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COFINITELY RADICAL SUPPLEMENTED AND COFINITELY WEAK RADICAL SUPPLEMENTED LATTICES

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Abstract. In this work, cofinitely radical supplemented and cofinitely weak radical supplemented lattices are defined and some properties of them are investigated. Let L be a lattice, I be a nonempty index set and $a_i \in L$ for every $i \in I$. If $1 = \underset{i\in I}{\vee} a_i$ and $a_i/0$ is cofinitely (weak) radical supplemented for every $i \in I$, then L is also cofinitely (weak) radical supplemented. Let L be a cofinitely (weak) radical supplemented lattice and $a \in L$. Then $1/a$ is also cofinitely (weak) radical supplemented. Let L be a lattice. Then L is cofinitely weak radical supplemented if and only if every cofinite element of $1/r(L)$ is a direct summand of $1/r(L)$.

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1. INTRODUCTION

Throughout this paper, all lattices are complete modular lattices with the smallest element 0 and the greatest element 1. Let L be a lattice, $a,b \in L$ and $a \leq b$. A sublattice $\{x \in L | a \leq x \leq b\}$ is called a *quotient sublattice*, denoted by b/a. An element a' of a lattice L is called a *complement* of a if $a \wedge a' = 0$ and $a \vee a' = 1$ (in this case a and a' are said to be *direct summands* of L and denoted by $1 = a \oplus a'$). A lattice L is said to be *complemented* if each element of L has at least one complement in L. An element c of L is said to be *compact* if for every subset X of L such that $c \leq \vee X$ there exists a finite $F \subset X$ such that $c \leq \vee F$. A lattice L is said to be *compactly generated* if each of its elements is a join of compact elements. A lattice L is said to be *compact* if 1 is a compact element of L. An element a of a lattice L is said to be *cofinite* if $1/a$ is compact. An element a of L is said to be *small* or *superfluous* if $a \vee b \neq 1$ holds for every $b \neq 1$ and denoted by $a \ll L$. The meet of all the maximal $(\neq 1)$ elements of a lattice L is called the *radical* of L and denoted by $r(L)$. An element c of L is called a *supplement* of b in L if it is minimal for $b \vee c = 1$. a is a supplement of b in a lattice L if and only if $a \vee b = 1$ and $a \wedge b \ll a/0$. L is called a *supplemented lattice* if every element of L has a supplement in L. We say that an element b of L *lies above* an element a of L if $a \leq b$ and $b \ll 1/a$. L is said to be *hollow* if every element $(\neq 1)$

is superfluous in L and L is said to be *local* if L has the greatest element $(\neq 1)$. An element a of L is called a *weak supplement* of b in L if $a \vee b = 1$ and $a \wedge b \ll L$. L is called a *weakly supplemented lattice*, if every element of L has a weak supplement in L. It is clear that every supplemented lattice is weakly supplemented. An element a of L is called a *generalized* (*radical*) *supplement* (or briefly, *Rad-supplement*) of b in L if $a \vee b = 1$ and $a \wedge b \leq r(a/0)$. L is said o be *radical* (*generalized*) *supplemented* if every element of L has a Rad-supplement in L.

More information about supplemented lattices are in [1, 2] and [5]. More results about supplemented modules are in [9] and [10]. The definitions of generalized supplemented modules and some properties of them are in [8]. More information about cofinitely Rad-supplemented modules are in [7]. We generalize cofinitely Rad-supplemented modules to lattices.

2. Cofinitely Radical Supplemented Lattices

In this part, cofinitely radical supplemented lattices are defined and some properties of them are given.

Definition 1. Let L be a lattice. If every cofinite element of L has a Rad-supplement in L, then L is called a cofinitely radical supplemented (or cofinitely Rad-supplemented) lattice.

Clearly we can see that every cofinitely supplemented lattice is cofinitely Rad-supplemented. Hollow and local lattices are cofinitely Rad-supplemented.

Proposition 1. *Let L be a compact lattice. Then L is cofinitely Rad-supplemented if and only if L is Rad-supplemented.*

Proof. Clear, since every element of L is cofinite. □

Lemma 1. *Let L be a lattice, $a \in L$ and x be a cofinite element of L. If $x \vee a$ has a Rad-supplement in L and $a/0$ cofinitely Rad-supplemented, then x has a Rad-supplement in L.*

Proof. Let b be a Rad-supplement of $x \vee a$ in L. Then $x \vee a \vee b = 1$ and $(x \vee a) \wedge b \leq r(b/0)$. Since x is a cofinite element of L, we clearly see that $x \vee b$ is a cofinite element of L. Then by $\frac{1}{x \vee b} = \frac{x \vee a \vee b}{x \vee b} \simeq \frac{a}{a \wedge (x \vee b)}$, $a \wedge (x \vee b)$ is a cofinite element of $a/0$. Since $a/0$ is cofinitely Rad-supplemented, $a \wedge (x \vee b)$ has a Rad-supplement c in $a/0$. Here $(a \wedge (x \vee b)) \vee c = a$ and $c \wedge (x \vee b) = c \wedge a \wedge (x \vee b) \leq r(c/0)$. Hence $1 = x \vee a \vee b = x \vee b \vee (a \wedge (x \vee b)) \vee c = x \vee b \vee c$ and $x \wedge (b \vee c) \leq (b \wedge (x \vee c)) \vee (c \wedge (x \vee b)) \leq (b \wedge (x \vee a)) \vee r(c/0) \leq r(b/0) \vee r(c/0) \leq r((b \vee c)/0)$. Thus $b \vee c$ is a Rad-supplement of x in L. □

Corollary 1. *Let L be a lattice, $a_1, a_2, \ldots, a_n \in L$ and x be a cofinite element of L. If $x \vee a_1 \vee a_2 \vee \cdots \vee a_n$ has a Rad-supplement in L and $a_i/0$ is cofinitely Rad-supplemented for every $i = 1, 2, ..., n$, then x has a Rad-supplement in L.*

Proof. Clear from Lemma 1. □

Lemma 2. *Let L be a lattice, I be a nonempty index set and $a_i \in L$ for every $i \in I$. If $1 = \bigvee_{i\in I} a_i$ and $a_i/0$ is cofinitely Rad-supplemented for every $i \in I$, then L is also cofinitely Rad-supplemented.*

Proof. Let a be any cofinite element of L. By hypothesis, $1 = \bigvee_{i\in I} a_i = \bigvee_{i\in I} (a \vee a_i)$. Since $1/a$ is compact and $a \vee a_i \in 1/a$ for every $i \in I$, there exist $i_1, i_2, ..., i_n \in I$ such that $1 = a \vee a_{i_1} \vee a_{i_2} \vee ... \vee a_{i_n}$. Since 0 is a Rad-supplement of $1 = a \vee a_{i_1} \vee a_{i_2} \vee ... \vee a_{i_n}$ and $a_{i_t}/0$ is cofinitely Rad-supplemented for every $t = 1, 2, ..., n$, by Corollary 1, a has a Rad-supplement in L. Hence L is cofinitely Rad-supplemented. □

Corollary 2. *Let L be a lattice and $1 = a_1 \vee a_2 \vee ... \vee a_n$ in L. If $a_i/0$ is cofinitely Rad-supplemented for every $i = 1, 2, ..., n$, then L is also cofinitely Rad-supplemented.*

Proof. Clear from Lemma 2. □

Proposition 2. *Let L be a cofinitely Rad-supplemented lattice and $a \in L$. Then $1/a$ is also cofinitely Rad-supplemented.*

Proof. Let x be any cofinite element of $1/a$. Then $1/x$ is compact and x is a cofinite element of L. Since L is cofinitely Rad-supplemented, x has a Rad-supplement y in L. Since $a \leq x$, by [3, Lemma 5], $a \vee y$ is a Rad-supplement of x in $1/a$. Hence $1/a$ is cofinitely Rad-supplemented. □

Proposition 3. *Let L be a cofinitely Rad-supplemented lattice. Then every cofinite element of $1/r(L)$ is a direct summand of $1/r(L)$.*

Proof. Let x be any cofinite element of $1/r(L)$. Then $1/x$ is compact and x is a cofinite element of L. Since L is cofinitely Rad-supplemented, x has a Rad-supplement y in L. Here $1 = x \vee y$ and $x \wedge y \leq r(y/0) \leq r(L)$. Then $1 = x \vee y \vee r(L)$ and since $r(L) \leq x$, $x \wedge (y \vee r(L)) = (x \wedge y) \vee r(L) = r(L)$. Hence $1 = x \oplus (y \vee r(L))$ in $1/r(L)$ and x is a direct summand of $1/r(L)$. □

3. Cofinitely Weak Radical Supplemented Lattices

In this part, cofinitely weak radical supplemented lattices are defined and some properties of them are given.

Definition 2. Let L be a lattice and $a, b \in L$. If $a \vee b = 1$ and $a \wedge b \leq r(L)$, then b is called a weak radical supplement (or briefly, weak Rad-supplement) of a in L.

Definition 3. Let L be a lattice. If every element of L has a weak Rad-supplement in L, then L is called a weakly radical supplemented (or weakly Rad-supplemented) lattice. If every cofinite element of L has a weak Rad-supplement in L, then L is called a cofinitely weak radical supplemented (or cofinitely weak Rad-supplemented) lattice.

It is clear that every cofinitely weak supplemented lattice is cofinitely weak Rad-supplemented. It is also clear that every cofinitely Rad-supplemented lattice is cofinitely weak Rad-supplemented.

Proposition 4. *Let L be a cofinitely weak Rad-supplemented lattice. If $r(L) \ll L$, then L is cofinitely weak supplemented.*

Proof. Clear from definitions. □

Proposition 5. *Let L be a compact lattice. Then L is cofinitely weak Rad-supplemented if and only if L is weakly Rad-supplemented.*

Proof. Clear, since every element of L is cofinite. □

Lemma 3. *Let L be a lattice, $a \in L$ and x be a cofinite element of L. If $x \vee a$ has a weak Rad-supplement in L and $a/0$ is cofinitely weak Rad-supplemented, then x has a weak Rad-supplement in L.*

Proof. Let b be a weak Rad-supplement of $x \vee a$ in L. Then $x \vee a \vee b = 1$ and $(x \vee a) \wedge b \leq r(L)$. Since x is a cofinite element of L, we clearly see that $x \vee b$ is a cofinite element of L. Then by $\frac{1}{x \vee b} = \frac{x \vee a \vee b}{x \vee b} \cong \frac{a}{a \wedge (x \vee b)}$, $a \wedge (x \vee b)$ is a cofinite element of $a/0$. Since $a/0$ is cofinitely weak Rad-supplemented, $a \wedge (x \vee b)$ has a weak Rad-supplement c in $a/0$. Here $(a \wedge (x \vee b)) \vee c = a$ and $c \wedge (x \vee b) = c \wedge a \wedge (x \vee b) \leq r(a/0) \leq r(L)$. Hence $1 = x \vee a \vee b = x \vee b \vee (a \wedge (x \vee b)) \vee c = x \vee b \vee c$ and $x \wedge (b \vee c) \leq (b \wedge (x \vee c)) \vee (c \wedge (x \vee b)) \leq (b \wedge (x \vee a)) \vee r(L) \leq r(L) \vee r(L) = r(L)$. Thus $b \vee c$ is a weak Rad-supplement of x in L. □

Corollary 3. *Let L be a lattice, $a_1, a_2, \ldots, a_n \in L$ and x be a cofinite element of L. If $x \vee a_1 \vee a_2 \vee \cdots \vee a_n$ has a weak Rad-supplement in L and $a_i/0$ is cofinitely weak Rad-supplemented for every $i = 1, 2, ..., n$, then x has a weak Rad-supplement in L.*

Proof. Clear from Lemma 3. □

Lemma 4. *Let L be a lattice, I be a nonempty index set and $a_i \in L$ for every $i \in I$. If $1 = \bigvee_{i \in I} a_i$ and $a_i/0$ is cofinitely weak Rad-supplemented for every $i \in I$, then L is also cofinitely weak Rad-supplemented.*

Proof. Let a be any cofinite element of L. By hypothesis, $1 = \bigvee_{i \in I} a_i = \bigvee_{i \in I} (a \vee a_i)$. Since $1/a$ is compact and $a \vee a_i \in 1/a$ for every $i \in I$, there exist $i_1, i_2, ..., i_n \in I$ such that $1 = a \vee a_{i_1} \vee a_{i_2} \vee ... \vee a_{i_n}$. Since 0 is a weak Rad-supplement of $1 = a \vee a_{i_1} \vee a_{i_2} \vee ... \vee a_{i_n}$ and $a_{i_t}/0$ is cofinitely weak Rad-supplemented for every $t = 1, 2, ..., n$, by Corollary 3, a has a weak Rad-supplement in L. Hence L is cofinitely weak Rad-supplemented. □

Corollary 4. *Let L be a lattice and $1 = a_1 \vee a_2 \vee ... \vee a_n$ in L. If $a_i/0$ is cofinitely weak Rad-supplemented for every $i = 1, 2, ..., n$, then L is also cofinitely weak Rad-supplemented.*

Proof. Clear from Lemma 4. □

Lemma 5. *Let L be a lattice, $a,b,x \in L$ and $x \leq a$. If b is a weak Rad-supplement of a in L, then $x \vee b$ is a weak Rad-supplement of a in $1/x$.*

Proof. Since b is a weak Rad-supplement of a in L, $a \vee b = 1$ and $a \wedge b \leq r(L)$. Then $a \vee x \vee b = 1$ and $a \wedge (x \vee b) \leq x \vee (a \wedge b) \leq x \vee r(L) \leq r(1/x)$. Hence $x \vee b$ is a weak Rad-supplement of a in $1/x$. □

Proposition 6. *Let L be a cofinitely weak Rad-supplemented lattice and $a \in L$. Then $1/a$ is also cofinitely weak Rad-supplemented.*

Proof. Let x be any cofinite element of $1/a$. Then $1/x$ is compact and x is a cofinite element of L. Since L is cofinitely weak Rad-supplemented, x has a weak Rad-supplement y in L. Since $a \leq x$, by Lemma 5, $a \vee y$ is a weak Rad-supplement of x in $1/a$. Hence $1/a$ is cofinitely weak Rad-supplemented. □

Proposition 7. *Let L be a lattice. Then L is cofinitely weak Rad-supplemented if and only if every cofinite element of $1/r(L)$ is a direct summand of $1/r(L)$.*

Proof. ($\Longrightarrow$) Let x be any cofinite element of $1/r(L)$. Then $1/x$ is compact and x is a cofinite element of L. Since L is cofinitely weak Rad-supplemented, x has a weak Rad-supplement y in L. Here $1 = x \vee y$ and $x \wedge y \leq r(L)$. Then $1 = x \vee y \vee r(L)$ and since $r(L) \leq x$, $x \wedge (y \vee r(L)) = (x \wedge y) \vee r(L) = r(L)$. Hence $1 = x \oplus (y \vee r(L))$ in $1/r(L)$ and x is a direct summand of $1/r(L)$.

($\Longleftarrow$) Let x be any cofinite element of L. Here clearly we can see that $x \vee r(L)$ is a cofinite element of $1/r(L)$. By hypothesis, $x \vee r(L)$ is a direct summand of $1/r(L)$. Then there exists $y \in 1/r(L)$ such that $1 = x \vee r(L) \vee y = x \vee y$ and $(x \vee r(L)) \wedge y = r(L)$. Since $r(L) \leq y$, by modularity, $r(L) = (x \vee r(L)) \wedge y = (x \wedge y) \vee r(L)$ and $x \wedge y \leq r(L)$. Hence y is a weak Rad-supplement of x in L. Therefore, L is cofinitely weak Rad-supplemented. □

Proposition 8. *Let L be a lattice and $a \ll L$. If $1/a$ is cofinitely weak Rad-supplemented, then L is also cofinitely weak Rad-supplemented.*

Proof. Let x be any cofinite element of L. Clearly we see that $x \vee a$ is a cofinite element of $1/a$. Since $1/a$ is cofinitely weak Rad-supplemented, $x \vee a$ has a weak Rad-supplement y in $1/a$. Here $x \vee a \vee y = 1$ and $(x \vee a) \wedge y \leq r(1/a)$. Since $x \vee a \vee y = 1$ and $a \leq y$, $x \vee y = 1$. Since $a \ll L$, clearly we see that $r(1/a) = r(L)$. Hence $x \vee y = 1$ and $x \wedge y \leq (x \vee a) \wedge y \leq r(1/a) = r(L)$. Thus L is cofinitely weak Rad-supplemented. □

Let $x, y \in L$. It is defined a relation β_* on the elements of L by $x \beta_* y$ if and only if for every $t \in L$ with $x \vee t = 1$ then $y \vee t = 1$ and for every $k \in L$ with $y \vee k = 1$ then $x \vee k = 1$. The definition of β_* relation and some properties of this relation are in [6]. The definition of β^* relation on modules and some properties of this relation are in [4].

Lemma 6. *Let L be a lattice. If every cofinite element of L is β_* equivalent to a weak Rad-supplement element in L, then L is cofinitely weak Rad-supplemented.*

Proof. Let x be a cofinite element of L. By hypothesis, there exists a weak Rad-supplement element y in L such that $x\beta_* y$. Let y is a weak Rad-supplement of a in L. Here $y \vee a = 1$ and $y \wedge a \leq r(L)$. Since $x\beta_* y$ and $y \vee a = 1$, $x \vee a = 1$. Assume $x \wedge a \nleq r(L)$. Then there exists a maximal $(\neq 1)$ element t of L with $x \wedge a \nleq t$. Here $(x \wedge a) \vee t = 1$. By [6, Lemma 2], $x \vee (a \wedge t) = 1$ and since $x\beta_* y$, $y \vee (a \wedge t) = 1$. Since $a \vee t = 1$, by [6, Lemma 2], $(y \wedge a) \vee t = 1$. Since $y \wedge a \leq r(L) \leq t$, $t = (y \wedge a) \vee t = 1$. This contradicts with $t \neq 1$. Hence $x \wedge a \leq r(L)$. Therefore, a is a weak Rad-supplement of x in L and L is cofinitely weak Rad-supplemented. □

Corollary 5. *Let L be a lattice. If every cofinite element of L lies above a weak Rad-supplement element in L, then L is cofinitely weak Rad-supplemented.*

Proof. Clear from Lemma 6. □

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