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INTEGRAL AND LIMIT REPRESENTATIONS OF THE CMP INVERSE

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Abstract. We develop various integral and limit representations for the CMP inverse of a complex square matrix, which do not require any restriction on the spectrum of a corresponding matrix. Also, we present integral and limit representations for the DMP and MPD inverses.

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1. INTRODUCTION

Let $\mathbb{C}^{m\times n}$ be the set of all $m\times n$ complex matrices. We use $rank(A)$, A^*, $R(A)$ and $N(A)$ to denote the rank, the conjugate transpose, the range (column space) and the null space of $A\in\mathbb{C}^{m\times n}$, respectively. The index of $A\in\mathbb{C}^{n\times n}$, denoted by $\operatorname{ind}(A)$, is the smallest nonnegative integer k for which $rank(A^k)=rank(A^{k+1})$. By I will be denoted the identity matrix of corresponding size. If M and N are two complementary subspaces of $\mathbb{C}^{m\times 1}$ (that is, $\mathbb{C}^{m\times 1}$ is direct sum of M and N), we denote by $P_{M,N}$ the projector onto M along N. In the case that N is the subspace orthogonal to M, this notation will be reduced to P_M.

The Drazin inverse of $A\in\mathbb{C}^{n\times n}$ is the unique matrix $A^D=X\in\mathbb{C}^{n\times n}$ such that

$$A^{k+1}X=A^k,\quad XAX=X,\quad AX=XA.$$

where $k=\operatorname{ind}(A)$. If $\operatorname{ind}(A)=1$, then A^D is the group inverse of A, which is denoted by $A^\#$. For basic properties of the Drazin inverse and its various applications see [1, 3].

The Moore–Penrose inverse of $A\in\mathbb{C}^{m\times n}$ is the unique matrix $A^\dagger=X\in\mathbb{C}^{n\times m}$ which satisfies the Penrose equations

$$AXA=A,\qquad XAX=X,\qquad (AX)^*=AX,\qquad (XA)^*=XA.$$

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The Moore-Penrose inverse is a powerful tool in computing polar decomposition, the areas of electrical networks, control theory, filtering, estimation theory and pattern recognition.

Let $A \in \mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s \le r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m-s$. If a matrix $X \in \mathbb{C}^{n\times m}$ satisfies

$$XAX = X, \quad R(X) = T, \quad N(X) = S,$$

then X is called the outer inverse of A with the range T and the null-space S, and the notation $X = A^{(2)}_{T,S}$ is commonly used. Drazin [6] introduced a new class of outer inverses, called the (B,C)-inverses. For $A \in \mathbb{C}^{m\times n}$ and $B,C \in \mathbb{C}^{n\times m}$, if a matrix $X \in \mathbb{C}^{n\times m}$ satisfies $XAB = B$, $CAX = C$, $R(X) \subseteq R(B)$ and $N(C) \subseteq N(X)$, then X is called the (B,C)-inverse of A. In the case when X exists, it is unique and denoted by $X = A^{||(B,C)}$ [2,6]. By [2, Theorem 7.1], it follows that $A^{||(B,C)} = A^{(2)}_{R(B),N(C)}$.

Using the Drazin inverse and the Moore–Penrose inverse, Malik and Thome [9] defined a new generalized inverse of a square matrix of an arbitrary index, which is called the DMP inverse and defined as $A^{D,\dagger} = A^DAA^\dagger$, for $A \in \mathbb{C}^{n\times n}$. The DMP inverse for a Hilbert space operator was investigated in [13,17,19] as a generalization of the DMP inverse for a square matrix. For $A \in \mathbb{C}^{n\times n}$, the MPD inverse, as the dual DMP inverse, was given by $A^{\dagger,D} = A^\dagger AA^D$ [9].

Mehdipour and Salemi [10] introduced a new inverse of a square matrix A named CMP inverse, since they used the core part AA^DA of A and the Moore–Penrose inverse of A. The CMP inverse of $A \in \mathbb{C}^{n\times n}$ is defined as $A^{c,\dagger} = A^\dagger AA^DAA^\dagger$ and it is the unique solution of the following equations:

$$XAX = X, \qquad AXA = AA^DA, \qquad AX = AA^DAA^\dagger, \qquad XA = A^\dagger AA^DA.$$

For more details about the CMP inverse see [12, 18].

It is well-known that if the eigenvalues of $A \in \mathbb{C}^{n\times n}$ lie in the open right halfplane, then the inverse of A can be presented by

$$A^{-1} = \int_0^\infty \exp(-tA)dt.$$

Many integral representations of various generalized inverse such as Moore-Penrose inverse, Drazin inverse and DMP inverse were presented in papers [4,5,7,20]. Several of these integral representations have some restriction on the eigenvalues of A and the other holds without any restrictions on the eigenvalues.

Notice that investigation of the limit representations of different kinds of generalized inverses are hot topics many years. One limit representation of the Drazin inverse was proved by Meyer [11] in 1974. Some limit representations of the outer inverse are given in [8, 15, 16].

The above mentioned results motivate us to investigate the integral and limit representations of the CMP inverse of a square matrix, without any restriction on the spectrum of a certain matrix. Firstly, we develop these representations based on the full-rank decomposition of a given matrix. Then we establish integral and limit representations of the CMP inverse which depend on corresponding projections and expressions for the Moore-Penrose, Drazin and outer inverses. Various integral and limit representations of the DMP and MPD inverses are also derived.

2. Integral representations of the CMP inverse

In this section, we will establish integral representations of the CMP inverse for a square complex matrix without any restriction on the spectrum of matrix. If $A \in \mathbb{C}^{n\times n}$ is nilpotent, then $A^D = 0$ and so $A^{c,\dagger} = 0$. Since this case is trivial, we consider the matrix A to be non-nilpotent in this paper.

Lemma 1 ([1]). *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. If $A = B_1G_1$ is a full-rank decomposition and $G_iB_i = B_{i+1}G_{i+1}$ are also full-rank decompositions, $i = 1,2,\dots,k-1$. Then the following statements hold:*

(i) *G_kB_k is invertible;*
(ii) *$A^k = B_1B_2\dots B_kG_k\dots G_2G_1$;*
(iii) *$A^D = B_1B_2\dots B_k(G_kB_k)^{-k-1}G_k\dots G_2G_1$;*
(iv) *$A^\dagger = G_1^*(G_1G_1^*)^{-1}(B_1^*B_1)^{-1}B_1^*$.*

In particular, for $k = 1$, then G_1B_1 is invertible and $A^\# = B_1(G_1B_1)^{-2}G_1$.

Lemma 2. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and the full-rank decomposition of A as in Lemma 1. Then*

$$G_1A^{c,\dagger}B_1 = B_2\dots B_k(G_kB_k)^{-(k-1)}G_k\dots G_2.$$

Proof. By [20, Lemma 3.1], we have that $A^{D,\dagger}B_1 = B_1B_2\dots B_k(G_kB_k)^{-k}G_k\dots G_2$ which implies

$$A^{c,\dagger}B_1 = A^\dagger AA^{D,\dagger}B_1 = A^\dagger AB_1\dots B_k(G_kB_k)^{-k}G_k\dots G_2.$$

Therefore, by Lemma 1,

$$\begin{aligned}
G_1A^{c,\dagger}B_1 &= G_1A^\dagger AB_1\dots B_k(G_kB_k)^{-k}G_k\dots G_2\\
&= G_1G_1^*(G_1G_1^*)^{-1}(B_1^*B_1)^{-1}B_1^*B_1G_1B_1\dots B_k(G_kB_k)^{-k}G_k\dots G_2\\
&= G_1B_1\dots B_k(G_kB_k)^{-k}G_k\dots G_2 = B_2G_2B_2\dots B_k(G_kB_k)^{-k}G_k\dots G_2\\
&= B_2\dots G_kB_k(G_kB_k)^{-k}G_k\dots G_2 = B_2\dots B_k(G_kB_k)^{-(k-1)}G_k\dots G_2.
\end{aligned}$$

□

Theorem 1. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and the full-rank decomposition of A as in Lemma 1. Then*

$$A^{c,\dagger} = \int_0^\infty G_1^*\exp(-G_1G_1^*t)dt\int_0^\infty MB_1^*\exp(-B_1B_1^*u)du,$$

where $M = B_2\dots B_k(G_kB_k)^{-(k-1)}G_k\dots G_2$.

Proof. Set $X = G_1^\dagger M B_1^\dagger$. Recall that, by [7],

$$A^\dagger = \int_0^\infty A^* \exp(-AA^* t) dt. \tag{2.1}$$

It is enough to prove that $X = A^{c,\dagger}$. Because B_1 is a full-column rank matrix, then $B_1^\dagger = (B_1^* B_1)^{-1} B_1^*$ and so $B_1^\dagger B_1 = I$. Similarly, we have that $G_1^\dagger = G_1^*(G_1 G_1^*)^{-1}$ and $G_1 G_1^\dagger = I$. Notice that, using

$$G_k \dots G_2 B_2 \dots B_k = G_k \dots G_3 B_3 G_3 B_3 \dots B_k = \dots = (G_k B_k)^{k-1},$$

we get

$$\begin{aligned} XAX &= G_1^\dagger B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 B_1^\dagger B_1 G_1 G_1^\dagger M B_1^\dagger \\ &= G_1^\dagger B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 M B_1^\dagger \\ &= G_1^\dagger B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 B_1^\dagger \\ &= G_1^\dagger B_2 \dots B_k (G_k B_k)^{-(k-1)} (G_k B_k)^{k-1} (G_k B_k)^{-(k-1)} G_k \dots G_2 B_1^\dagger \\ &= G_1^\dagger B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 B_1^\dagger \\ &= X. \end{aligned}$$

Applying Lemma 1, we observe that

$$\begin{aligned} AA^D A &= B_1 G_1 B_1 B_2 \dots B_k (G_k B_k)^{-k-1} G_k \dots G_2 G_1 B_1 G_1 \\ &= B_1 B_2 G_2 B_2 \dots B_k (G_k B_k)^{-k-1} G_k \dots G_2 B_2 G_2 G_1 \\ &= B_1 B_2 \dots B_k G_k B_k (G_k B_k)^{-k-1} G_k B_k G_k \dots G_2 G_1 \\ &= B_1 B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 G_1. \end{aligned}$$

Therefore,

$$\begin{aligned} XA &= G_1^\dagger B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 G_1 \\ &= G_1^* (G_1 G_1^*)^{-1} (B_1^* B_1)^{-1} B_1^* B_1 B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 G_1 \\ &= A^\dagger AA^D A \end{aligned}$$

and

$$\begin{aligned} AX &= B_1 B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 B_1^\dagger \\ &= B_1 B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 G_1 G_1^* (G_1 G_1^*)^{-1} (B_1^* B_1)^{-1} B_1^* \\ &= AA^D AA^\dagger. \end{aligned}$$

By [12, Corollary 2.2], we deduce that $X = A^{c,\dagger}$. □

Notice that we represent the CMP inverse by two integrals in Theorem 1. In order to simplify integral representation of the CMP inverse, we firstly use the DMP inverse, MPD inverse and orthogonal projections.

Theorem 2. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and the full-rank decomposition of A as in Lemma 1. Then*

$$A^{c,\dagger} = \int_0^\infty P_{R(A^*)} M_1 B_1^* \exp(-B_1 B_1^* u) du = \int_0^\infty G_1^* \exp(-G_1 G_1^* t) M_2 P_{R(A)} dt,$$

where $M_1 = B_1 B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2$ and $M_2 = B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2 G_1$.

Proof. Based on $A^{c,\dagger} = P_{R(A^*)} A^{D,\dagger} = A^{\dagger,D} P_{R(A)}$ and [20, Theorem 3.2], we obtain this result. □

Applying an integral representation for the Drazin inverse showed in [4], which does not require any restriction on its eigenvalues, we give the following integral representations for the CMP inverse.

Theorem 3. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then*

$$A^{c,\dagger} = \int_0^\infty P_{R(A^*)} \exp\left[-tA^k (A^{2k+1})^* A^{k+1}\right] A^k (A^{2k+1})^* A^k P_{R(A)}\, dt.$$

Proof. It follows by the equality $A^{c,\dagger} = P_{R(A^*)} A^D P_{R(A)}$ and the next integral representation for the Drazin inverse proved in [4, Theorem 2.1]:

$$A^D = \int_0^\infty \exp\left[-tA^k (A^{2k+1})^* A^{k+1}\right] A^k (A^{2k+1})^* A^k\, dt.$$

□

As Theorem 3, new integral representations for the DMP and MPD inverses are obtained.

Corollary 1. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then*

$$A^{D,\dagger} = \int_0^\infty P_{R(A^*)} \exp\left[-tA^k (A^{2k+1})^* A^{k+1}\right] A^k (A^{2k+1})^* A^k\, dt$$

and

$$A^{\dagger,D} = \int_0^\infty \exp\left[-tA^k (A^{2k+1})^* A^{k+1}\right] A^k (A^{2k+1})^* A^k P_{R(A)}\, dt.$$

We present more expressions for the CMP inverse involving one integral.

Theorem 4. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then*

$$A^{c,\dagger} = \int_0^\infty A^* \exp(-AA^* t) P_{R(A^k),N(A^k)} P_{R(A)} dt = \int_0^\infty P_{R(A^*)} P_{R(A^k),N(A^k)} A^* \exp(-AA^* t) dt.$$

Proof. The equalities $A^{c,\dagger} = A^\dagger P_{R(A^k),N(A^k)} P_{R(A)} = P_{R(A^*)} P_{R(A^k),N(A^k)} A^\dagger$ and (2.1) yield these formulae. □

Similarly as Theorem 4, we show some formulae for the DMP inverse and MPD inverse.

Corollary 2. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\operatorname{ind}(A) = k$. *Then*

$$A^{D,\dagger} = \int_0^\infty A^* \exp(-AA^*t) P_{R(A^k),N(A^k)} dt$$

and

$$A^{\dagger,D} = \int_0^\infty P_{R(A^*)} P_{R(A^k),N(A^k)} A^* \exp(-AA^*t) dt.$$

Theorem 5. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\operatorname{ind}(A) = k$. *If* $G \in \mathbb{C}^{n\times n}$ *such that* $R(G) = R(A^\dagger A^k)$ *and* $N(G) = N(A^k A^\dagger)$, *then*

$$A^{c,\dagger} = \int_0^\infty \exp\left[-G(GAG)^* GAt\right] G(GAG)^* G dt.$$

Proof. Using [14, Corollary 3.7], we have $A^{c,\dagger} = A^{(2)}_{R(A^\dagger A^D),N(A^D A^\dagger)} = A^{(2)}_{R(A^\dagger A^k),N(A^k A^\dagger)}$. By [17, Theorem 2.2] (or [2, Corollary 7.6]), we obtain

$$A^{(2)}_{R(A^\dagger A^k),N(A^k A^\dagger)} = \int_0^\infty \exp\left[-G(GAG)^* GAt\right] G(GAG)^* G dt.$$

□

Using the integral representation for the (B,C)-inverse proved in [2], we obtain the next integral representation for the CMP inverse based on some restriction on the eigenvalues of corresponding matrix.

Theorem 6. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\operatorname{ind}(A) = k$ *and let* $G \in \mathbb{C}^{n\times n}$ *such that* $R(G) = R(A^\dagger A^k)$ *and* $N(G) = N(A^k A^\dagger)$. *If the nonzero spectrum of GA lies in the open left half plane, then*

$$A^{c,\dagger} = -\int_0^\infty \exp(GAt) G dt.$$

Proof. It follows by $A^{c,\dagger} = A^{(2)}_{R(A^\dagger A^k),N(A^k A^\dagger)} = A^{||(A^\dagger A^k, A^k A^\dagger)}$ and [2, Corollary 7.7]. □

3. Limit representations of the CMP inverse

In the beginning of this section, we present the limit representation of the CMP inverse based on the full-rank decomposition of A given in Lemma 1.

Theorem 7. *Let* $A \in \mathbb{C}^{n\times n}$ *with* $\operatorname{ind}(A) = k$ *and the full-rank decomposition of* A *as in Lemma 1. Then*

$$A^{c,\dagger} = \lim_{\lambda\to 0} G_1^*(\lambda I + G_1 G_1^*)^{-1} \lim_{t\to 0} M(tI + B_1^* B_1)^{-1} B_1^*,$$

where $M = B_2 \dots B_k (G_k B_k)^{-(k-1)} G_k \dots G_2$.

Proof. We have, by [15],

$$A^{\dagger} = \lim_{\lambda \to 0} A^*(\lambda I + AA^*)^{-1} = \lim_{\lambda \to 0} (\lambda I + A^*A)^{-1}A^*.$$

For $X = G_1^{\dagger} M B_1^{\dagger}$, we check that $X = A^{c,\dagger}$ as in the proof of Theorem 1. □

To avoid two limits, we included orthogonal projections in limit representations of CMP inverse. Similarly as Theorem 7 and Theorem 2, we verify the following result.

Theorem 8. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and the full-rank decomposition of A as in Lemma 1. Then*

$$A^{c,\dagger} = \lim_{\lambda \to 0} P_{R(A^*)} M_1 B_1^*(\lambda I + B_1 B_1^*)^{-1} = \lim_{\lambda \to 0} G_1^*(\lambda I + G_1 G_1^*)^{-1} M_2 P_{R(A)} dt,$$

where $M_1 = B_1 B_2 \ldots B_k (G_k B_k)^{-(k-1)} G_k \ldots G_2$ and $M_2 = B_2 \ldots B_k (G_k B_k)^{-(k-1)} G_k \ldots G_2 G_1$.

Analogously, we can prove the limit representations of DMP and MPD inverses.

Corollary 3. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$ and the full-rank decomposition of A as in Lemma 1. Then*

$$A^{D,\dagger} = \lim_{\lambda \to 0} M_1 B_1^*(\lambda I + B_1 B_1^*)^{-1}$$

and

$$A^{\dagger,D} = \lim_{\lambda \to 0} G_1^*(\lambda I + G_1 G_1^*)^{-1} M_2 dt,$$

where $M_1 = B_1 B_2 \ldots B_k (G_k B_k)^{-(k-1)} G_k \ldots G_2$ and $M_2 = B_2 \ldots B_k (G_k B_k)^{-(k-1)} G_k \ldots G_2 G_1$.

By the limit representation for the Drazin inverse proved in [11], we get the next limit representation for the CMP inverse.

Theorem 9. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. If $k \le l$, then*

$$A^{c,\dagger} = \lim_{\lambda \to 0} P_{R(A^*)} A^l (A^{l+1} + \lambda I)^{-1} P_{R(A)}.$$

Proof. This expressions can be verified using the following limit representation for the Drazin inverse presented in [11]:

$$A^D = \lim_{\lambda \to 0} A^l (A^{l+1} + \lambda I)^{-1}.$$

□

Also, the corresponding limit representations of DMP and MPD inverses can be showed.

Corollary 4. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. If $k \le l$, then*

$$A^{D,\dagger} = \lim_{\lambda\to 0} P_{R(A^*)} A^l (A^{l+1} + \lambda I)^{-1}$$

and

$$A^{\dagger,D} = \lim_{\lambda\to 0} A^l (A^{l+1} + \lambda I)^{-1} P_{R(A)}.$$

As Theorem 4 and Theorem 5, we obtain some limit representations of CMP inverse which involve one limit.

Theorem 10. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then*

$$A^{c,\dagger} = \lim_{\lambda\to 0} A^*(\lambda I + AA^*)^{-1} P_{R(A^k),N(A^k)} P_{R(A)} = \lim_{\lambda\to 0} P_{R(A^*)} P_{R(A^k),N(A^k)} A^*(\lambda I + AA^*)^{-1}.$$

For DMP and MPD inverses, the following limit representations hold.

Corollary 5. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. Then*

$$A^{D,\dagger} = \lim_{\lambda\to 0} A^*(\lambda I + AA^*)^{-1} P_{R(A^k),N(A^k)}$$

and

$$A^{\dagger,D} = \lim_{\lambda\to 0} P_{R(A^k),N(A^k)} A^*(\lambda I + AA^*)^{-1}.$$

We need one auxiliary result to prove new expressions for the CMP, DMP and MPD inverses.

Lemma 3 ([16]). *Let $A \in \mathbb{C}^{m\times n}$ be of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s \le r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. In addition, suppose that $G \in \mathbb{C}^{n\times m}$ satisfies $R(G) = T$ and $N(G) = S$. If $A^{(2)}_{T,S}$ exists, then it possesses the limit representations*

$$A^{(2)}_{T,S} = \lim_{\lambda\to 0} (GA + \lambda I)^{-1} G = \lim_{\lambda\to 0} G(AG + \lambda I)^{-1}. \tag{3.1}$$

Theorem 11. *Let $A \in \mathbb{C}^{n\times n}$ with $\operatorname{ind}(A) = k$. If $G \in \mathbb{C}^{n\times n}$ such that $R(G) = R(A^\dagger A^k)$ and $N(G) = N(A^k A^\dagger)$, then*

$$A^{c,\dagger} = \lim_{\lambda\to 0} (GA + \lambda I)^{-1} G = \lim_{\lambda\to 0} G(AG + \lambda I)^{-1}.$$

Proof. By Lemma 3 (or [2, Corollary 7.5]), we have

$$A^{c,\dagger} = A^{(2)}_{R(A^\dagger A^k),N(A^k A^\dagger)} = \lim_{\lambda\to 0} (GA + \lambda I)^{-1} G = \lim_{\lambda\to 0} G(AG + \lambda I)^{-1}.$$

□

Theorem 12. *Let $A \in \mathbb{C}^{n\times n}$ be of rank r and $\operatorname{ind}(A) = k$, $B \in \mathbb{C}^{n\times s}_s$ and $C \in \mathbb{C}^{s\times n}_s$.*

(i) *Suppose that $R(B) = R(A^\dagger A^k)$ is a subspace of $\mathbb{C}^n$ of dimension $s \le r$ and $N(C) = N(A^k A^\dagger)$ is a subspace of $\mathbb{C}^n$ of dimension $n - s$. Then*

$$A^{c,\dagger} = \lim_{t\to 0} B(tI + CAB)^{-1} C.$$

(ii) *Suppose that* $R(B_1) = R(A^k)$ *is a subspace of* $\mathbb{C}^n$ *of dimension* $s \leq r$ *and* $N(C_1) = N(A^k A^\dagger)$ *is a subspace of* $\mathbb{C}^n$ *of dimension* $n-s$. *If* $(C_1)^{(2)}_{R(AB_1),N(AB_1)}$ *exists, then*

$$A^{D,\dagger} = \lim_{t\to 0} B_1(tI + C_1AB_1)^{-1}C_1$$

and

$$A^{c,\dagger} = A^\dagger (C_1)^{(2)}_{R(AB_1),N(AB_1)} C_1.$$

(iii) *Suppose that* $R(B_2) = R(A^\dagger A^k)$ *is a subspace of* $\mathbb{C}^n$ *of dimension* $s \leq r$ *and* $N(C_2) = N(A^k)$ *is a subspace of* $\mathbb{C}^n$ *of dimension* $n-s$. *If* $(B_2)^{(2)}_{R(C_2A),N(C_2A)}$ *exists, then*

$$A^{\dagger,D} = \lim_{t\to 0} B_2(tI + C_2AB_2)^{-1}C_2$$

and

$$A^{c,\dagger} = B_2(B_2)^{(2)}_{R(C_2A),N(C_2A)} A^\dagger.$$

Proof. (i) Applying [8, Theorem 7], we have that

$$A^{(2)}_{R(A^\dagger A^k),N(A^k A^\dagger)} = \lim_{t\to 0} B(tI + CAB)^{-1}C.$$

(ii) We firstly observe that $A^{D,\dagger} = A^{(2)}_{R(A^k),N(A^k A^\dagger)}$ and then, by [8, Theorem 7],

$$A^{D,\dagger} = \lim_{t\to 0} B_1(tI + C_1AB_1)^{-1}C_1.$$

Therefore, by Lemma 3,

$$\begin{aligned} A^{c,\dagger} &= A^\dagger A A^{D,\dagger} = A^\dagger \lim_{t\to 0} AB_1(tI + C_1AB_1)^{-1}C_1 \\ &= A^\dagger (C_1)^{(2)}_{R(AB_1),N(AB_1)} C_1. \end{aligned}$$

(iii) This part can be proved in an analogy way as part (ii). □

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