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L^p-REGULARITY RESULTS FOR $2m$-TH ORDER PARABOLIC EQUATIONS IN TIME-VARYING DOMAINS

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Abstract. This paper is devoted to the analysis of the following linear $2m$-th order parabolic equation $\partial_t u+(-1)^m \sum_{k=1}^N \partial_{x_k}^{2m} u = f$, subject to Dirichlet type condition $\partial_\nu^l u = 0$, $l = 0,1,\ldots,m-1$, on the lateral boundary, where m is a positive integer. The right-hand side f of the equation is taken in the Lebesgue space L^p, $1 < p < +\infty$. The problem is set in a domain of the form $Q = \left\{(t,x_1,\ldots,x_N) \in \mathbb{R}^{N+1} : 0 \leq \sqrt{x_1^2+\ldots+x_N^2} < t^\alpha\right\}$ with $\alpha > 1/2m$.

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1. INTRODUCTION

Let Q be an open set of $\mathbb{R}^{N+1}$ defined by

$$Q = \left\{(t,x_1,x_2,\ldots,x_N) \in \mathbb{R}^{N+1} : (x_1,x_2,\ldots,x_N) \in \Omega_t, 0 < t < 1\right\}$$

where for a fixed t in the interval $]0,1[$, Ω_t is a bounded domain of $\mathbb{R}^N$, $N > 1$, defined by

$$\Omega_t = \left\{(x_1,x_2,\ldots,x_N) \in \mathbb{R}^N : 0 \leq \sqrt{x_1^2+x_2^2+\ldots+x_N^2} < t^\alpha\right\}$$

with $\alpha > 1/2m$ and m belongs to the set of all nonzero natural numbers $\mathbb{N}^*$. In Q, consider the boundary value problem

$$\begin{cases} \partial_t u + Mu &= f \in L^p(Q), \\ \partial_\nu^l u\big|_{\partial Q \setminus \Gamma_1} &= 0,\ l = 0,1,\ldots,m-1, \end{cases} \tag{1.1}$$

where $M = (-1)^m \sum_{k=1}^N \partial_{x_k}^{2m}$, ∂Q is the boundary of Q, Γ_1 is the part of the boundary of Q where $t = 1$ and ∂_ν^l stands for the derivative of order l throughout the normal

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vector ν on $(\partial Q \setminus \Gamma_1)$. Here, $L^p(Q)$, $1 < p < +\infty$, denotes the space of p-integrable functions on Q with the measure $dtdx_1...dx_N$.

If the domain under investigation is a cylinder, the solvability of the corresponding problem is known over the scales of anisotropic Sobolev-Slobodetskii or Hölder spaces since the mid of the last century. Indeed, classical results on the resolution of Problem (1.1) can be found in [20] and [21] and in the references therein. Some recent regularity results are given in [15], [10], [8], [11], [26], [27], [25] and [2].

Besides being interesting in itself, the study of Problem (1.1) is motivated by the interest of researchers for many mathematical questions related to non-regular domains. During the last decades and since many applied problems lead directly to boundary-value problems in "bad" domains, numerous authors studied partial differential equations in non-smooth domains. Among these which are related to higher order parabolic equations we can cite Baderko [1], Cherepova [3], Sadallah [29], Galaktionov [12], Mikhailov [23], [24], Cherfaoui *et al.* [5], Grimaldi [13] and the references therein.

The L^2-solvability of Problem (1.1) has been investigated in [6] by the domain decomposition method, see also [16] and [4]. The difficulty with the space L^p, $p \neq 2$, is that this space is not a Hilbert space. So, the domain decomposition method used in [6] does not seem to be appropriate for our study and cannot be generalized in this sense. An idea for this extension (to the case L^p, $p \in (1,\infty)$) can be found in [8] and [17], in which the operators sum method was used. This method is interesting because it may be generalized to Banach spaces instead of Hilbert spaces. For more details and recent results concerning this method, see [7], [28] and the references therein.

In this work, we are especially interested in the question of what sufficient conditions, as weak as possible, the dimension N, the exponent p and the type of the domain Q must verify in order that Problem (1.1) has a solution with optimal regularity, that is a solution u belonging to the anisotropic Sobolev space

$$H_{0,p}^{1,2m}(Q) := \left\{ u \in H_p^{1,2m}(Q) : \partial_\nu^l u|_{\partial Q \setminus \Gamma_1} = 0,\ l = 0, 1, ..., m-1 \right\},$$

with

$$H_p^{1,2m}(Q) = \{u : \partial_t u, \partial^\alpha u \in L^p(Q), |\alpha| \leq 2m\}$$

where $\alpha = (i_1, i_2, ..., i_N) \in \mathbb{N}^N, |\alpha| = i_1 + i_2 + ... + i_N$ and $\partial^\alpha u = \partial_{x_1}^{i_1} \partial_{x_2}^{i_2} ... \partial_{x_N}^{i_N} u$. The space $H_p^{1,2m}(Q)$ is equipped with the natural norm, that is

$$\|u\|_{H_p^{1,2m}(Q)} = \left(\|\partial_t u\|_{L^p(Q)}^p + \sum_{|\alpha| \leq 2m} \|\partial^\alpha u\|_{L^p(Q)}^p \right)^{1/p}.$$

The main assumption is

$$\frac{1}{2m} < \alpha < \frac{p-1}{N}. \tag{1.2}$$

The outline of this paper is as follows. In Section 2 we recall the essential of the sum theory we will have to apply. In Section 3 we perform a change of variables conserving (modulo a weight) the spaces L^p and $H_p^{1,2m}$, and transforming Problem (1.1) into a degenerate parabolic problems in a cylindrical domain. Section 4 is concerned with the application of the sum operators method to the transformed problem. We can find in the Favini-Yagi book [9] an important study of abstract problems of parabolic type with degenerated terms in the time derivative. They used the notion of multi-valued linear operators and constructed fundamental solutions when the right-hand side has a Hölder regularity with respect to the time. Our approach is based on the direct use of operators sums in a weighted L^p-Sobolev space. Finally, in Section 5 we give results concerning the transformed problem and we return to our initial problem by using the inverse change of variables.

Note that this approach may be extended at least in the following directions:

(1) The high order operator M may be replaced by the following constant coefficient operator:
$$L = \sum_{|\delta|=|\beta|=m} (-1)^m a_{\delta\beta} \partial^\delta \partial^\beta$$
with $a_{\delta\beta} = a_{\beta\delta}$ and there exists a constant $C > 0$ such that
$$a_{\delta\beta} \xi^\delta \xi^\beta > C|\xi|^{2m},\ \xi \in \mathbf{R}^N.$$

(2) The function f on the right-hand side of the equation of Problem (1.1), may be taken in Hölder or little Hölder spaces.

These questions will be developed in forthcoming works.

2. ON THE NON-COMMUTATIVE SUM OF LINEAR OPERATORS

Let Λ be a closed linear operator in a complex Banach space E. Then, Λ is said to be sectorial if

(i) $D(\Lambda)$ and $Im(\Lambda)$ are dense in E,

(ii) $\ker(\Lambda) = \{0\}$,

(iii) $]-\infty, 0[\subset \rho(\Lambda)$ ($\rho(\Lambda)$ is the resolvent set of Λ) and there exists a constant $K \geq 1$ such that $\forall t > 0$, $\left\| t (\Lambda + tI)^{-1} \right\|_{L(E)} \leq K$. If Λ is sectorial it follows easily that $\rho(-\Lambda)$ contains an open sector $\Sigma_\varphi := \{z \in \mathbb{C} : z \neq 0, |\arg z| < \varphi\}$, with $\varphi \in]0, \pi[$.

Consider two closed linear operators A and B with dense domains $D(A)$ and $D(B)$ respectively in E. Assume that both operators satisfy the following assumptions of Da Prato-Grisvard's type [7].

There exist positive numbers r, C_A, C_B, θ_A, θ_B such that
$$\theta_A + \theta_B < \pi, \tag{2.1}$$
$$\rho(-A) \supset \Sigma_{\pi-\theta_A} := \{z \in \mathbb{C} : |z| \geq r, |\arg z| < \pi - \theta_A\} \tag{2.2}$$

$$\begin{aligned}
&\text{and } \forall\lambda \in \Sigma_{\pi-\theta_A}, \left\|(A+\lambda I)^{-1}\right\|_{L(E)} \leq \frac{C_A}{|\lambda|}, \\
&\qquad\qquad \rho(-B) \supset \Sigma_{\pi-\theta_B} := \{z \in \mathbb{C} : |z| \geq r, |\arg z| < \pi - \theta_B\} \\
&\text{and } \forall\mu \in \Sigma_{\pi-\theta_B}, \left\|(B+\mu I)^{-1}\right\|_{L(E)} \leq \frac{C_B}{|\lambda|}.
\end{aligned} \tag{2.3}$$

We also assume that there are constants $C > 0$, $\lambda_0 > 0$, (with $\lambda_0 \in \rho(-A)$), τ and ρ such that

$$\begin{cases} (i)\ \left\|(A+\lambda_0 I)(A+\lambda I)^{-1}\left[(A+\lambda_0 I)^{-1};(B+\mu I)^{-1}\right]\right\|_{L(E)} \\ \qquad \leq \dfrac{C}{|\lambda|^{1-\tau}.|\mu|^{1+\rho}}\ \forall\lambda \in \rho(-A),\ \forall\mu \in \rho(-B), \\ (ii)\ 0 \leq \tau < \rho \leq 1. \end{cases} \tag{2.4}$$

For more details concerning this last Labbas-Terreni commutator assumption, see [18], [19].

For any $\sigma \in]0,1[$ and $1 \leq p \leq +\infty$, let us introduce the real Banach interpolation spaces $D_A(\sigma,p)$ between $D(A)$ and E (or $D_B(\sigma,p)$ between $D(B)$ and E) which are characterized (for $1 \leq p < +\infty$) by

$$D_A(\sigma,p) = \left\{\xi \in E : t \longmapsto \left\|t^{\sigma}A(A+tI)^{-1}\xi\right\|_E \in L^p_*\right\},$$

where L^p_* denotes the space of p-integrable functions on $(0,+\infty)$ with the measure dt/t. For $p = +\infty$,

$$D_A(\sigma,+\infty) = \left\{\xi \in E : \sup_{t>0}\left\|t^{\sigma}A(A+tI)^{-1}\xi\right\|_E < \infty\right\}.$$

For these spaces, see [14]. Then the main result proved in Labbas-Terreni [18] is

Theorem 1. *Under assumptions (2.1), (2.2), (2.3) and (2.4), there exists $\lambda^* > 0$ such that for any $\lambda \geq \lambda^*$ and for any $h \in D_A(\sigma,p)$, equation*

$$Aw + Bw + \lambda w = h,$$

has a unique solution $w \in D(A) \cap D(B)$ with the regularities $Aw, Bw \in D_A(\theta,p)$ and $Aw \in D_B(\theta,p)$ for any θ verifying $\theta \leq \min(\sigma, \rho-\tau)$.

3. Change of variables and operational setting of the problem

3.1. *Change of variables*

We make the following change of variables and functions

$$\begin{aligned} \Pi\ \ : Q &\longrightarrow G \\ (t,x_1,x_2,\ldots,x_N) &\longmapsto (t,y_1,y_2,\ldots,y_N) = \left(t,\frac{x_1}{t^{\alpha}},\frac{x_2}{t^{\alpha}},\ldots,\frac{x_N}{t^{\alpha}}\right) \end{aligned}$$

where $G =]0,1[\times B(0,1)$, with $B(0,1)$ is the unit ball of $\mathbb{R}^N$ centered at the origin. Set $u(t,x_1,x_2,...,x_N) = v(t,y_1,y_2,...,y_N)$ and $f(t,x_1,x_2,...,x_N) = g(t,y_1,y_2,...,y_N)$, then Problem (1.1) is transformed, in G, into the following degenerate evolution problem

$$\left\{ \begin{array}{lcl} t^{2m\alpha}\partial_t v + Mv - \alpha t^{2m\alpha-1}\sum_{k=1}^{N} y_k \partial_{y_k} v & = & t^{2m\alpha} g = h, \\ v|_{\Sigma_0} & = & 0, \\ \partial_\nu^l v|_{\partial G \setminus (\Sigma_0 \cup \Sigma_1)} & = & 0,\ l = 0,1...,m-1, \end{array} \right. \tag{3.1}$$

where $M = \sum_{k=1}^{N} \partial_{y_k}^{2m}$, $\Sigma_j,\ j = 0,\ 1$ is the part of the boundary of G where $t = j$. It is easy to see that $f \in L^p(Q)$ if and only if $t^{N\alpha/p} g \in L^p(G)$. Indeed,

$$\begin{aligned} f \in L^p(Q) &\Leftrightarrow \int_0^1 \int_{\Omega_t} |f(t,x_1,...,x_N)|^p \, dt\, dx_1...dx_N & < +\infty \\ &\Leftrightarrow \int_0^1 \int_{B(0,1)} \left| t^{N\alpha/p} g(t,y_1,...,y_N) \right|^p dt\, dy_1...dy_N & < +\infty \\ &\Leftrightarrow t^{N\alpha/p} g \in L^p(G). \end{aligned}$$

Consequently, $f \in L^p(Q)$ if and only if $t^{-2m\alpha+(N\alpha/p)} h \in L^p(G)$ which implies that $h \in L^p(G)$, since $h = \left(t^{-2m\alpha+(N\alpha/p)} h\right) t^{2m\alpha-(N\alpha/p)}$ and $2m\alpha - (N\alpha/p) > 0$. Then, the function $h = t^{2m\alpha} g$ lies in the closed subspace of $L^p(G)$ defined by

$$E = \left\{ h \in L^p(0,1; L^p(B(0,1))) : t^{-2m\alpha+(N\alpha/p)} h \in L^p(0,1; L^p(B(0,1))) \right\}.$$

This space is equipped with the norm $\|h\|_E = \left\| t^{-2m\alpha+(N\alpha/p)} h \right\|_{L^p(0,1;L^p(B(0,1)))}$.

3.2. *Operational formulation of Problem (3.1)*

Recall that $\alpha > 1/2m$ and assume

$$p > 1 + N\alpha. \tag{3.2}$$

Set $X = L^p(B(0,1))$ and define the functions

$$\begin{array}{llll} v & : [0,1] \longrightarrow X; & t \longmapsto v(t); & v(t)(y_1,y_2,...,y_N) = v(t,y_1,y_2,...,y_N), \\ h & : [0,1] \longrightarrow X; & t \longmapsto h(t); & h(t)(y_1,y_2,...,y_N) = h(t,y_1,y_2,...,y_N). \end{array}$$

Consider the family of operators $(L(t))_{t\in[0,1]}$ defined by

$$\left\{ \begin{array}{lcl} D(L(t)) & = & \left\{ \psi \in W^{2m,p}(B(0,1)) : \partial_\nu^l \psi|_{\partial B(0,1)} = 0,\ l = 1,...,m-1 \right\}, \\ (L(t)\psi) & = & M\psi - \alpha t^{2m\alpha-1} \sum_{k=1}^{N} y_k \partial_{y_k} \psi \text{ for a.e. } t \in (0,1), \end{array} \right.$$

then Problem (3.1) is equivalent to the following operational degenerate Cauchy problem in X

$$\begin{cases} t^{2m\alpha}v'(t)+L(t)v(t) & =h(t), \qquad t\in(0,1), \\ v(0) & =0. \end{cases} \tag{3.3}$$

Observe that $\overline{D(L(t))}=X$. Set

$$\begin{cases} w(t) & =e^{\lambda\frac{t^{1-2m\alpha}}{1-2m\alpha}}v(t) \\ k(t) & =e^{\lambda\frac{t^{1-2m\alpha}}{1-2m\alpha}}h(t), \end{cases}$$

where λ is some positive number. Then, w verifies

$$\begin{cases} t^{2m\alpha}w'(t)+L(t)w(t)+\lambda w(t) & =k(t), \qquad t\in(0,1), \\ w(0) & =0, \end{cases}$$

where k belongs to the space

$$E=\left\{h\in L^p(0,1;X):t^{-2m\alpha+(N\alpha/p)}h\in L^p(0,1;X)\right\}.$$

We obtain then the new operational form of the previous problem, mainly

$$Aw+Bw+\lambda w=k,$$

where

$$(Aw)(t)=L(t)w(t),\ t\in[0,1],$$

with domain

$$D(A)=\left\{w\in E:t^{-2m\alpha+(N\alpha/p)}w\in L^p\left(0,1;W^{2m,p}(B(0,1))\cap W_0^{m,p}(B(0,1))\right)\right\}$$

and

$$(Bw)(t)=t^{2m\alpha}(t)w'(t),\ t\in[0,1],$$

with domain

$$D(B)=\left\{w\in E:t^{(N\alpha/p)}w'\in L^p(0,1;X)\text{ and }w(0)=0\right\}.$$

Note that the trace $w(0)$ is well defined in $D(B)$. In fact, we have

$$t^{N\alpha/p}w\in L^p(0,1;X),\ t^{N\alpha/p}w'\in L^p(0,1;X),$$

and in virtue of (3.2) $(N\alpha/p)+(1/p)<1$. Then, w is continuous on $[0,1]$, (see [31, Lemma, p. 42]).

4. Application of the sums

Now we are in position to apply the result of the sums of operators. For this purpose, we must verify the assumptions of Theorem 1. The spectral properties of A and B are as follows.

Proposition 1. *A and B are linear closed operators and their domains are dense in E. Moreover, they satisfy assumptions (2.1), (2.2) and (2.3).*

Proof. 1. Let us study the spectral equation related to the operator B

$$Bw + zw = k.$$

Fix some positive μ_0 and let z such that $Re\, z \geq \mu_0$. Then the general solution of the problem

$$\begin{cases} t^{2m\alpha} w'(t) + zw(t) & = & k(t)\,,\ t \in [0,1]\,, \\ w(0) = 0, & & \end{cases}$$

is given by

$$w(t) = d \exp\left(z \int_t^1 \frac{ds}{s^{2m\alpha}} \right) + \int_0^t \left(\frac{k(\sigma)}{\sigma^{2m\alpha}} \exp\left(-z \int_\sigma^t \frac{ds}{s^{2m\alpha}} \right) \right) d\sigma,$$

where d is an arbitrary constant. The hypothesis $p > 1 + N\alpha$ implies that the function

$$t \mapsto t^{-2m\alpha + (N\alpha/p)} \exp\left(z \int_t^1 \frac{ds}{s^{2m\alpha}} \right)$$

does not belong to $L^p(B(0,1))$. So, we will take $d = 0$ to obtain $w \in E$. Consequently

$$\begin{array}{rcl} w(t) & = & \left((B + zI)^{-1} k \right)(t) \\ & = & \int_0^t \left(\dfrac{k(\sigma)}{\sigma^{2m\alpha}} \exp\left(-z \int_\sigma^t \dfrac{ds}{s^{2m\alpha}} \right) \right) d\sigma \\ & = & \exp\left(\frac{z}{(2m\alpha - 1) t^{2m\alpha - 1}} \right) \int_0^t \dfrac{k(\sigma)}{\sigma^{2m\alpha}} \exp\left(\frac{-z}{(2m\alpha - 1)\sigma^{2m\alpha - 1}} \right) d\sigma \end{array}$$

Let us check that this formula is well defined on $[0,1]$ and gives $w(0) = 0$. Set $\mu = \frac{z}{(2m\alpha - 1)}$, then

$$\begin{array}{rcl} \|w(t)\| & \leq & \exp\left(\frac{Re\,\mu}{t^{2m\alpha - 1}} \right) \int_0^t \left\| \sigma^{-2m\alpha + (N\alpha/p)} k(\sigma) \right\| \sigma^{-N\alpha/p} \exp\left(\frac{-Re\,\mu}{\sigma^{2m\alpha - 1}} \right) d\sigma \\ & \leq & \left(\displaystyle\int_0^t \left\| \sigma^{-2m\alpha + (N\alpha/p)} k(\sigma) \right\|^p d\sigma \right)^{1/p} \left(\displaystyle\int_0^t \sigma^{-qN\alpha/p} d\sigma \right)^{1/q} \\ & \leq & \left(\frac{1}{1 - (qN\alpha)/p} \right)^{\frac{1}{q}} t^{(1/q) - (N\alpha/p)} \|k\|_E \end{array}$$

where $(1/p) + (1/q) = 1$. Hence $w(t)$ is defined and $w(0) = 0$ since

$$\frac{1}{q} - \frac{N\alpha}{p} = 1 - \frac{1}{p} - \frac{N\alpha}{p}$$

means $p > 1 + N\alpha$. On the other hand we can write

$$\begin{aligned}
&t^{-2m\alpha+(N\alpha/p)} w(t) \\
&= \int_0^t \left(\frac{k(\sigma)}{t^{2m\alpha-(N\alpha/p)}\sigma^{2m\alpha}} \exp\mu \left(t^{-2m\alpha+1} - \sigma^{-2m\alpha+1} \right) \right) d\sigma \\
&= \int_0^t \left(\frac{k(\sigma)}{\sigma^{2m\alpha-(N\alpha/p)}} \left(\frac{1}{t^{2m\alpha-(N\alpha/p)}\sigma^{N\alpha/p}} \exp\mu \left(t^{-2m\alpha+1} - \sigma^{-2m\alpha+1} \right) \right) \right) d\sigma.
\end{aligned}$$

Putting

$$K_\mu(t,\sigma) = \begin{cases} \dfrac{1}{t^{2m\alpha-(N\alpha/p)}\sigma^{N\alpha/p}} \exp\mu \left(t^{-2m\alpha+1} - \sigma^{-2m\alpha+1} \right) & \text{if } t > \sigma, \\ 0 & \text{if } t < \sigma, \end{cases}$$

we deduce that

$$t^{-2m\alpha+(N\alpha/p)} w(t) = \int_0^1 \frac{k(\sigma)}{\sigma^{2m\alpha-(N\alpha/p)}} K_\mu(t,\sigma)\, d\sigma.$$

We need the following classical interpolation result, the so-called Schur's Lemma.

Lemma 1. *If there exists a constant C such that*

a) $\left| \int_0^1 K_\mu(t,\sigma)\, d\sigma \right| \leq C$ *for every* $t \in]0,1[$,

b) $\left| \int_0^1 K_\mu(t,\sigma)\, dt \right| \leq C$ *for every* $\sigma \in]0,1[$,

then

$$\left\| t^{-2m\alpha+(N\alpha/p)} w \right\|_{L^p(0,1;X)} \leq C \left\| t^{-2m\alpha+(N\alpha/p)} k \right\|_{L^p(0,1;X)}.$$

Now, we have to check that the conditions a) and b) are satisfied.

Condition a)

We have

$$\begin{aligned}
\int_0^1 \left| K_\mu(t,\sigma) \right| d\sigma &= \tfrac{1}{t^{2m\alpha-(N\alpha/p)}} \exp\left(t^{-2m\alpha+1}.Re\,\mu \right) \int_0^t \frac{\exp\left(-\sigma^{-2m\alpha+1}.Re\,\mu \right)}{\sigma^{N\alpha/p}} d\sigma \\
&\leq \tfrac{1}{2m\alpha-1} \exp\left(t^{-2m\alpha+1}.Re\,\mu \right) \int_{t^{-2m\alpha+1}}^{+\infty} \exp(-s.Re\,\mu)\, ds \\
&\leq \tfrac{1}{Re\,z}.
\end{aligned}$$

Consequently

$$\max_{t\in[0,1]} \int_0^1 \left| K_\mu(t,\sigma) \right| d\sigma \leq \frac{1}{Re\,z}. \tag{4.1}$$

This shows that the condition a) of Lemma 1 holds true.

Condition b)

We have

$$\begin{aligned}\int_0^1 \left|K_\mu(t,\sigma)\right| dt &= \tfrac{1}{\sigma^{\frac{N\alpha}{p}}} \exp\left(-\sigma^{-2m\alpha+1}.Re\,\mu\right) \int_\sigma^1 \frac{\exp\left(t^{-2m\alpha+1}.Re\,\mu\right)}{t^{2m\alpha-(N\alpha/p)}} dt \\ &= \tfrac{1}{2m\alpha-1} \tfrac{\exp\left(-\sigma^{-2m\alpha+1}.Re\,\mu\right)}{\sigma^{\frac{N\alpha}{p}}} \int_1^{\sigma^{-2m\alpha+1}} \frac{1}{s^{\frac{N\alpha}{p(2m\alpha-1)}}} \exp(s.Re\,\mu)\, ds,\end{aligned}$$

and

$$\begin{aligned}\int_1^{\sigma^{-2m\alpha+1}} \frac{1}{s^{\frac{N\alpha}{p(2m\alpha-1)}}} \exp(s.Re\,\mu)\, ds &= \int_1^{\frac{1+\sigma^{-2m\alpha+1}}{2}} \frac{1}{s^{\frac{N\alpha}{p(2m\alpha-1)}}} \exp(s.Re\,\mu)\, ds \\ &\quad + \int_{\frac{1+\sigma^{-2m\alpha+1}}{2}}^{\sigma^{-2m\alpha+1}} \frac{1}{s^{\frac{N\alpha}{p(2m\alpha-1)}}} \exp(s.Re\,\mu)\, ds \\ &\leq \int_1^{\frac{1+\sigma^{-2m\alpha+1}}{2}} \exp(s.Re\,\mu)\, ds \\ &\quad + \frac{1}{\left(\frac{1+\sigma^{-2m\alpha+1}}{2}\right)^{\frac{N\alpha}{p(2m\alpha-1)}}} \int_{\frac{1+\sigma^{-2m\alpha+1}}{2}}^{\sigma^{-2m\alpha+1}} \exp(s.Re\,\mu)\, ds, \\ &= I_1 + I_2.\end{aligned}$$

Then

$$\begin{aligned} I_1 &\leq \tfrac{1}{Re\,\mu}\left[\exp\left(Re\,\mu \tfrac{(1+\sigma^{-2m\alpha+1})}{2}\right) - \exp(Re\,\mu)\right] \\ &\leq \tfrac{1}{Re\,\mu} \exp\left(Re\,\mu \tfrac{(1+\sigma^{-2m\alpha+1})}{2}\right),\end{aligned}$$

and

$$\begin{aligned}\tfrac{1}{2m\alpha-1} \tfrac{\exp\left(-\sigma^{-2m\alpha+1}.Re\,\mu\right)}{\sigma^{\frac{N\alpha}{p}}} I_1 &\leq \tfrac{1}{Re\,z} \frac{\exp\left(\frac{-(\sigma^{-2m\alpha+1}-1)}{2}.Re\,\mu\right)}{\sigma^{\frac{N\alpha}{p}}} \\ &\leq \tfrac{1}{Re\,z} \frac{\exp\left(\frac{-(\sigma^{-2m\alpha+1}-1)}{2}.\mu_0\right)}{\sigma^{\frac{N\alpha}{p}}} \\ &\leq \frac{C_1(\alpha,p)}{Re\,z},\end{aligned}$$

since the function

$$\sigma \mapsto \frac{\exp\left(\frac{-(\sigma^{-2m\alpha+1}-1)}{2}.\mu_0\right)}{\sigma^{\frac{N\alpha}{p}}}$$

is continuous on $[0,1]$. Moreover

$$\begin{aligned}
\frac{1}{2m\alpha-1}\frac{\exp\left(-\sigma^{-2m\alpha+1}.Re\,\mu\right)}{\sigma^{\frac{N\alpha}{p}}}I_2 &\leq \frac{1}{2m\alpha-1}\frac{\exp\left(-\sigma^{-2m\alpha+1}.Re\,\mu\right)}{\sigma^{\frac{N\alpha}{p}}\left(\frac{1+\sigma^{-2m\alpha+1}}{2}\right)^{\frac{N\alpha}{p(2m\alpha-1)}}}\int_{\frac{1+\sigma^{-2m\alpha+1}}{2}}^{\sigma^{-2m\alpha+1}}\exp\left(s.Re\,\mu\right)ds \\
&\leq \frac{1}{Re\,z}\frac{C_2(\alpha,p)}{\sigma^{\frac{N\alpha}{p}}\left(\frac{1+\sigma^{-2m\alpha+1}}{2}\right)^{\frac{N\alpha}{p(2m\alpha-1)}}} \\
&\leq \frac{C_3(\alpha,p)}{Re\,z}
\end{aligned}$$

in virtue of the fact that

$$\lim_{\sigma\to 0}\frac{1}{\sigma^{\frac{N\alpha}{p}}\left(1+\sigma^{-2m\alpha+1}\right)^{\frac{N\alpha}{p(2m\alpha-1)}}}=1.$$

Consequently, there exists some constant $C(\alpha,p)>0$ such that

$$\max_{\sigma\in[0,1]}\int_0^1\left|K_z(t,\sigma)\right|dt\leq\frac{C(\alpha,p)}{Re\,z}. \tag{4.2}$$

This shows that the condition b) of Lemma 1 holds also true. Now, using Lemma 1 together with (4.1) and (4.2), we obtain

$$\left\|t^{-2m\alpha+(N\alpha/p)}w\right\|_{L^p(0,1;X)}\leq\frac{C(\alpha,p)}{Re\,z}\left\|t^{-2m\alpha+(N\alpha/p)}k\right\|_{L^p(0,1;X)},$$

from which it follows

$$\left\|(B+zI)^{-1}\right\|_{L(E)}\leq\frac{C(\alpha,p)}{Re\,z}.$$

Thus, we can take $\theta_B=\frac{\pi}{2}-\theta_0$, (for each $\theta_0\in\left]0,\frac{\pi}{2}\right[$).

2. Now, we are concerned with the operator A which has the same properties as its realization $L(t)$. The study uses the following perturbation result due to Lunardi ([22, Proposition 2.4.3, p. 65]).

Proposition 2. *Let L_0 be a linear operator of domain $D(L_0)$ dense in E. Assume that L_0 is sectorial and P a linear continuous operator on $D(L_0)$ which is compact. Then operator $L_0+P:D(L_0)\to X$ is sectorial.*

For each $t\in[0,1]$, we write

$$L(t)\psi = L_0\psi+P(t)\psi$$

with

$$\begin{cases} D(L_0) &= \left\{\psi\in W^{2m,p}(B(0,1)):\partial_\nu^l\psi|_{\partial B(0,1)}=0,\ l=0,1,\ldots,m-1\right\} \\ L_0\psi &= M\psi=\sum_{k=1}^N\partial_{y_k}^{2m}\psi\end{cases}$$

and

$$\begin{cases} D(P(t)) & = & W^{1,p}(B(0,1)) \\ P(t)\psi & = & -\alpha t^{2m\alpha-1}\sum_{k=1}^{N} y_k \partial_{y_k}\psi, \end{cases}$$

It is well known that $\overline{D(L_0)} = L^p(B(0,1))$. The fact that L_0 is sectorial can be proved as in [8, Lemma 5.2 and Lemma 5.3 , pp. 18-19]. Observe that

$$\psi^{(l)}(y) = \int_0^y (-s\psi^{(l+1)}(s))ds + \int_y^1 (1-s)\psi^{(l+1)}(s)ds - \int_0^1 \psi^{(l)}(s)ds,$$

$l = 1, 2, ..., 2m-1$, where $\psi^{(l)}$ denotes the derivative of order l of ψ. Thanks to Hölder inequality, for $\psi \in D(L_0) \subset D(P(t))$ and by using the previous equality we have

$$\begin{array}{l} \|P(t)\psi\|_{L^p(B(0,1))} \\ \quad = \left(\int_{B(0,1)} \left|-\alpha t^{2m\alpha-1}\sum_{k=1}^{N} y_k\partial_{y_k}\psi(y_1,y_2,...,y_N)\right|^p dy_1 dy_2...dy_N\right)^{\frac{1}{p}} \\ \quad \leq \sum_{k=1}^{N}\left(\int_{B(0,1)} \left|-\alpha t^{2m\alpha-1} y_k(\int_0^{y_k} s_k\partial_{s_k}^2\psi ds_k + \int_{y_k}^1 (1-s_k)\partial_{s_k}^2\psi ds_k)\right|^p dy_1...dy_N\right)^{\frac{1}{p}} \\ \quad \leq \alpha t^{2m\alpha-1}\left[C_1(p)\|M\psi\|_{L^p(B(0,1))} + C_2(p)\|M\psi\|_{L^p(B(0,1))}\right] \\ \quad \leq C_3(\alpha,p)\|\psi\|_{D(L_0)}. \end{array}$$

On the other hand, let us set

$$\begin{array}{llll} m_k(t): & L^p(B(0,1)) & \to & L^p(B(0,1)) \\ & \psi & \mapsto & (m_k(t)\psi) = -\alpha t^{2m\alpha-1} y_k\psi,\ k = 1,...,N, \\ & & & \\ i: & W^{1,p}(B(0,1)) & \to & L^p(B(0,1)) \\ & \psi & \mapsto & \psi, \\ & & & \\ d_k: & W^{2m,p}(B(0,1)) & \to & W^{1,p}(B(0,1)) \\ & \psi & \mapsto & d_k(\psi) = \partial_{y_k}\psi,\ k = 1,...,N. \end{array}$$

Then one has

$$P(t) = \sum_{k=1}^{N} P_k(t) = \sum_{k=1}^{N} m_k(t) \circ i \circ d_k.$$

Thus, $P(t)$ is compact from $D(L_0)$ into E since i is compact and $d_k, m_k(t), k = 1,...,N$ are continuous. So for any $t \in [0,1]$, the operator $L(t)$ is sectorial and consequently there exist some $r_0 > 0$ and $\theta_1 \in \left]0, \frac{\pi}{2}\right[$ such that

$$\rho(-L(t)) \supset \Sigma_{\pi-\theta_1} = \{z : |z| \geq r_0,\ |\arg z| < \pi - \theta_1\}.$$

Now, for $k \in E$ and $z \in \Sigma_{\pi-\theta_1}$ the spectral equation

$$Aw + zw = k,$$

is equivalent to

$$L(t)w(t) + zw(t) = k(t),\ t \in [0,1],$$

which admits a unique solution

$$w(t) = (L(t) + zI)^{-1} k(t).$$

Hence

$$\|w(t)\|_{L^p(B(0,1))} \leq \frac{K}{|z|} \|k(t)\|_{L^p(B(0,1))},$$

which implies

$$\|w\|_E = \left(\int_0^1 \left\| t^{-2m\alpha+(N\alpha/p)} w(t) \right\|_X^p d\tau \right)^{1/p} \leq \frac{K}{|z|} \|k\|_E.$$

This ends the proof of Proposition 1. □

Proposition 3. *Operators A and B satisfy the Labbas-Terreni condition (2.4).*

Proof. In our case, since the domains $D(L(t))$ are constant, the condition (2.4) holds whenever the following so-called estimate of Sobolevskii [30] is fulfilled: There exists $K > 0$ such that for all $t, \sigma \in [0, 1]$,

$$\left\| (L(t)L(\sigma)^{-1} - I) \right\|_{L(X)} \leq K |t - \sigma|^{\rho} \tag{4.3}$$

For $g \in X = L^p(B(0,1))$, the equation $\psi = L(\sigma)^{-1} g$ is equivalent to

$$\begin{cases} (L(t)\psi)(y) &= M\psi - \alpha t^{2m\alpha-1} \sum_{k=1}^N y_k \partial_{y_k} \psi(y) = g(y), \\ \partial_\nu^l \psi|_{\partial B(0,1)} &= 0, \ l = 0, 1, \ldots, m-1, \end{cases}$$

and

$$[(L(t) - L(\sigma))L(\sigma)^{-1} g](y) = \alpha(\sigma^{2m\alpha-1} - t^{2m\alpha-1}) \sum_{k=1}^N y_k \partial_{y_k} \psi(y).$$

where $y = (y_1, y_2, \ldots, y_N)$. Then, we get

$$\begin{aligned} \left\| [(L(t) - L(\sigma))L(\sigma)^{-1} g] \right\|_X &\leq \alpha \left| t^{2m\alpha-1} - \sigma^{2m\alpha-1} \right| \left\| \textstyle\sum_{k=1}^N y_k \partial_{y_k} \psi \right\|_X \\ &\leq M_1 |t - \sigma|^{\min(1, 2m\alpha-1)} \|M\psi\|_{L^p(B(0,1))} \\ &\leq M_2 |t - \sigma|^{\min(1, 2m\alpha-1)} \|\psi\|_{W^{2m,p}(B(0,1))} \\ &\leq K |t - \sigma|^{\min(1, 2m\alpha-1)} \|g\|_{L^p(B(0,1))}. \end{aligned}$$

So, the condition (4.3) is satisfied with $\rho = \min(1, 2m\alpha - 1)$. To prove (2.4), it is sufficient to estimate

$$\left\| A(A+\lambda)^{-1} \left[A^{-1}; (B+z)^{-1} \right] \right\|_{L(E)}$$

where $\lambda \in \rho(-A)$ and $z \in \rho(-B)$. Let $k \in E$, then

$$
\begin{aligned}
D &= \left(t^{-2m\alpha+(N\alpha/p)} A (A+\lambda)^{-1} \left[A^{-1}; (B+z)^{-1}\right] k\right)(t) \\
&= t^{-2m\alpha+(N\alpha/p)} (A (A+\lambda)^{-1} \left(A^{-1} (B+z)^{-1} - (B+z)^{-1} A^{-1}\right) k)(t) \\
&= t^{-2m\alpha+(N\alpha/p)} L(t) (L(t)+\lambda)^{-1} \\
&\quad \times \left[L(t)^{-1} \left((B+z)^{-1} k\right)(t) - \left((B+z)^{-1} L(t)^{-1} k\right)(t)\right] \\
&= L(t) (L(t)+\lambda)^{-1} \int_0^1 \sigma^{-2m\alpha+(N\alpha/p)} K_\mu(t,\sigma) \left(L(t)^{-1} - L(\sigma)^{-1}\right) k(\sigma)\, d\sigma \\
&= \int_0^1 \sigma^{-2m\alpha+(N\alpha/p)} K_\mu(t,\sigma) L(t) (L(t)+\lambda)^{-1} \left(L(t)^{-1} - L(\sigma)^{-1}\right) k(\sigma)\, d\sigma \\
&= \int_0^1 \sigma^{-2m\alpha+(N\alpha/p)} K_\mu(t,\sigma) (L(t)+\lambda)^{-1} \left(I - L(t) L(\sigma)^{-1}\right) k(\sigma)\, d\sigma
\end{aligned}
$$

since the domains $D(L(t))$ are constant, where (we recall)

$$
K_\mu(t,\sigma) = \begin{cases} \dfrac{1}{t^{2m\alpha-(N\alpha/p)} \sigma^{N\alpha/p}} \exp \mu \left(t^{-2m\alpha+1} - \sigma^{-2m\alpha+1}\right) & \text{if } t > \sigma, \\ 0 & \text{if } t < \sigma, \end{cases}
$$

with $\mu = \frac{z}{(2m\alpha-1)}$. Then

$$
\|D\|_X \leq \frac{K}{|\lambda|} \int_0^1 \left|K_\mu(t,\sigma)\right| |t-\sigma|^\rho \sigma^{-2m\alpha+(N\alpha/p)} \|k(\sigma)\|_X\, d\sigma,
$$

with $\rho = \min(1, 2m\alpha - 1)$. We have

$$
\begin{aligned}
\int_0^1 \left|K_\mu(t,\sigma)\right| |t-\sigma|^\rho\, d\sigma &= \frac{1}{t^{2m\alpha-(N\alpha/p)}} \exp\left(t^{-2m\alpha+1}.Re\, \mu\right) \\
&\quad \times \int_0^t \sigma^{-N\alpha/p} (t-\sigma)^\rho \exp\left(-\sigma^{-2m\alpha+1}.Re\, \mu\right) d\sigma.
\end{aligned}
$$

Then by Hölder inequality, one has

$$
\begin{aligned}
&\int_0^t \sigma^{-N\alpha/p} (t-\sigma)^\rho \exp\left(-\sigma^{-2m\alpha+1}.Re\, \mu\right) d\sigma \\
&\qquad \leq \left(\int_0^t \sigma^{-N\alpha/p} \exp\left(-\sigma^{-2m\alpha+1}.Re\, \mu\right) d\sigma\right)^{1-\rho} \\
&\qquad\quad \times \left(\int_0^t \sigma^{-N\alpha/p} (t-\sigma) \exp\left(-\sigma^{-2m\alpha+1}.Re\, \mu\right) d\sigma\right)^\rho
\end{aligned}
$$

and

$$
\begin{aligned}
J_1 &= \left(\int_0^t \sigma^{2m\alpha-(N\alpha/p)} \sigma^{-2m\alpha} \exp\left(-\sigma^{-2m\alpha+1}.Re\, \mu\right) d\sigma\right)^{1-\rho} \\
&\leq \frac{\left(t^{2m\alpha-(N\alpha/p)}\right)^{1-\rho}}{(2m\alpha-1)^{1-\rho}} \frac{1}{(Re\, \mu)^{1-\rho}} \left(\exp\left(-t^{-2m\alpha+1}.Re\, \mu\right)\right)^{1-\rho} \\
J_2 &= \left(\int_0^t \sigma^{2m\alpha-(N\alpha/p)} \sigma^{-2m\alpha} (t-\sigma) \exp\left(-\sigma^{-2m\alpha+1}.Re\, \mu\right) d\sigma\right)^\rho \\
&\leq \frac{\left(t^{2m\alpha-(N\alpha/p)}\right)^\rho}{(2m\alpha-1)^\rho} \frac{1}{(Re\, \mu)^\rho} \left(\int_0^t (t-\sigma) \chi'(\sigma)\, d\sigma\right)^\rho
\end{aligned}
$$

where $\chi(\sigma) = \exp\left(-\sigma^{-2m\alpha+1}.Re\,\mu\right)$. Using an integration by parts, we obtain

$$\begin{array}{rcl} \int_0^t (t-\sigma)\chi'(\sigma)\,d\sigma & = & \int_0^t \exp\left(-\sigma^{-2m\alpha+1}.Re\,\mu\right)d\sigma \\ & = & \int_0^t \sigma^{2m\alpha}\sigma^{-2m\alpha}\exp\left(-\sigma^{-2m\alpha+1}.Re\,\mu\right)d\sigma \\ & \leq & \frac{t^{2m\alpha}}{2m\alpha-1}\dfrac{1}{Re\,\mu}\exp\left(-t^{-2m\alpha+1}.Re\,\mu\right) \end{array}$$

from which we deduce that

$$J_2 \leq \frac{\left(t^{2m\alpha-(N\alpha/p)}\right)^{\rho}}{(2m\alpha-1)^{\rho}}\frac{1}{(Re\,\mu)^{\rho}}\frac{\left(t^{2m\alpha}\right)^{\rho}}{(2m\alpha-1)^{\rho}}\frac{1}{(Re\,\mu)^{\rho}}\left(\exp\left(-t^{-2m\alpha+1}.Re\,\mu\right)\right)^{\rho}.$$

Finally we have

$$\begin{array}{l} \int_0^1 \left|K_\mu(t,\sigma)\right| |t-\sigma|^{\rho}\,d\sigma \\ \quad \leq \frac{\exp\left(t^{-2m\alpha+1}.Re\,\mu\right)}{t^{2m\alpha-(N\alpha/p)}}\frac{\left(t^{2m\alpha-(N\alpha/p)}\right)^{1-\rho}}{(2m\alpha-1)^{1-\rho}}\frac{\left(\exp\left(-t^{-2m\alpha+1}.Re\,\mu\right)\right)^{1-\rho}}{(Re\mu)^{1-\rho}} \\ \quad\quad \times\frac{\left(t^{2m\alpha-(N\alpha/p)}\right)^{\rho}}{(2m\alpha-1)^{\rho}}\frac{1}{(Re\,\mu)^{\rho}}\frac{\rho}{(2m\alpha-1)^{\rho}}\frac{1}{(Re\,\mu)^{\rho}}\left[\exp\left(-t^{-2m\alpha+1}Re\,.\mu\right)\right]^{\rho} \\ \quad \leq \frac{\left(t^{2m\alpha}\right)^{\rho}}{(2m\alpha-1)^{1+\rho}}\frac{1}{(Re\,\mu)^{1+\rho}}, \end{array}$$

and

$$\max_{t\in[0,1]}\int_0^1 \left|K_\mu(t,\sigma)\right| |t-\sigma|^{\rho}\,d\sigma \leq \frac{C}{(Re\,\mu)^{1+\rho}}. \tag{4.4}$$

In a similar manner we obtain

$$\max_{\sigma\in[0,1]}\int_0^1 \left|K_\mu(t,\sigma)\right| |t-\sigma|^{\rho}\,dt \leq \frac{C}{(Re\,\mu)^{1+\rho}}. \tag{4.5}$$

Now, using Schur interpolation Lemma together with (4.4) and (4.5), we obtain

$$\left\|A(A+\lambda)^{-1}\left[A^{-1};(B+z)^{-1}\right]\right\|_{L(E)} \leq \frac{C}{|\lambda|\,(Re\mu)^{1+\rho}} = \frac{C}{|\lambda|\,(Re\,z)^{1+\rho}}$$

which implies

$$\left\|A(A+\lambda)^{-1}\left[A^{-1};(B+z)^{-1}\right]\right\|_{L(E)} \leq \frac{C}{|\lambda|\,|z|^{1+\rho}}$$

for any $\lambda\in\rho(-A)$ and any z belonging to a simple path γ joining $\infty e^{-i\theta_2}$ to $\infty e^{i\theta_2}$ for some $\theta_2\in]\pi-\theta_B,\theta_1[$, γ lies to $\Sigma_{\pi-\theta_1}\cap\Sigma_{\pi-\theta_B}$. Then (2.4) is verified with $(\tau,\rho) = (0,\min(1,2m\alpha-1))$. $\square$

5. Regularity results for the original problem

5.1. *Regularity results for the transformed problem (3.1)*

Using Theorem 1, we deduce the following result

Proposition 4. *There exists λ^* such that for all $\lambda \geq \lambda^*$ and for all $k \in D_A(\sigma, p)$ (respectively, $k \in D_B(\sigma, p)$), Problem (3.2) admits a unique solution $w \in D(A) \cap D(B)$ such that for all $\theta \leq \min(\sigma, 2m\alpha - 1)$*

i) $L(.)w \in D_A(\theta, p)$,
ii) $t^\alpha w' \in D_A(\theta, p)$,
iii) $L(.)w \in D_B(\theta, p)$

(respectively,

i) $L(.)w \in D_B(\theta, p)$,
ii) $t^\alpha w' \in D_B(\theta, p)$,
iii) $L(.)w \in D_A(\theta, p)$*).*

Now, let us specify the space $D_A(\sigma, p)$. One has

$$D_A(\sigma, p) = \begin{cases} \left\{ w \in E : t^{-2m\alpha + (N\alpha/p)} w \in L^p\left(0, 1; W^{2m\sigma, p}(B(0,1))\right), \partial_\nu^l w\big|_{\partial B(0,1)} = 0 \right\} \\ \text{where } l = 0, 1, \ldots, m-1; \text{ if } 2m\sigma > 1/p, \\ \\ \left\{ w \in E : t^{-2m\alpha + (N\alpha/p)} w \in L^p\left(0, 1; W^{2m\sigma, p}(B(0,1))\right) \right\} \text{ if } 2m\sigma < 1/p. \end{cases}$$

Indeed, we know that

$$D_A(\sigma, p) = \left\{ w \in E : \left\| \zeta^{1-\sigma} A e^{-\zeta A} w \right\|_E \in L_*^p \right\},$$

because $(-A)$ is a generator of the analytic semigroup $\left\{ e^{-\zeta A} \right\}_{\zeta \geq 0}$. Now, $w \in D_A(\sigma, p)$ implies

$$\left\| \zeta^{1-\sigma} A e^{-\zeta A} w \right\|_E \in L_*^p.$$

Or $\left\| \zeta^{1-\sigma} A e^{-\zeta A} w \right\|_E \in L_*^p$ is equivalent to

$$\begin{aligned} &\int_0^\infty \left\| \zeta^{1-\sigma} A e^{-\zeta A} w \right\|_E^p \frac{d\zeta}{\zeta} \\ &= \int_0^\infty \left\| t^{-2m\alpha + (N\alpha/p)} \zeta^{1-\sigma} A e^{-\zeta A} w \right\|_{L^p(0,1; L^p(B(0,1)))}^p \frac{d\zeta}{\zeta} \\ &= \int_0^\infty \left(\int_0^1 \left\| t^{-2m\alpha + (N\alpha/p)} \zeta^{1-\sigma} \left(A e^{-\zeta A} w \right)(t) \right\|_{L^p(B(0,1))}^p dt \right) \frac{d\zeta}{\zeta} < +\infty. \end{aligned}$$

On the other hand, thanks to the Dunford representation of the semigroup $\left\{e^{-\zeta A}\right\}_{\zeta\geq 0}$, we have

$$e^{-\zeta A}=\frac{1}{2i\pi}\int_{\gamma}e^{\zeta\lambda}\left(A+\lambda I\right)^{-1}d\lambda,$$

where γ is a sectorial curve lying in $\rho\left(-A\right)$ such that $Re\left(-\lambda\right)<0$ for a larger $\lambda\in\gamma$. Moreover

$$\left(Ae^{-\zeta A}w\right)\left(t\right)=L\left(t\right)e^{\zeta L(t)}\left(w\left(t\right)\right).$$

Then, by Fubini's Theorem, we obtain

$$\begin{aligned}
&\int_0^\infty\left\|\zeta^{1-\sigma}Ae^{-\zeta A}w\right\|_E^p\frac{d\zeta}{\zeta}\\
&=\int_0^\infty\left[\int_0^1\left\|t^{-2m\alpha+(N\alpha/p)}\zeta^{1-\sigma}L\left(t\right)e^{\zeta L(t)}\left(w\left(t\right)\right)\right\|_{L^p(B(0,1))}^p dt\right]\frac{d\zeta}{\zeta}\\
&=\int_0^1\left\|t^{-2m\alpha+(N\alpha/p)}\right\|^p\left[\int_0^\infty\left\|\zeta^{1-\sigma}L\left(t\right)e^{\zeta L(t)}w\left(t\right)\right\|_{L^p(B(0,1))}^p\frac{d\zeta}{\zeta}\right]dt<+\infty,
\end{aligned}$$

which means that, for almost every t, the function

$$\left(y_1,y_2,...,y_N\right)\quad\mapsto\quad t^{-2m\alpha+(N\alpha/p)}(t)w\left(t\right)\left(y_1,y_2,...,y_N\right)$$

is in $D_{L(t)}\left(\sigma,p\right)$. It is well known that this last space is the following:

$$D_{L(t)}\left(\sigma,p\right)=\left(W^{2m,p}\left(B(0,1)\right)\cap W_0^{m,p}\left(B(0,1)\right);L^p\left(B(0,1)\right)\right)_{1-\sigma,p}$$

and

$$\begin{aligned}
&\left(W^{2m,p}\left(B(0,1)\right)\cap W_0^{2,p}\left(B(0,1)\right);L^p\left(B(0,1)\right)\right)_{1-\sigma,p}\\
&=\begin{cases}\left\{w\in W^{2m\sigma,p}\left(B(0,1)\right):\partial_\nu^l w|_{\partial B(0,1)}=0,\ l=0,1,...,m-1\right\}\\ \text{if } 2m\sigma>1/p,\\ \\ W^{2m\sigma,p}\left(B(0,1)\right)\ \text{if } 2m\sigma<1/p.\end{cases}
\end{aligned}$$

Let σ be a fixed positive number satisfying $\sigma<1/2mp$ and $\sigma\leq 2m\alpha-1$. From the above proposition, we deduce the following result.

Proposition 5. *For all h such that $t^{-2m\alpha+(N\alpha/p)}h\in L^p\left(0,1;W^{2m\sigma,p}\left(B(0,1)\right)\right)$, Problem (3.3) admits a unique solution fulfilling the following regularity properties:*

(i) $w\in L^p\left(]0,1[\times B(0,1)\right)$, $t^{-2m\alpha+(N\alpha/p)}w\in L^p\left(]0,1[\times B(0,1)\right)$, $w\left(0\right)=0$,
(ii) $t^{-2m\alpha+(N\alpha/p)}Mw\in L^p\left(]0,1[\times B(0,1)\right)$,
(iii) $t^{N\alpha/p}\partial_t w\in L^p\left(]0,1[\times B(0,1)\right)$,
(iv) $t^{-2m\alpha+(N\alpha/p)}Mw\in L^p\left(0,1;W^{2m\sigma,p}\left(B(0,1)\right)\right)$,
(v) $t^{N\alpha/p}\partial_t w\in L^p\left(0,1;W^{2m\sigma,p}\left(B(0,1)\right)\right)$.

5.2. *Going back to the original problem (1.1)*

We go back to our original domain Q by using the inverse change of variables

$$\begin{array}{llll}\Pi^{-1} & : G =]0,1[\times B(0,1) & \longrightarrow & Q \\ & (t,y_1,y_2,...,y_N) & \longmapsto & (t,x_1,x_2,...,x_N) = (t,t^{\alpha}y_1,t^{\alpha}y_2,...,t^{\alpha}y_N).\end{array}$$

Let us recall that

$$\begin{cases} h(t,y_1,y_2,...,y_N) = t^{2m\alpha} g(t,y_1,y_2,...,y_N) \\ g(t,y_1,y_2,...,y_N) = f(t,x_1,x_2,...,x_N), \\ w(t,y_1,y_2,...,y_N) = u(t,x_1,x_2,...,x_N). \end{cases}$$

First, we see that

$$\begin{cases} \partial_{y_k} w = t^{\alpha}\partial_{x_k} u,\ k = 1,2,...,N, \\ \partial_{y_k}^{2m} w = t^{2m\alpha}\partial_{x_k}^{2m} u,\ k = 1,2,...,N, \\ \partial_t w = \partial_t u + (\alpha/t)\sum_{k=1}^{N} x_k \partial_{x_k} u. \end{cases}$$

The assumption $t^{-2m\alpha+(N\alpha/p)}h \in L^p\left(0,1; W^{2m\sigma,p}(B(0,1))\right)$ means that

$$\int_0^1 \left\| t^{-2m\alpha+(N\alpha/p)}(t)h(t,.)\right\|^p_{W^{2m\sigma}(B(0,1))} dt < \infty.$$

So, by setting

$$y = (y_1,y_2,...,y_N),\ y' = \left(y_1',y_2',...,y_N'\right),\ dy = dy_1...dy_N,\ dy' = dy_1'...dy_N',$$

we have

$$\begin{aligned} &\textstyle\int_0^1 \left\| t^{-2m\alpha+(N\alpha/p)}(t)h(t,.)\right\|^p_{W^{2m\sigma}(B(0,1))} dt \\ &\quad = \int_0^1 t^{N\alpha-2m\alpha p} \int_{B(0,1)}\int_{B(0,1)} \frac{|h(t,y)-h(t,y')|^p}{\|y-y'\|^{2m\sigma p+N}} dy dy' dt \\ &\quad = \int_0^1 t^{2m\sigma\alpha p} \int_{\Omega_t}\int_{\Omega_t} \frac{|f(t,x)-f(t,x')|^p}{\|x-x'\|^{2m\sigma p+N}} dx dx' dt, \end{aligned}$$

where

$x = (t^{\alpha}y_1,t^{\alpha}y_2,...,t^{\alpha}y_N)$, $x' = \left(t^{\alpha}y_1',t^{\alpha}y_2',...,t^{\alpha}y_N'\right)$, $dx = dx_1...dx_N$, $dx' = dx_1'...dx_N'$

and

$$\Omega_t = \left\{(x_1,x_2,...,x_N) \in \mathbb{R}^N : 0 \leq \sqrt{x_1^2+x_2^2+...+x_N^2} < t^{\alpha}\right\}.$$

Let us introduce the following subspace of $L^p(Q)$:

$$L^p_{t^{2m\sigma\alpha}}\left(0,1; W^{2m\sigma,p}_{t^{\alpha}}\right) =$$

$$\left\{ f \in L^p(Q) : \int_0^1 t^{2m\alpha\sigma p} \int_{\Omega_t}\int_{\Omega_t} \frac{|f(t,x)-f(t,x')|^p}{\|x-x'\|^{2m\sigma p+N}} dx dx' dt < \infty \right\}.$$

Then, we are in position to prove the main result of this work.

Theorem 2. *For given $\sigma \in \left]0,1\right[$ such that $0 < \sigma < \dfrac{1}{2mp}$ (such that p verifies (1.2)), and for any $f \in L^p_{t^{2m\alpha\sigma}}\left(0,1;W^{2m\sigma,p}_{t^\alpha}\right)$, Problem (1.1) has a unique solution $u \in H^{1,2m}_p(Q)$ with the regularities: u, $\partial_t u$, $\partial_{x_k} u$, $k = 1,\ldots,N$ and Mu belong to $L^p_{t^{2m\alpha\sigma}}\left(0,1;W^{2m\sigma,p}_{t^\alpha}\right)$.*

The proof of Theorem 2 can be easily deduced from the following equivalences.

Proposition 6.

(i) $t^{-2m\alpha+(N\alpha/p)}h \in L^p\left(0,1;W^{2m\sigma,p}(B(0,1))\right)$ *if and only if*

$$f \in L^p_{t^{2m\alpha\sigma}}\left(0,1;W^{2m\sigma,p}_{t^\alpha}\right),$$

(ii) $t^{-2m\alpha+(N\alpha/p)}w \in L^p\left(0,1;L^p(B(0,1))\right)$ *if and only if* $u \in L^p(Q)$,

(iii) $t^{-2m\alpha+(N\alpha/p)}Mw \in L^p\left(0,1;W^{2m\sigma,p}(B(0,1))\right)$ *if and only if*

$$Mu \in L^p_{t^{2m\alpha\sigma}}\left(0,1;W^{2m\sigma,p}_{t^\alpha}\right),$$

(iv) $t^{N\alpha/p}\partial_t w \in L^p\left(0,1;W^{2m\sigma,p}(B(0,1))\right)$ *if and only if*

$$\partial_t u \in L^p_{t^{2m\alpha\sigma}}\left(0,1;W^{2\sigma,p}_{t^\alpha}\right).$$

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