forall x, y \in X \text{ with } (x, y) \in \mathcal{R}.$$

Then T has a fixed point. Moreover, if X is \mathcal{R}^s -connected, then T has a unique fixed point.

In view of symmetry of d , we can have the following.

Proposition 5. *If (X, d) is a metric space, \mathcal{R} is a binary relation on X , T is a self-mapping on X and ϕ is a comparison function, then the following contractivity conditions are equivalent:*

- (I) $d(Tx, Ty) \leq \phi(d(x, y)) \quad \forall x, y \in X \text{ with } (x, y) \in \mathcal{R}$,
- (II) $d(Tx, Ty) \leq \phi(d(x, y)) \quad \forall x, y \in X \text{ with } [x, y] \in \mathcal{R}$.

Now, we propose the two main properties of the comparison function:

Proposition 6. [17] *Let ϕ be a comparison function, then ϕ is control function.*

Proof. Let there exists $t_0 > 0$ such that $t_0 \leq \phi(t_0)$. As ϕ is increasing $\phi(t_0) \leq \phi^2(t_0)$, it follows that $t_0 \leq \phi(t_0) \leq \phi^2(t_0)$. Thus, inductively for all $n \in \mathbb{N}$, we have $t_0 \leq \phi^n(t_0)$ which on letting $n \rightarrow \infty$, gives rise, $t_0 \leq 0$, which is a contradiction. \square

Proposition 7. *Let ϕ be a comparison function, then $\phi(0) = 0$.*

Proof. Suppose on contrary that $\phi(0) = t$ for some $t > 0$. As $0 < t$ and ϕ is increasing, $\phi(0) \leq \phi(t)$, it follows that $t \leq \phi(t) < t$, which is contradiction, hence $\phi(0) = 0$. \square

4. MAIN RESULTS

Now, we state and prove the existence and uniqueness of fixed point results via a locally T -transitive binary relation using comparison functions, besides deducing some special cases.

Theorem 3. *Let (X, d) be a metric space endowed with a binary relation \mathcal{R} and T a self-mapping on X . Suppose that the following conditions hold:*

- (a) (X, d) is \mathcal{R} -complete,
- (b) \mathcal{R} is T -closed and locally T -transitive,
- (c) either T is \mathcal{R} -continuous or \mathcal{R} is d -self-closed,
- (d) $X(T, \mathcal{R})$ is nonempty,
- (e) there exists a comparison function ϕ such that

$$d(Tx, Ty) \leq \phi(d(x, y)) \quad \forall x, y \in X \text{ with } (x, y) \in \mathcal{R}.$$

Then T has a fixed point.

Proof. As $X(T, \mathcal{R}) \neq \emptyset$, choose $x_0 \in X(T, \mathcal{R})$. Construct a sequence $\{x_n\}$ of iteration based at the initial point x_0 , i.e.,

$$x_n = T^n(x_0) \quad \forall n \in \mathbb{N}_0. \tag{4.1}$$

As $(x_0, Tx_0) \in \mathcal{R}$, using T -closedness of \mathcal{R} and Proposition 4, we get

$$(T^n x_0, T^{n+1} x_0) \in \mathcal{R}$$

so that

$$(x_n, x_{n+1}) \in \mathcal{R} \quad \forall n \in \mathbb{N}_0. \quad (4.2)$$

Therefore the sequence $\{x_n\}$ is \mathcal{R} -preserving. Now, if $d(x_{n_0+1}, x_{n_0}) = 0$ for some $n_0 \in \mathbb{N}_0$, then in view of (4.1), we have $T(x_{n_0}) = x_{n_0}$ so that x_{n_0} is a fixed point of T and hence we are done.

On the other hand, if $d(x_{n+1}, x_n) > 0 \quad \forall n \in \mathbb{N}_0$, then applying the contractivity condition (e) to (4.2), we deduce, for all $n \in \mathbb{N}_0$ that

$$d(x_{n+1}, x_n) \leq \phi(d(x_n, x_{n-1})),$$

which on using (4.2), contractive condition (e) and increasing property of ϕ , reduces to

$$d(x_{n+1}, x_n) \leq \phi^n(d(x_1, x_0)). \quad (4.3)$$

Making $n \rightarrow \infty$, in (4.3) and using the definition of comparison function, we get

$$\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = 0. \quad (4.4)$$

Fix $\varepsilon > 0$. Then in view of (4.4), we can choose $n \in \mathbb{N}_0$ such that

$$d(x_{n+1}, x_n) < \varepsilon - \phi(\varepsilon). \quad (4.5)$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence. To substantiate the claim, using increasing property of ϕ , (4.2) and (4.5), we obtain

$$\begin{aligned} d(x_{n+2}, x_n) &\leq d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n) < d(Tx_{n+1}, Tx_n) + \varepsilon - \phi(\varepsilon) \\ &\leq \phi(d(x_{n+1}, x_n)) + \varepsilon - \phi(\varepsilon) \leq \phi(\varepsilon - \phi(\varepsilon)) + \varepsilon - \phi(\varepsilon) \\ &\leq \phi(\varepsilon) + \varepsilon - \phi(\varepsilon) = \varepsilon. \end{aligned}$$

Now, using the increasing property of ϕ , (4.2) and locally T -transitivity of \mathcal{R} , we obtain

$$\begin{aligned} d(x_{n+3}, x_n) &\leq d(x_{n+3}, x_{n+1}) + d(x_{n+1}, x_n) < d(Tx_{n+2}, Tx_n) + \varepsilon - \phi(\varepsilon) \\ &\leq \phi(d(x_{n+2}, x_n)) + \varepsilon - \phi(\varepsilon) \leq \phi(\varepsilon - \phi(\varepsilon)) + \varepsilon - \phi(\varepsilon) \\ &\leq \phi(\varepsilon) + \varepsilon - \phi(\varepsilon) = \varepsilon \end{aligned}$$

so that inductively yields,

$$d(x_{n+k}, x_n) < \varepsilon \quad \text{for all } k \in \mathbb{N},$$

which shows that the sequence $\{x_n\}$ is Cauchy, which is also \mathcal{R} -preserving. By \mathcal{R} -completeness of (X, d) , $\exists x \in X$ such that $x_n \xrightarrow{d} x$.

Now, we show that x is a fixed point of T . To do this, assume that T is \mathcal{R} -continuous. As $\{x_n\}$ is \mathcal{R} -preserving with $x_n \xrightarrow{d} x$, \mathcal{R} -continuity of T implies that $x_{n+1} = T(x_n) \xrightarrow{d} T(x)$. Using the uniqueness of limit, we obtain $T(x) = x$, i.e., x is a fixed point of T .

Alternatively, assume that \mathcal{R} is d -self-closed. As $\{x_n\}$ is \mathcal{R} -preserving such that $x_n \xrightarrow{d} x$, the d -self-closedness of \mathcal{R} guarantees the existence of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in \mathcal{R}$ ($\forall k \in \mathbb{N}_0$). On using assumption (e), Proposition 5, $[x_{n_k}, x] \in \mathcal{R}$ with $x_{n_k} \xrightarrow{d} x$ and Propositions 6 and 7 (whether $d(x_{n_k}, x)$ is zero or nonzero), we obtain

$$\begin{aligned} d(x_{n_{k+1}}, Tx) &= d(Tx_{n_k}, Tx) \leq \phi(d(x_{n_k}, x)) \leq d(x_{n_k}, x) \\ &\rightarrow 0 \text{ as } k \rightarrow \infty \end{aligned}$$

so that $x_{n_{k+1}} \xrightarrow{d} T(x)$. Owing to the uniqueness of limit, we obtain $T(x) = x$ so that x is a fixed point of T . Hence proof is completed. \square

Combining with Proposition 1, we deduce the following consequence of Theorem 3.

Corollary 1. *Theorem 3 remains true if locally T -transitivity of \mathcal{R} (utilized in assumption (b)) is replaced by any one of the following conditions besides retaining rest of the hypotheses:*

- (i) \mathcal{R} is transitive,
- (ii) \mathcal{R} is T -transitive,
- (iii) \mathcal{R} is locally transitive.

Now, we prove a uniqueness result corresponding to Theorem 3.

Theorem 4. *In addition to the hypotheses of Theorem 3, assume that the following condition holds:*

- (u) $T(X)$ is \mathcal{R}^s -connected.

Then T has a unique fixed point.

Proof. Due to Theorem 3, $F(T) \neq \emptyset$. Choose $x, y \in F(T)$, then for all $n \in \mathbb{N}_0$, we have

$$T^n(x) = x \text{ and } T^n(y) = y. \quad (4.6)$$

By hypothesis (u), there exists a path (say $\{z_0, z_1, z_2, \dots, z_k\}$) of some finite length k in \mathcal{R}^s from x to y so that

$$z_0 = x, z_k = y \text{ and } [z_i, z_{i+1}] \in \mathcal{R} \text{ for each } i \ (0 \leq i \leq k-1). \quad (4.7)$$

Since \mathcal{R} is T -closed, using Propositions 3 and 4, we have

$$[T^n z_i, T^n z_{i+1}] \in \mathcal{R} \text{ for each } i \ (0 \leq i \leq k-1) \text{ and for each } n \in \mathbb{N}_0. \quad (4.8)$$

Now, for each $n \in \mathbb{N}_0$ and for each $i \ (0 \leq i \leq k-1)$, write $t_n^i := d(T^n z_i, T^n z_{i+1})$. We assert that

$$\lim_{n \rightarrow \infty} t_n^i = 0. \quad (4.9)$$

For each fixed $i \ (0 \leq i \leq k-1)$, we differentiate two cases. Firstly, assume that $t_{n_0}^i = d(T^{n_0} z_i, T^{n_0} z_{i+1}) = 0$ for some $n_0 \in \mathbb{N}_0$, i.e., $T^{n_0}(z_i) = T^{n_0}(z_{i+1})$, which yields

that $T^{n_0+1}(z_i) = T^{n_0+1}(z_{i+1})$. Consequently, we get $t_{n_0+1}^i = d(T^{n_0+1}z_i, T^{n_0+1}z_{i+1}) = 0$. Hence by induction on n , we get $t_n^i = 0 \forall n \geq n_0$, so that $\lim_{n \rightarrow \infty} t_n^i = 0$. Secondly, assume that $t_n^i > 0 \forall n \in \mathbb{N}_0$. Then on using (4.8), assumption (e), Proposition 5 and increasing property of ϕ , we have

$$t_{n+1}^i = d(T^{n+1}z_i, T^{n+1}z_{i+1}) \leq \phi(d(T^n z_i, T^n z_{i+1})) = \phi(t_n^i) \leq \phi^2(t_{n-1}^i) \leq \dots \leq \phi^n(t_1^i)$$

so that

$$t_{n+1}^i \leq \phi^n(t_1^i). \quad (4.10)$$

On making $n \rightarrow \infty$ in (4.10) and using the definition of ϕ , we have

$$\lim_{n \rightarrow \infty} t_{n+1}^i \leq \lim_{n \rightarrow \infty} \phi^n(t_1^i) = 0.$$

Thus in each case, (4.9) is proved.

In view of (4.6), (4.7), (4.9) and the triangular inequality, we have

$$d(x, y) = d(T^n z_0, T^n z_k) \leq t_n^0 + t_n^1 + \dots + t_n^{k-1} \rightarrow 0 \text{ as } n \rightarrow \infty$$

so that $x = y$. Hence T has a unique fixed point. \square

Corollary 2. *Theorem 4 remains true if we replace condition (u) by one of the following conditions besides retaining rest of the hypotheses:*

- (u') $\mathcal{R}|_{T(X)}$ is complete,
- (u'') $T(X)$ is \mathcal{R}^s -directed.

The proof of above corollary can be completed on the lines of the proof of Corollary 3.4 contained in [6].

Now, we deduce some special cases, which are noted fixed point theorems of the existing literature.

- (1) Under the universal relation (i.e., $\mathcal{R} = X^2$), Theorem 4 deduces to Matkowski fixed point theorem.
- (2) On choosing \mathcal{R} to be a partial order \preceq in Theorem 3, we obtain Theorem 1. Clearly, T -closedness of \preceq is equivalent to increasing property of T .
- (3) Taking $\phi(t) = \alpha t$ (where $\alpha \in [0, 1)$) in Theorem 4, we obtain Theorem 2. In this case, the requirement of locally T -transitivity on a binary relation is not necessary.

5. ILLUSTRATIVE EXAMPLES

Now, we furnish some examples to demonstrate the utility of Theorems 3 and 4 over corresponding earlier known results.

Example 1. Let $X = [0, 2]$ equipped with usual metric d . Let

$$\mathcal{R} = \{(0, 0), (1, 0), (0, 1), (1, 1), (0, 2)\}$$

be a binary relation, then $\mathcal{R}^s = \{(0, 0), (0, 1), (1, 0), (1, 1), (0, 2), (2, 0)\}$. On X , define a self-mapping T by

$$T(x) = \begin{cases} 0, & x \in [0, 2] \cap \mathbb{Q}; \\ 1, & x \in [0, 2] \cap \mathbb{Q}^c, \end{cases}$$

Next, define a comparison function ϕ by $\phi(t) = \log(1+t) \forall t \in [0, \infty)$. It can be easily seen that, for all $(x, y) \in \mathcal{R}$ hypothesis (e) satisfied. Notice that \mathcal{R} is locally T -transitive but not transitive. Choose any \mathcal{R} -preserving sequence $\{x_n\}$ in X , *i.e.*,

$$(x_n, x_{n+1}) \in \mathcal{R}, \text{ for all } n \in \mathbb{N}_0 \text{ with } x_n \xrightarrow{d} x.$$

Now, if $(x_n, x_{n+1}) \in \mathcal{R}$, for all $n \in \mathbb{N}_0$, then there exists $N \in \mathbb{N}$ such that $x_n = x \in \{0, 1\}$, for all $n \geq N$. Therefore, we choose a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $x_{n_k} = x$, for all $k \in \mathbb{N}$, which amounts to saying that $[x_{n_k}, x] \in \mathcal{R}$, for all $k \in \mathbb{N}$. Hence, \mathcal{R} is d -self-closed. Further, remaining hypotheses of Theorem 4 can be easily verified. Notice that T has a unique fixed point (namely $x = 0$). As \mathcal{R} is not transitive; therefore \mathcal{R} is not a partial order so this example can not be covered by Theorem 1 (due to Agarwal *et al.* [1]), which substantiate the utility of our results.

Example 2. Consider $X = [0, \infty)$ endowed with usual metric d . Define a mapping $T : X \rightarrow X$ by $T(x) = \frac{x}{1+2x} \forall x \in X$. Let $\mathcal{R} := \{(x, y) \in X^2 : x - y > 0 \text{ and } x \in \mathbb{Q}^+\}$, then \mathcal{R} is locally T -transitive binary relation on X . Clearly, X is \mathcal{R} -complete and \mathcal{R} is T -closed. Now, define a comparison function ϕ by $\phi(t) = \frac{t}{t+1} \forall t \in [0, \infty)$. Now, for all $(x, y) \in \mathcal{R}$, we have

$$\begin{aligned} d(Tx, Ty) &= \left| \frac{x}{1+2x} - \frac{y}{1+2y} \right| = \left| \frac{(x-y)}{1+2x+2y+4xy} \right| \\ &\leq \frac{(x-y)}{1+(x-y)} = \frac{d(x,y)}{1+d(x,y)} = \phi(d(x,y)). \end{aligned}$$

Hence T and ϕ satisfy the assumption (e) of Theorem 3. Further, rest of the conditions of Theorem 4 can be satisfied easily. Observe T has a unique fixed point (namely $x = 0$).

Notice that \mathcal{R} is not partially ordered and T is not linear contraction, therefore Example 2 can not be covered by corresponding Theorems 1 (due to Agarwal *et al.* [1]) and 2 (due to Alam and Imdad [4]), respectively.

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