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ANALYTICAL INVESTIGATION FOR MODIFIED RIEMANN-LIOUVILLE FRACTIONAL EQUAL-WIDTH EQUATION TYPES BASED ON $(G'/G)-$ EXPANSION TECHNIQUE

BAHAR KARAMAN

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Abstract. The present investigation studies exact solutions of modified Riemann-Liouville fractional Equal-Width (MRLFEW) equation types with the help of the $(G'/G)-$ expansion method. Firstly, the MRLFEW equation is converted into an ordinary differential equation via fractional complex transform. Then, the proposed method has applied this equation to construct the exact solutions.

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1. Introduction

In recent years, real-world phenomena can be modeled successfully by the partial differential equations with fractional derivatives (FPDEs). These equations have significant applications in various areas such as fluid mechanics, viscoelastic materials, finance, control theory, fractional dynamics, biology, physics, applied mathematics and engineering [1, 3, 4, 6, 8, 10, 13, 18]. Many studies have been introduced by many researchers [5, 7, 15, 17]. To seek exact solutions of the FPDEs, many well-known methods have been proposed such as the exp-function method [16], modified simple method [21], the fractional sub-equation method [2, 23], (G'/G)-expansion method [20, 22].

In this manuscript, we are mainly concerned to explore traveling wave solutions of MRLFEW equation types. So far, various analytical methods have been suggested for the equation types by the references therein [11, 14, 19]. To our knowledge, (G'/G)-expansion technique is not used to generate the traveling wave solutions of MRLFEW equation types. Adding to this, the advantages of this technique are ease of implementation and obtaining new solutions. That's why we implement the technique to construct exact solutions to the mentioned equations. The layout of this work is as follows: In Section 1, a brief of the MRLF derivative is given. The next Section

2 gives the description of the (G'/G)-expansion methodology. In Section 3, we construct the implementation of the suggested method for the MRLFEW equation types. The results are also illustrated in this section. Finally, we end with a short conclusion in Section 4.

1.1. *A Brief of the MRLF derivative*

Jumarie is expressed $\alpha-$th order fractional derivative in sense of MRL of a given function $y = v(t)$ as

$$D_t^{\alpha} v(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha}(v(\xi)-v(0))d\xi, & 0<\alpha<1, \\ \\ (v^{(m)}(t))^{\alpha-m}, & m \leq \alpha < m+1, \quad m \geq 1 \end{cases} \tag{1.1}$$

This definition has some prominent properties. They are given [12] following as

$$D_t^{\alpha} t^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} t^{\alpha-p}, \quad p>0 \tag{1.2}$$

$$D_t^{\alpha}(f(t)g(t)) = f(t)D_t^{\alpha} g(t) + g(t) D_t^{\alpha} f(t). \tag{1.3}$$

2. The (G'/G)- expansion methodology

We outline the basic idea of the (G'/G)- expansion technique. Firstly, a general nonlinear fractional differential equation with two independent variables x and t is regarded as

$$R(v, D_t^{\alpha} v, D_x^{\beta} v, D_t^{\alpha} D_t^{\alpha} v, D_t^{\alpha} D_x^{\beta} v, D_x^{\beta} D_x^{\beta} v, \cdots) = 0, \quad 0<\alpha,\beta<1, \tag{2.1}$$

in here v is an unknown function, R is a polynomial of v and its various partial derivatives containing the highest order derivatives and nonlinear terms.

There are main steps to implement the proposed method. These are as follows.

Stage 1: A nonlinear fractional complex transformation proposed by Li and He [9] is used to reduce fractional differential equations into ordinary differential equations (ODEs). This transformation is described as

$$v(x,t) = V(\xi), \quad \xi = \frac{c_1 x^{\beta}}{\Gamma(1+\beta)} - \frac{c_2 t^{\alpha}}{\Gamma(1+\alpha)} \tag{2.2}$$

where c_1 and c_2 are arbitrary constants. We would also like to note that the chain rule can be computed as

$$D^{\alpha}{}_t v = \sigma_1 \frac{dV}{d\xi} D_t^{\alpha} \xi \tag{2.3}$$

$$D^{\beta}{}_x v = \sigma_2 \frac{dV}{d\xi} D_x^{\beta} \xi \tag{2.4}$$

where σ_1, σ_2 are fractional indexes [9]. Eqn. (2.1) can be rewritten by using Eqns. (2.2) and the chain rule (2.3)-(2.4) as the following ODE:

$$F(V, V', V'', V''', \cdots) = 0 \tag{2.5}$$

in here the prime denotes the derivation with respect to ξ.

Stage 2: Assuming the exact solution of Eqn. (2.5) can be represented by a polynomial in (G'/G) as the following form:

$$v(\xi) = \sum_{j=0}^{p} b_j \left(\frac{G'}{G}\right)^j, \quad b_p \neq 0 \tag{2.6}$$

where $b_0, b_1, \cdots, b_p$ are constants. Also, $G(\xi)$ satisfies the second-order linear ODE which is defined by

$$G''(\xi) + \lambda_1 G'(\xi) + \lambda_2 G(\xi) = 0 \tag{2.7}$$

with $\lambda_{1,2}$ being arbitrary constants. Balancing between the highest order derivatives and the nonlinear term rising in Eqn. (2.5) is used to determine the balancing number p.

Stage 3: In this step, we first substitute Eqn. (2.6) into Eqn. (2.5) and use Eqn. (2.7). Then, we gather up all the coefficients with the same power of (G'/G). Equalizing each term of the obtaining polynomial to zero yields a set of algebraic equations for $b_0, b_1, \cdots, b_p, \lambda_1, \lambda_2, c_1$, and c_2.

Stage 4: We construct the constants $b_0, b_1, \cdots, b_p, \lambda_1, \lambda_2, c_1$ and c_2 by solving the obtained system. Substituting these parameters along with the general solution of Eqn. (2.7) into Eqn. (2.6), a variety exact solutions of Eqn. (2.1) is obtained.

3. Implementation of the suggested method for MRLFEW equation types

The main aim of this part is to solve MRLFEW equation types based on the above-mentioned methodology. We consider the equation types that is defined as follows

$$D_t^{\alpha} v(x,t) + \varepsilon D_x^{\alpha} v^2(x,t) - \delta D_{xxt}^{3\alpha} v(x,t) = 0 \tag{3.1}$$

and

$$D_t^{\alpha} v(x,t) + \varepsilon D_x^{\alpha} v^3(x,t) - \delta D_{xxt}^{3\alpha} v(x,t) = 0 \tag{3.2}$$

where ε and δ are real parameters and α is the order of MRLF derivative. The first equation is known as *fractional EW equation*. The second is called *modified fractional EW equation*.

3.1. *Fractional EW equation*

By substituting the transformation (2.2) into Eqn. (3.1), the following ODE can be found

$$-cV' + \varepsilon k(V^2)' + \delta ck^2 V''' = 0 \tag{3.3}$$

where $c = -\sigma_1 c_2$ and $k = \sigma_2 c_1$. Let's integrate Eqn. (3.3) once and setting the integration constant to zero yield

$$-cV + \varepsilon kV^2 + \delta ck^2 V'' = 0 \tag{3.4}$$

Eqn. (3.4) gives the balancing number,

$$m + 2 = 2m,$$

so

$$m = 2.$$

We assume that the solution of Eqn. (3.4) can be described by a polynomial in (G'/G) as follows:

$$V(\xi) = b_0 + b_1\left(\frac{G'}{G}\right) + b_2\left(\frac{G'}{G}\right)^2, \quad b_2 \neq 0 \tag{3.5}$$

Using Eqn. (2.7) and Eqn. (3.5), we have

$$\begin{aligned} V^2(\xi) &= b_0^2 + b_1^2\left(\frac{G'}{G}\right)^2 + b_2^2\left(\frac{G'}{G}\right)^4 + 2b_0b_1\left(\frac{G'}{G}\right) \\ &\quad + 2b_0b_2\left(\frac{G'}{G}\right)^2 + 2b_1b_2\left(\frac{G'}{G}\right)^3, \end{aligned} \tag{3.6}$$

$$\begin{aligned} V''(\xi) &= (b_1\lambda_1 + 2b_2\lambda_2)\left[\left(\frac{G'}{G}\right)^2 + \lambda_1\left(\frac{G'}{G}\right) + \lambda_2\right] \\ &\quad + 2(b_1 + 2b_2\lambda_1)\left[\left(\frac{G'}{G}\right)^3 + \lambda_1\left(\frac{G'}{G}\right)^2 + \lambda_2\left(\frac{G'}{G}\right)\right] \\ &\quad + 6b_2\left[\left(\frac{G'}{G}\right)^4 + \lambda_1\left(\frac{G'}{G}\right)^3 + \lambda_2\left(\frac{G'}{G}\right)^2\right]. \end{aligned} \tag{3.7}$$

Let's substitute Eqns. (3.5)-(3.6) into Eqn. (3.4). Then collecting the coefficients of $(G'/G)^j, (j = 0,1,2,3,4)$ and setting them to be zero, the system is found as form:

$$-cb_0 + ekb_0^2 + b_1\lambda_1\lambda_2\delta ck^2 + 2b_2\lambda_2^2\delta ck^2 + \chi_0 = 0, \tag{3.8}$$

$$\begin{aligned} &-cb_1 + 2\varepsilon kb_0b_1 + b_1\lambda_1^2\delta ck^2 + 2b_2\lambda_1\lambda_2\delta ck^2 \\ &\quad +2b_1\lambda_2\delta ck^2 + 4b_2\lambda_1\lambda_2\delta ck^2 = 0, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &-cb_2 + \varepsilon kb_1^2 + 2\varepsilon kb_0b_2 + b_1\lambda_1\delta ck^2 + 2b_2\lambda_2\delta ck^2 \\ &\quad +2b_1\lambda_1\delta ck^2 + 4b_2\lambda_1^2\delta ck^2 + 6b_2\lambda_2\delta ck^2 = 0, \end{aligned} \tag{3.10}$$

$$2\varepsilon k b_1 b_2 + 2b_1 \delta c k^2 + 4b_2 \lambda_1 \delta c k^2 + 6b_2 \lambda_1 \delta c k^2 = 0, \tag{3.11}$$

$$\varepsilon k b_2^2 + 6\delta c k^2 b_2 = 0. \tag{3.12}$$

This system is solved by using Maple, we obtain

$$b_0 = -\frac{c(k^2\lambda_1^2\delta + 8k^2\lambda_2\delta - 1)}{2k\varepsilon}, \quad b_1 = -\frac{6ck\delta\lambda_1}{\varepsilon}, \quad b_2 = -\frac{6ck\delta}{\varepsilon}$$
$$\delta = \delta, \quad \varepsilon = \varepsilon, \quad c = c, \quad k = k. \tag{3.13}$$

where λ_1 and λ_2 are arbitrary constants. Eqn. (3.5) can be rewritten by using Eqn. (3.13) as follows:

$$V(\xi) = -\frac{c(k^2\lambda_1^2\delta + 8k^2\lambda_2\delta - 1)}{2k\varepsilon} - \frac{6ck\delta\lambda_1}{\varepsilon}\left(\frac{G'}{G}\right) - \frac{6ck\delta}{\varepsilon}\left(\frac{G'}{G}\right)^2 \tag{3.14}$$

The general solution of Eqn. (2.7) is substituted into Eqn. (3.14), we get three types of travelling wave solutions of the Eqn. (3.1) as follows:

When $\lambda_1^2 - 4\lambda_2 > 0$,

$$V_1(\xi) = -\frac{c(k^2\lambda_1^2\delta + 8k^2\lambda_2\delta - 1)}{2k\varepsilon} - \frac{6ck\delta\lambda_1}{\varepsilon}\left(\frac{G'}{G}\right) - \frac{6ck\delta}{\varepsilon}\left(\frac{G'}{G}\right)^2, \tag{3.15}$$

where

$$\left(\frac{G'}{G}\right) = -\frac{\lambda_1}{2} + \frac{\sqrt{\lambda_1^2 - 4\lambda_2}}{2}\left(\frac{C_1\cosh(\frac{\sqrt{\lambda_1^2-4\lambda_2}}{2})\xi + C_2\sinh(\frac{\sqrt{\lambda_1^2-4\lambda_2}}{2})\xi}{C_1\sinh(\frac{\sqrt{\lambda_1^2-4\lambda_2}}{2})\xi + C_2\cosh(\frac{\sqrt{\lambda_1^2-4\lambda_2}}{2})\xi}\right)$$

When $\lambda_1^2 - 4\lambda_2 < 0$,

$$V_2(\xi) = -\frac{c(k^2\lambda_1^2\delta + 8k^2\lambda_2\delta - 1)}{2k\varepsilon} - \frac{6ck\delta\lambda_1}{\varepsilon}\left(\frac{G'}{G}\right) - \frac{6ck\delta}{\varepsilon}\left(\frac{G'}{G}\right)^2, \tag{3.16}$$

where

$$\left(\frac{G'}{G}\right) = -\frac{\lambda_1}{2} + \frac{\sqrt{4\lambda_2 - \lambda_1^2}}{2}\left(\frac{C_1\cos(\frac{\sqrt{4\lambda_2-\lambda_1^2}}{2})\xi - C_2\sin(\frac{\sqrt{4\lambda_2-\lambda_1^2}}{2})\xi}{C_1\sin(\frac{\sqrt{4\lambda_2-\lambda_1^2}}{2})\xi - C_2\cos(\frac{\sqrt{4\lambda_2-\lambda_1^2}}{2})\xi}\right)$$

When $\lambda_1^2 - 4\lambda_2 = 0$,

$$V_3(\xi) = -\frac{c(k^2\lambda_1^2\delta + 8k^2\lambda_2\delta - 1)}{2k\varepsilon} - \frac{6ck\delta\lambda_1}{\varepsilon}\left(\frac{G'}{G}\right) - \frac{6ck\delta}{\varepsilon}\left(\frac{G'}{G}\right)^2, \tag{3.17}$$

where

$$\left(\frac{G'}{G}\right) = -\frac{\lambda_1}{2} + \frac{C_2}{C_1 + C_2\xi}.$$

3.2. *Modified fractional EW equation*

By substituting the transformation (2.2) into Eqn. (3.2), the following ODE can be found

$$-cV' + \varepsilon k(V^3)' + \delta ck^2 V''' = 0 \tag{3.18}$$

where $c = \sigma_1 c_2$ and $k = \sigma_2 c_1$. Let's integrate Eqn. (3.18) once, we get

$$-cV + \varepsilon kV^3 + \delta ck^2 V'' + \chi_1 = 0 \tag{3.19}$$

where χ_1 is integration constant. Eqn. (3.19) gives the balancing number,

$$m + 2 = 3m,$$

so

$$m = 1.$$

We assume that the solution of Eqn. (3.19) can be described by a polynomial in (G'/G) as follows:

$$V(\xi) = b_0 + b_1 \left(\frac{G'}{G}\right), \quad b_1 \neq 0 \tag{3.20}$$

Using Eqn. (2.7) and Eqn. (3.20), we have

$$\begin{aligned} V^3(\xi) = & \quad b_0^3 + 3b_0^2 b_1 \left(\frac{G'}{G}\right) + 3b_0 b_1^2 \left(\frac{G'}{G}\right)^2 + b_1^3 \left(\frac{G'}{G}\right)^3 \\ V''(\xi) = & \quad 2b_1 \left(\frac{G'}{G}\right)^3 + 3b_1\lambda_1 \left(\frac{G'}{G}\right)^2 \\ + & \quad (b_1\lambda_1^2 + 2b_1\lambda_2)\left(\frac{G'}{G}\right) + b_1\lambda_1\lambda_2 \end{aligned} \tag{3.21}$$

Let's substitute Eqns. (3.20)-(3.21) into Eqn. (3.19). Then collecting the coefficients of $(G'/G)^j, (j = 0, 1, 2, 3)$ and setting them to be zero, the algebraic equation system is found as form:

$$-cb_0 + \varepsilon kb_0^3 + \delta ck^2 b_1\lambda_1\lambda_2 + \chi_1 = 0, \tag{3.22}$$

$$-cb_1 + 3\varepsilon kb_0^2 b_1 + 2\delta ck^2 b_1\lambda_2 + \delta ck^2 b_1\lambda_1^2 = 0, \tag{3.23}$$

$$3\varepsilon kb_0 b_1^2 + 3\delta ck^2 b_1\lambda_1 = 0, \tag{3.24}$$

$$\varepsilon kb_1^3 + 2b_1\delta ck^2 = 0. \tag{3.25}$$

This system is solved by using Maple, we obtain

$$\begin{aligned} b_0 = \pm\frac{\sqrt{2\varepsilon ck\delta}\lambda_1 i}{2\varepsilon}, \quad b_1 = \pm\frac{\sqrt{2\varepsilon ck\delta}i}{2\varepsilon}, \\ \chi_1 = 0, \quad k = \pm\frac{2}{\sqrt{-2\lambda_1^2\delta + 8\lambda_2\delta}}, \\ \delta = \delta, \quad \varepsilon = \varepsilon, \quad c = c \end{aligned} \tag{3.26}$$

where λ_1 and λ_2 are arbitrary constants. Eqn. (3.20) can be rewritten by using Eqn. (3.26) as follows:

$$V(\xi) = \pm\frac{\sqrt{2\varepsilon ck\delta}\lambda_1 i}{2\varepsilon} \pm \frac{\sqrt{2\varepsilon ck\delta}i}{2\varepsilon}\left(\frac{G'}{G}\right) \tag{3.27}$$

The general solution of Eqn. (2.7) is substituted into Eqn. (3.14), we get three types of travelling wave solutions of the Eqn. (3.1) as follows:

When $\lambda_1^2 - 4\lambda_2 > 0$,

$$V_1(\xi) = \pm\frac{\sqrt{2\varepsilon ck\delta}\lambda_1 i}{2\varepsilon} \pm \frac{\sqrt{2\varepsilon ck\delta}i}{2\varepsilon}\left(\frac{G'}{G}\right) \tag{3.28}$$

where

$$\left(\frac{G'}{G}\right) = -\frac{\lambda_1}{2} + \frac{\sqrt{\lambda_1^2 - 4\lambda_2}}{2}\left(\frac{C_1\cosh(\frac{\sqrt{\lambda_1^2-4\lambda_2}}{2})\xi + C_2\sinh(\frac{\sqrt{\lambda_1^2-4\lambda_2}}{2})\xi}{C_1\sinh(\frac{\sqrt{\lambda_1^2-4\lambda_2}}{2})\xi + C_2\cosh(\frac{\sqrt{\lambda_1^2-4\lambda_2}}{2})\xi}\right)$$

When $\lambda_1^2 - 4\lambda_2 < 0$,

$$V_2(\xi) = \pm\frac{\sqrt{2\varepsilon ck\delta}\lambda_1 i}{2\varepsilon} \pm \frac{\sqrt{2\varepsilon ck\delta}i}{2\varepsilon}\left(\frac{G'}{G}\right) \tag{3.29}$$

where

$$\left(\frac{G'}{G}\right) = -\frac{\lambda_1}{2} + \frac{\sqrt{4\lambda_2 - \lambda_1^2}}{2}\left(\frac{C_1\cos(\frac{\sqrt{4\lambda_2-\lambda_1^2}}{2})\xi - C_2\sin(\frac{\sqrt{4\lambda_2-\lambda_1^2}}{2})\xi}{C_1\sin(\frac{\sqrt{4\lambda_2-\lambda_1^2}}{2})\xi - C_2\cos(\frac{\sqrt{4\lambda_2-\lambda_1^2}}{2})\xi}\right)$$

When $\lambda_1^2 - 4\lambda_2 = 0$,

$$V_3(\xi) = \pm\frac{\sqrt{2\varepsilon ck\delta}\lambda_1 i}{2\varepsilon} \pm \frac{\sqrt{2\varepsilon ck\delta}i}{2\varepsilon}\left(\frac{G'}{G}\right) \tag{3.30}$$

where

$$\left(\frac{G'}{G}\right) = -\frac{\lambda_1}{2} + \frac{C_2}{C_1 + C_2\xi}.$$

4. Conclusion

In this study, we have obtained three traveling wave solutions of FEW and MFEW equations by using (G'/G)-expansion method. Modified Riemann-Liouville derivative is preferred for time-space fractional derivatives. Besides, fractional complex transform that is simple and effective, is implemented to convert FPDE into an ODE. We would also like to say that the method can be used for many other nonlinear fractional differential equations. Another point that is worthy of being emphasized is that the acquired solutions in this work have not been reported in the literature up to now.

As a final remark that the exactness of the obtained solutions is verified by substituting them back into the original equation. It can be said that they satisfy the FEW and MFEW equations under consideration.

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Author's address

Bahar Karaman
E-mail address: bahar_korkmaz@eskisehir.edu.tr

Department of Mathematics, Eskişehir Technical University, Eskişehir 26470, Turkey