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NOTES ON THE COMMUTATIVITY OF PRIME NEAR-RINGS

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Abstract. Let N be a $3-$prime right near-ring and let f be a generalized $(\theta,\theta)-$ derivation on N with associated $(\theta,\theta)-$derivation d. It is proved that N must be a commutative ring if $d \neq 0$ and one of the following conditions is satisfied for all $x, y \in N$: (i) $f([x,y]) = 0$; (ii) $f([x,y]) = \theta([x,y])$; (iii) $f(xoy) = 0$; (iv) $f(xoy) = \theta(xoy)$; (v) $f([x,y]) = \theta(xoy)$; (vi) $f(xoy) = \theta([x,y])$. We also prove theorems which assert that N is commutative, but not necessarily a ring.

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1. INTRODUCTION

An additively written group $(N,+)$ equipped with a binary operation $\cdot : N \times N \to N$, $(x,y) \to xy$ such that $(xy)z = x(yz)$ and $(x+y)z = xz + yz$ for all $x,y,z \in N$ is called a right near-ring. Recall that a near-ring N is called $3-$prime if for any $x,y \in N$, $xNy = 0$ implies that $x = 0$ or $y = 0$. For $x,y \in N$ the symbol $[x,y]$ will denote $xy - yx$, while the symbol xoy will denote $xy + yx$. Z is the multiplicative center of N. An additive mapping $d : N \to N$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x,y \in N$, or equivalently, as noted in [12], that $d(xy) = d(x)y + xd(y)$ for all $x,y \in N$. Recently, in [7], Bresar defined the following concept. An additive mapping $F : N \to N$ is called a generalized derivation if there exists a derivation $d : N \to N$ such that

$$F(xy) = F(x)y + xd(y), \text{ for all } x,y \in N.$$

Basic examples are derivations and generalized inner derivations (i.e., maps of type $x \to ax + xb$ for some $a,b \in N$). One may observe that the concept of generalized derivations includes the concept of derivations and of left multipliers (i.e., $F(xy) = F(x)y$, for all $x,y \in N$).

Inspired by the definition of derivation (resp. generalized derivation), we define the notion of $(\theta,\phi)-$derivation (resp. generalized $(\theta,\phi)-$derivation) as follows: Let θ,ϕ be two near-ring automorphisms of N. An additive mapping $d : N \to N$ is called a $(\theta,\phi)-$derivation (resp. generalized $(\theta,\phi)-$derivation) if $d(xy) = \phi(x)d(y) +$

$d(x)\theta(y)$ (resp. $f(xy) = f(x)\theta(y) + \phi(x)d(y)$, where d is a $(\theta,\phi)-$derivation) holds for all $x, y \in N$. It is noted that $d(xy) = d(x)\theta(y) + \phi(x)d(y)$, for all $x, y \in N$ in [9, Lemma 1]. Of course a $(1,1)-$derivation (resp. generalized $(1,1)-$derivation) is a derivation (resp. generalized derivation) on N, where 1 is the identity on N.

Many authors have investigated the properties of derivations of prime and semiprime rings. The study of derivations of near-rings was initiated by H. E. Bell and G. Mason in 1987 [5]. Some recent results on rings deal with commutativity on prime and semiprime rings admitting suitably constrained derivations. It is natural to look for comparable results on near-rings and this has been done in [3], [5], [6], [4], [2], [9].

In [8], Daif and Bell showed that the ideal I of a semiprime ring is contained in the center of R if

$$d([x,y]) = [x,y] \text{ for all } x,y \in I \text{ or } d([x,y]) = -[x,y] \text{ for all } x,y \in I. \quad (1.1)$$

Several authors have obtained commutativity results for prime or semiprime rings admitting derivations or generalized derivations d satisfying (1.1) or similar conditions (see [1], [11], [10]). The first purpose of this paper is to show that $3-$prime near-rings must be commutative rings if they admit appropriate generalized $(\theta,\theta)-$derivations satisfying conditions related to (1.1). The second aim is to prove some commutativity theorems for $3-$prime near-rings with $(\theta,\theta)-$derivations.

2. Results on generalized $(\theta,\theta)-$derivations

Lemma 1. *[9, Theorem 2] Let N be a $3-$prime near-ring admitting a non trivial $(\sigma,\tau)-$derivation d. If $d(N) \subset Z$, then $(N,+)$ is abelian. Moreover, if N is 2-torsion free and σ,τ commute with d, then N is a commutative ring.*

Theorem 1. *Let N be a 2-torsion free $3-$prime near-ring, (f,d) a generalized $(\theta,\theta)-$ derivation of N and $d\theta = \theta d$. If $f([x,y]) = 0$ for all $x,y \in N$ and $d \neq 0$, then N is a commutative ring.*

Proof. By the hyphothesis, we have

$$f([x,y]) = 0, \text{ for all } x,y \in N. \quad (2.1)$$

Replacing y by yx in (2.1) and using $[x,yx] = [x,y]x$, we obtain that

$$f([x,y])\theta(x) + \theta([x,y])d(x) = 0, \text{ for all } x,y \in N.$$

By (2.1), we get

$$\theta([x,y])d(x) = 0, \text{ for all } x,y \in N,$$

and so

$$\theta(x)\theta(y)d(x) = \theta(y)\theta(x)d(x), \text{ for all } x,y \in N. \quad (2.2)$$

Taking zy, $z \in N$ instead of y in (2.2) and using (2.2), we arrive at

$$\theta([x,z])\theta(y)d(x) = 0, \text{ for all } x,y,z \in N.$$

Since θ is an automorphism of N, we have

$$\theta([x,z])\,Nd(x) = 0, \text{ for all } x,z \in N.$$

By the primeness of N, we get either $\theta([x,z]) = 0$ or $d(x) = 0$ for each $x \in N$. Again using $\theta \in AutN$, we conclude that

$$x \in Z \text{ or } d(x) = 0 \text{ for each } x \in N.$$

If $x \in Z$, then $d(x) \in Z$. Indeed, for all $y \in N$, we get

$$xy = yx,$$

and so

$$d(xy) = d(yx), \text{ for all } y \in N,$$

$$\theta(x)d(y) + d(x)\theta(y) = d(y)\theta(x) + \theta(y)d(x), \text{ for all } y \in N.$$

Using $x \in Z$ in this equation, we obtain that

$$d(x)\theta(y) = \theta(y)d(x), \text{ for all } y \in N$$

and so

$$d(x)y = yd(x), \text{ for all } y \in N.$$

Thus $d(x) \in Z$, for all $x \in N$.

By Lemma 1, we conclude that N is a commutative ring. This completes the proof. □

Theorem 2. *Let N be a 2-torsion free 3−prime near-ring, (f,d) a generalized $(\theta,\theta)-$ derivation of N and $d\theta = \theta d$. If $f([x,y]) = \pm\theta([x,y])$ for all $x,y \in N$ and $d \neq 0$, then N is a commutative ring.*

Proof. Replacing y by yx in the hypothesis yields that

$$f([x,y]x) = \pm\theta([x,y]x), \text{ for all } x,y \in N,$$

and so

$$f([x,y])\theta(x) + \theta([x,y])d(x) = \pm\theta([x,y])\theta(x), \text{ for all } x,y \in N.$$

Using our hypothesis, the above relation yields that

$$\pm\theta([x,y])\theta(x) + \theta([x,y])d(x) = \pm\theta([x,y])\theta(x), \text{ for all } x,y \in N,$$

and so

$$\theta([x,y])d(x) = 0, \text{ for all } x,y \in N.$$

Arguing in the similar manner as we have done in the proof of Theorem 1, we find N is a commutative ring. □

Theorem 3. *Let N be a 2-torsion free 3−prime near-ring, (f,d) a generalized $(\theta,\theta)-$ derivation of N and $d\theta = \theta d$. If $f(xoy) = 0$ for all $x,y \in N$ and $d \neq 0$, then N is a commutative ring.*

Proof. Assume that

$$f(xoy) = 0, \text{ for all } x, y \in N. \tag{2.3}$$

Substituting yx for y in (2.3), we get

$$f(xoy)\theta(x) + \theta(xoy)d(x) = 0, \text{ for all } x, y \in N.$$

By (2.3), we obtain that

$$\theta(xoy)d(x) = 0, \text{ for all } x, y \in N,$$

and so

$$\theta(x)\theta(y)d(x) = -\theta(y)\theta(x)d(x), \text{ for all } x, y \in N.$$

Taking zy, $z \in N$ instead of y in this relation and using this equation, we have

$$\theta(x)\theta(z)\theta(y)d(x) = \theta(z)\theta(x)\theta(y)d(x), \text{ for all } x, y, z \in N,$$

and so

$$\theta([x,z])\theta(y)d(x) = 0, \text{ for all } x, y, z \in N.$$

Applying the same techniques in the proof of Theorem 1, we conclude that N is a commutative ring. □

Theorem 4. *Let N be a 2-torsion free 3−prime near-ring, (f,d) a generalized $(\theta,\theta)-$ derivation of N and $d\theta = \theta d$. If $f(xoy) = \pm\theta(xoy)$ for all $x, y \in N$ and $d \neq 0$, then N is commutative ring.*

Proof. We have

$$f(xoy) = \pm\theta(xoy), \text{ for all } x, y \in N. \tag{2.4}$$

Substituting yx for y in (2.4), we obtain that

$$f(xoy)\theta(x) + \theta(xoy)d(x) = \pm\theta(xoy)\theta(x), \text{ for all } x, y \in N.$$

Using (2.4), we get

$$\theta(xoy)d(x) = 0, \text{ for all } x, y \in N.$$

Replacing y by zy in the above relation, we arrive at

$$\theta([x,z])\theta(y)d(x) = 0, \text{ for all } x, y, z \in N.$$

Again using the same arguments in the proof of Theorem 1, we find the required result. □

Theorem 5. *Let N be a 2-torsion free 3−prime near-ring, (f,d) a generalized $(\theta,\theta)-$ derivation of N and $d\theta = \theta d$. If $f([x,y]) = \pm\theta(xoy)$, for all $x, y \in N$ and $d \neq 0$, then N is a commutative ring.*

Proof. Writing yx by y in the hypothesis, we have

$$f([x,y])\theta(x)+\theta([x,y])d(x)=\pm\theta(xoy)\theta(x), \text{ for all } x,y\in N,$$

and so

$$\pm\theta(xoy)\theta(x)+\theta([x,y])d(x)=\pm\theta(xoy)\theta(x), \text{ for all } x,y\in N.$$

That is

$$\theta([x,y])d(x)=0, \text{ for all } x,y\in N.$$

Using the same arguments in the proof of Theorem 1, we arrive at the required result. □

Theorem 6. *Let N be a 2-torsion free 3−prime near-ring, (f,d) a generalized $(\theta,\theta)-$ derivation of N and $d\theta=\theta d$. If $f(xoy)=\pm\theta([x,y])$ for all $x,y\in N$ and $d\neq 0$, then N is a commutative ring.*

Proof. Suppose that

$$f(xoy)=\pm\theta([x,y]), \text{ for all } x,y\in N.$$

Replacing y by yx in this equation gives that

$$f(xoy)\theta(x)+\theta(xoy)d(x)=\pm\theta([x,y])\theta(x), \text{ for all } x,y\in N.$$

By the hypothesis, we have

$$\pm\theta([x,y])\theta(x)+\theta(xoy)d(x)=\pm\theta([x,y])\theta(x), \text{ for all } x,y\in N,$$

and so

$$\theta(xoy)d(x)=0, \text{ for all } x,y\in N.$$

Arguing in the similar manner as we have done in the proof of Theorem 3, we conclude that N is a commutative ring. □

Remark 1. Each of the above theorems yields on obvious result for $(\theta,\theta)-$derivations.

3. Results on $(\theta,\theta)-$derivations

Lemma 2. *Let N be a right near-ring, d a $(\theta,\theta)-$derivation of N and $a\in N$. Then*

$$a(d(x)\theta(y)+\theta(x)d(y))=ad(x)\theta(y)+a\theta(x)d(y), \textit{ for all } x,y\in N.$$

Proof. Given $x,y\in N$, obtain

$$\begin{aligned} d(a(xy)) &= d(a)\theta(xy)+\theta(a)d(xy) \\ &= d(a)\theta(x)\theta(y)+\theta(a)(d(x)\theta(y)+\theta(x)d(y)). \end{aligned} \tag{3.1}$$

On the other hand,

$$\begin{aligned} d((ax)y) &= d(ax)\theta(y)+\theta(ax)d(y) \\ &= d(a)\theta(x)\theta(y)+\theta(a)d(x)\theta(y)+\theta(a)\theta(x)d(y). \end{aligned} \tag{3.2}$$

Comparing (3.1) and (3.2), we conclude that

$$\theta(a)(d(x)\theta(y)+\theta(x)d(y)) = \theta(a)d(x)\theta(y)+\theta(a)\theta(x)d(y),$$

for all $x, y \in N$. Since θ is an automorphism of N, we can write this equation as

$$a(d(x)\theta(y)+\theta(x)d(y)) = ad(x)\theta(y)+a\theta(x)d(y), \text{ for all } x, y \in N.$$

□

Theorem 7. *Let N be a 2-torsion free 3−prime near-ring, d a (θ,θ)−derivation of N. If $d(x)d(y) = \theta([x,y])$ for all $x, y \in N$, then N is commutative.*

Proof. Assume that

$$d(x)d(y) = \theta([x,y]), \text{ for all } x, y \in N. \tag{3.3}$$

Replacing y by yx in (3.3), we obtain that

$$d(x)(d(y)\theta(x)+\theta(y)d(x)) = \theta([x,y])\theta(x), \text{ for all } x, y \in N.$$

By Lemma 2, we have

$$d(x)d(y)\theta(x)+d(x)\theta(y)d(x) = \theta([x,y])\theta(x), \text{ for all } x, y \in N.$$

Using equation (3.3), we find that

$$\theta([x,y])\theta(x)+d(x)\theta(y)d(x) = \theta([x,y])\theta(x), \text{ for all } x, y \in N,$$

and so

$$d(x)\theta(y)d(x) = 0, \text{ for all } x, y \in N.$$

Since θ is an automorphism of N, we get

$$d(x)Nd(x) = 0, \text{ for all } x \in N.$$

By the primeness of N, we arrive at $d(x) = 0$, for all $x \in N$. If $d = 0$, then we have $\theta([x,y]) = 0$ for all $x, y \in N$ by the hypothesis, and so N is commutative. □

Theorem 8. *Let N be a 2-torsion free 3−prime near-ring, d a (θ,θ)−derivation of N. If $d(x)d(y) = \theta(xoy)$ for all $x, y \in N$, then N is commutative.*

Proof. Replacing y by yx in the hypothesis, we have

$$d(x)(d(y)\theta(x)+\theta(y)d(x)) = \theta(xoy)\theta(x), \text{ for all } x, y \in N.$$

By Lemma 2, we get

$$d(x)d(y)\theta(x)+d(x)\theta(y)d(x) = \theta(xoy)\theta(x), \text{ for all } x, y \in N.$$

Using the hypothesis, we obtain that

$$\theta(xoy)\theta(x)+d(x)\theta(y)d(x) = \theta(xoy)\theta(x), \text{ for all } x, y \in N.$$

and so

$$d(x)\theta(y)d(x) = 0, \text{ for all } x, y \in N.$$

Since θ is an automorphism of N, we get

$$d(x)Nd(x) = 0, \text{ for all } x \in N.$$

By the primeness of N, we obtain that $d = 0$. If $d = 0$, then we have $\theta(xoy) = 0$, for all $x, y \in N$ by the hypothesis, and so $xy = -yx$, for all $x, y \in N$. Writing yz by y in this equation, we have

$$xyz = -yzx = yxz, \text{ for all } x, y, z \in N,$$

and so

$$[x, y]z = 0, \text{ for all } x, y, z \in N.$$

Since N is a $3-$prime near-ring, we get $[x, y] = 0$, for all $x, y \in N$, and so, N is commutative. □

Theorem 9. *Let N be a 2-torsion free $3-$prime near-ring and d, h be two $(\theta, \theta)-$derivations. If $d(x)\theta(y) = \theta(x)h(y)$ for all $x, y \in N$, then $d = h = 0$.*

Proof. We get

$$d(x)\theta(y) = \theta(x)h(y), \text{ for all } x, y \in N. \tag{3.4}$$

Replacing y by yz, $z \in N$ in (3.4), we arrive at

$$d(x)\theta(y)\theta(z) = \theta(x)(h(y)\theta(z) + \theta(y)h(z)), \text{ for all } x, y, z \in N.$$

By Lemma 2, we have

$$d(x)\theta(y)\theta(z) = \theta(x)h(y)\theta(z) + \theta(x)\theta(y)h(z), \text{ for all } x, y, z \in N.$$

Using (3.4), we find that

$$\theta(x)h(y)\theta(z) = \theta(x)h(y)\theta(z) + \theta(x)\theta(y)h(z), \text{ for all } x, y, z \in N,$$

and so

$$\theta(x)\theta(y)h(z) = 0, \text{ for all } x, y, z \in N.$$

That is

$$\theta(x)Nh(z) = 0, \text{ for all } x, z \in N.$$

By the primeness of N gives $h = 0$. If $h = 0$, then $d(x)\theta(y) = 0$, for all $x, y \in N$ by the hypothesis. Again using the primeness of N, we get $d = 0$. This completes the proof. □

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