Notes on the commutativity of prime near-rings

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NOTES ON THE COMMUTATIVITY OF PRIME NEAR-RINGS

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Abstract. Let \( N \) be a 3-prime right near-ring and let \( f \) be a generalized \((\theta, \theta)\)-derivation on \( N \) with associated \((\theta, \theta)\)-derivation \( d \). It is proved that \( N \) must be a commutative ring if \( d \neq 0 \) and one of the following conditions is satisfied for all \( x, y \in N \): (i) \( f ([x, y]) = 0 \); (ii) \( f ([x, y]) = \theta ([x, y]) \); (iii) \( f (xoy) = 0 \); (iv) \( f (xoy) = \theta (xoy) \); (v) \( f ([x, y]) = \theta (xoy) \); (vi) \( f (xoy) = \theta ([x, y]) \). We also prove theorems which assert that \( N \) is commutative, but not necessarily a ring.

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1. Introduction

An additively written group \((N, +)\) equipped with a binary operation \( \cdot : N \times N \to N \), \( (x, y) \to xy \) such that \((xy)z = x(yz)\) and \((x + y)z = xz + yz\) for all \( x, y, z \in N \) is called a right near-ring. Recall that a near-ring \( N \) is called 3-prime if for any \( x, y \in N \), \( xy = 0 \) implies that \( x = 0 \) or \( y = 0 \). For \( x, y \in N \) the symbol \([x, y]\) will denote \( xy - yx \), while the symbol \( xoy \) will denote \( xy + yx \). \( Z \) is the multiplicative center of \( N \). An additive mapping \( d : N \to N \) is said to be a derivation if \( d (xy) = xd (y) + d (x) y \) for all \( x, y \in N \), or equivalently, as noted in [12], that \( d (xy) = d (x) y + xd (y) \) for all \( x, y \in N \). Recently, in [7], Bresar defined the following concept. An additive mapping \( F : N \to N \) is called a generalized derivation if there exists a derivation \( d : N \to N \) such that

\[ F(xy) = F(x)y + xd(y), \text{ for all } x, y \in N. \]

Basic examples are derivations and generalized inner derivations (i.e., maps of type \( x \to ax + xb \) for some \( a, b \in N \)). One may observe that the concept of generalized derivations includes the concept of derivations and of left multipliers (i.e., \( F(xy) = F(x)y \), for all \( x, y \in N \)).

Inspired by the definition of derivation (resp. generalized derivation), we define the notion of \((\theta, \phi)\)-derivation (resp. generalized \((\theta, \phi)\)-derivation) as follows: Let \( \theta, \phi \) be two near-ring automorphisms of \( N \). An additive mapping \( d : N \to N \) is called a \((\theta, \phi)\)-derivation (resp. generalized \((\theta, \phi)\)-derivation) if \( d (xy) = \phi (x) d (y) + \)

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\[ d(x) \theta(y) \text{ (resp. } f(xy) = f(x) \theta(y) + \phi(x)d(y), \text{ where } d \text{ is a } (\theta, \phi)-\text{derivation}) \text{ holds for all } x, y \in N. \] It is noted that \[ d(xy) = d(x) \theta(y) + \phi(x)d(y), \text{ for all } x, y \in N \] in [9, Lemma 1]. Of course a \((1, 1)-\text{derivation (resp. generalized } (1, 1)-\text{derivation) is a derivation (resp. generalized derivation) on } N, \text{ where } 1 \text{ is the identity on } N.\]

Many authors have investigated the properties of derivations of prime and semiprime rings. The study of derivations of near-rings was initiated by H. E. Bell and G. Mason in 1987 [5]. Some recent results on rings deal with commutativity on prime and semiprime rings admitting suitably constrained derivations. It is natural to look for comparable results on near-rings and this has been done in [3], [5], [6], [4], [2], [9].

In [8], Daif and Bell showed that the ideal \( I \) of a semiprime ring is contained in the center of \( R \) if
\[ d([x, y]) = [x, y] \text{ for all } x, y \in I \text{ or } d([x, y]) = -[x, y] \text{ for all } x, y \in I. \] Several authors have obtained commutativity results for prime or semiprime rings admitting derivations or generalized derivations \( d \) satisfying (1.1) or similar conditions (see [1], [11], [10]). The first purpose of this paper is to show that \( 3\)-prime near-rings must be commutative rings if they admit appropriate generalized \((\theta, \theta)\)–derivations satisfying conditions related to (1.1). The second aim is to prove some commutativity theorems for \( 3\)-prime near-rings with \((\theta, \theta)\)–derivations.

2. RESULTS ON GENERALIZED \((\theta, \theta)\)-DERIVATIONS

**Lemma 1.** [9, Theorem 2] Let \( N \) be a \( 3\)-prime near-ring admitting a non trivial \((\sigma, \tau)\)-derivation \( d \). If \( d(N) \subset Z \), then \((N, +)\) is abelian. Moreover, if \( N \) is \( 2\)-torsion free and \( \sigma, \tau \) commute with \( d \), then \( N \) is a commutative ring.

**Theorem 1.** Let \( N \) be a \( 2\)-torsion free \( 3\)-prime near-ring, \((f, d)\) a generalized \((\theta, \theta)\)-derivation of \( N \) and \( d\theta = \theta d \). If \( f([x, y]) = 0 \) for all \( x, y \in N \) and \( d \neq 0 \), then \( N \) is a commutative ring.

**Proof.** By the hypothesis, we have
\[ f([x, y]) = 0, \text{ for all } x, y \in N. \] Replacing \( y \) by \( yx \) in (2.1) and using \([x, yx] = [x, y]x\), we obtain that
\[ f([x, y]) \theta(x) + \theta([x, y])d(x) = 0, \text{ for all } x, y \in N. \] By (2.1), we get
\[ \theta([x, y])d(x) = 0, \text{ for all } x, y \in N, \] and so
\[ \theta(x) \theta(y)d(x) = \theta(y) \theta(x)d(x), \text{ for all } x, y \in N. \] Taking \( z, y \in N \) instead of \( y \) in (2.2) and using (2.2), we arrive at
\[ \theta([x, z]) \theta(y)d(x) = 0, \text{ for all } x, y, z \in N. \]
Since $\theta$ is an automorphism of $N$, we have
$$\theta ([x, z]) N d (x) = 0, \text{ for all } x, z \in N.$$ 

By the primeness of $N$, we get either $\theta ([x, z]) = 0$ or $d (x) = 0$ for each $x \in N$. Again using $\theta \in Aut N$, we conclude that
$$x \in Z \text{ or } d (x) = 0 \text{ for each } x \in N.$$ 

If $x \in Z$, then $d (x) \in Z$. Indeed, for all $y \in N$, we get
$$xy = yx,$$
and so
$$d (xy) = d (yx), \text{ for all } y \in N,$$
$$\theta (x) d (y) + d (x) \theta (y) = d (y) \theta (x) + \theta (y) d (x), \text{ for all } y \in N.$$ 

Using $x \in Z$ in this equation, we obtain that
$$d (x) \theta (y) = \theta (y) d (x), \text{ for all } y \in N$$
and so
$$d (x) y = yd (x), \text{ for all } y \in N.$$ 

Thus $d (x) \in Z$, for all $x \in N$. 

By Lemma 1, we conclude that $N$ is a commutative ring. This completes the proof.

**Theorem 2.** Let $N$ be a 2-torsion free 3-prime near-ring, $(f, d)$ a generalized $\langle \theta, \tau \rangle$ derivation of $N$ and $d \tau = \theta d$. If $f ([x, y]) = \pm \theta ([x, y])$ for all $x, y \in N$ and $d \neq 0$, then $N$ is a commutative ring.

**Proof.** Replacing $y$ by $yx$ in the hypothesis yields that
$$f ([x, y] x) = \pm \theta ([x, y] x), \text{ for all } x, y \in N,$$
and so
$$f ([x, y]) = \pm \theta ([x, y]) \theta (x), \text{ for all } x, y \in N.$$ 

Using our hypothesis, the above relation yields that
$$\pm \theta ([x, y]) + \theta ([x, y] y) d (x) = \pm \theta ([x, y]) \theta (x), \text{ for all } x, y \in N,$$
and so
$$\theta ([x, y]) d (x) = 0, \text{ for all } x, y \in N.$$ 

Arguing in the similar manner as we have done in the proof of Theorem 1, we find $N$ is a commutative ring.

**Theorem 3.** Let $N$ be a 2-torsion free 3-prime near-ring, $(f, d)$ a generalized $\langle \theta, \tau \rangle$ derivation of $N$ and $d \tau = \theta d$. If $f (xoy) = 0$ for all $x, y \in N$ and $d \neq 0$, then $N$ is a commutative ring.
Proof. Assume that
\[ f(xoy) = 0, \text{ for all } x, y \in N. \] (2.3)
Substituting \(yx\) for \(y\) in (2.3), we get
\[ f(xoy) \theta(x) + \theta(xoy)d(x) = 0, \text{ for all } x, y \in N. \]
By (2.3), we obtain that
\[ \theta(xoy)d(x) = 0, \text{ for all } x, y \in N, \]
and so
\[ \theta(x)\theta(y)d(x) = -\theta(y)\theta(x)d(x), \text{ for all } x, y \in N. \]
Taking \(z, y, z \in N\) instead of \(y\) in this relation and using this equation, we have
\[ \theta(x)\theta(z)\theta(y)d(x) = \theta(z)\theta(x)\theta(y)d(x), \text{ for all } x, y, z \in N, \]
and so
\[ \theta([x, z])\theta(y)d(x) = 0, \text{ for all } x, y, z \in N. \]
Applying the same techniques in the proof of Theorem 1, we conclude that \(N\) is a commutative ring. □

Theorem 4. Let \(N\) be a 2-torsion free 3–prime near-ring, \((f, d)\) a generalized \((\theta, \theta)–\) derivation of \(N\) and \(d \theta = \theta d\). If \(f(xoy) = \pm \theta(xoy)\) for all \(x, y \in N\) and \(d \neq 0\), then \(N\) is commutative ring.

Proof. We have
\[ f(xoy) = \pm \theta(xoy), \text{ for all } x, y \in N. \] (2.4)
Substituting \(yx\) for \(y\) in (2.4), we obtain that
\[ f(xoy) \theta(x) + \theta(xoy)d(x) = \pm \theta(xoy)\theta(x), \text{ for all } x, y \in N. \]
Using (2.4), we get
\[ \theta(xoy)d(x) = 0, \text{ for all } x, y \in N. \]
Replacing \(y\) by \(z, y\) in the above relation, we arrive at
\[ \theta([x, z])\theta(y)d(x) = 0, \text{ for all } x, y, z \in N. \]
Again using the same arguments in the proof of Theorem 1, we find the required result. □

Theorem 5. Let \(N\) be a 2-torsion free 3–prime near-ring, \((f, d)\) a generalized \((\theta, \theta)–\) derivation of \(N\) and \(d \theta = \theta d\). If \(f([x, y]) = \pm \theta(xoy), \text{ for all } x, y \in N\) and \(d \neq 0\), then \(N\) is a commutative ring.
Proof. Writing \(yx\) by \(y\) in the hypothesis, we have
\[
f ([x, y]) \theta (x) + \theta ([x, y]) d (x) = \pm \theta (xoy) \theta (x), \quad \text{for all } x, y \in N,
\]
and so
\[
\pm \theta (xoy) \theta (x) + \theta ([x, y]) d (x) = \pm \theta (xoy) \theta (x), \quad \text{for all } x, y \in N.
\]
That is
\[
\theta ([x, y]) d (x) = 0, \quad \text{for all } x, y \in N.
\]
Using the same arguments in the proof of Theorem 1, we arrive at the required result.

\[\square\]

**Theorem 6.** Let \(N\) be a 2-torsion free 3–prime near-ring, \((f, d)\) a generalized \((\theta, \theta)\)–derivation of \(N\) and \(d \theta = \theta d\). If \(f (xoy) = \pm \theta ([x, y])\) for all \(x, y \in N\) and \(d \neq 0\), then \(N\) is a commutative ring.

**Proof.** Suppose that
\[
f (xoy) = \pm \theta ([x, y]), \quad \text{for all } x, y \in N.
\]
Replacing \(y\) by \(yx\) in this equation gives that
\[
f (xoy) \theta (x) + \theta (xoy) d (x) = \pm \theta ([x, y]) \theta (x), \quad \text{for all } x, y \in N.
\]
By the hypothesis, we have
\[
\pm \theta ([x, y]) \theta (x) + \theta (xoy) d (x) = \pm \theta ([x, y]) \theta (x), \quad \text{for all } x, y \in N,
\]
and so
\[
\theta (xoy) d (x) = 0, \quad \text{for all } x, y \in N.
\]
Arguing in the similar manner as we have done in the proof of Theorem 3, we conclude that \(N\) is a commutative ring.

\[\square\]

**Remark 1.** Each of the above theorems yields on obvious result for \((\theta, \theta)\)–derivations.

3. Results on \((\theta, \theta)\)–derivations

**Lemma 2.** Let \(N\) be a right near-ring, \(d\) a \((\theta, \theta)\)–derivation of \(N\) and \(a \in N\). Then
\[
a (d (x) \theta (y) + \theta (x) d (y)) = ad (x) \theta (y) + a \theta (x) d (y), \quad \text{for all } x, y \in N.
\]

**Proof.** Given \(x, y \in N\), obtain
\[
d (a (xy)) = d (a) \theta (xy) + \theta (a) d (xy)
= d (a) \theta (x) \theta (y) + \theta (a) (d (x) \theta (y) + \theta (x) d (y)). \tag{3.1}
\]
On the other hand,
\[
d ((ax) y) = d (ax) \theta (y) + \theta (ax) d (y)
= d (a) \theta (x) \theta (y) + \theta (a) d (x) \theta (y) + \theta (a) \theta (x) d (y). \tag{3.2}
\]
Comparing (3.1) and (3.2), we conclude that
\[ \theta (a) (d (x) \theta (y) + \theta (x) d (y)) = \theta (a) d (x) \theta (y) + \theta (a) \theta (x) d (y), \]
for all \( x, y \in N \). Since \( \theta \) is an automorphism of \( N \), we can write this equation as
\[ a (d (x) \theta (y) + \theta (x) d (y)) = ad (x) \theta (y) + a \theta (x) d (y), \]
for all \( x, y \in N \).

\[ \square \]

**Theorem 7.** Let \( N \) be a 2-torsion free 3–prime near-ring, \( d \) a \((\theta, \theta)\)–derivation of \( N \). If \( d (x) d (y) = \theta ([x, y]) \) for all \( x, y \in N \), then \( N \) is commutative.

**Proof.** Assume that
\[ d (x) d (y) = \theta ([x, y]), \]
for all \( x, y \in N \). (3.3)
Replacing \( y \) by \( y x \) in (3.3), we obtain that
\[ d (x) (d (y) \theta (x) + \theta (y) d (x)) = \theta ([x, y]) \theta (x), \]
for all \( x, y \in N \).
By Lemma 2, we have
\[ d (x) d (y) \theta (x) + d (x) \theta (y) d (x) = \theta ([x, y]) \theta (x), \]
for all \( x, y \in N \).
Using equation (3.3), we find that
\[ \theta ([x, y]) \theta (x) + d (x) \theta (y) d (x) = \theta ([x, y]) \theta (x), \]
for all \( x, y \in N \),
and so
\[ d (x) \theta (y) d (x) = 0, \]
for all \( x, y \in N \).
Since \( \theta \) is an automorphism of \( N \), we get
\[ d (x) N d (x) = 0, \]
for all \( x \in N \).
By the primeness of \( N \), we arrive at \( d (x) = 0 \), for all \( x \in N \). If \( d = 0 \), then we have \( \theta ([x, y]) = 0 \) for all \( x, y \in N \) by the hypothesis, and so \( N \) is commutative.

\[ \square \]

**Theorem 8.** Let \( N \) be a 2-torsion free 3–prime near-ring, \( d \) a \((\theta, \theta)\)–derivation of \( N \). If \( d (x) d (y) = \theta (x o y) \) for all \( x, y \in N \), then \( N \) is commutative.

**Proof.** Replacing \( y \) by \( y x \) in the hypothesis, we have
\[ d (x) (d (y) \theta (x) + \theta (y) d (x)) = \theta (x o y) \theta (x), \]
for all \( x, y \in N \).
By Lemma 2, we get
\[ d (x) d (y) \theta (x) + d (x) \theta (y) d (x) = \theta (x o y) \theta (x), \]
for all \( x, y \in N \).
Using the hypothesis, we obtain that
\[ \theta (x o y) \theta (x) + d (x) \theta (y) d (x) = \theta (x o y) \theta (x), \]
for all \( x, y \in N \),
and so
\[ d (x) \theta (y) d (x) = 0, \]
for all \( x, y \in N \).
Since $\theta$ is an automorphism of $N$, we get
\[ d(x) N d(x) = 0, \text{ for all } x \in N. \]

By the primeness of $N$, we obtain that $d = 0$. If $d = 0$, then we have $\theta(xo y) = 0$, for all $x, y \in N$ by the hypothesis, and so $xy = -yx$, for all $x, y \in N$. Writing $yz$ by $y$ in this equation, we have
\[ xyz = -yzx = yxz, \text{ for all } x, y, z \in N, \]
and so
\[ [x, y] z = 0, \text{ for all } x, y, z \in N. \]

Since $N$ is a 3–prime near-ring, we get $[x, y] = 0$, for all $x, y \in N$, and so, $N$ is commutative. □

**Theorem 9.** Let $N$ be a 2-torsion free 3–prime near-ring and $d, h$ be two $(\theta, \theta)–$  
derivations. If $d(x) \theta(y) = \theta(x) h(y)$ for all $x, y \in N$, then $d = h = 0$.

**Proof.** We get
\[ d(x) \theta(y) = \theta(x) h(y), \text{ for all } x, y \in N. \quad (3.4) \]
Replacing $y$ by $yz, z \in N$ in (3.4), we arrive at
\[ d(x) \theta(y) \theta(z) = \theta(x) (h(y) \theta(z) + \theta(y) h(z)), \text{ for all } x, y, z \in N. \]

By Lemma 2, we have
\[ d(x) \theta(y) \theta(z) = \theta(x) h(y) \theta(z) + \theta(x) \theta(y) h(z), \text{ for all } x, y, z \in N. \]

Using (3.4), we find that
\[ \theta(x) h(y) \theta(z) = \theta(x) h(y) \theta(z) + \theta(x) \theta(y) h(z), \text{ for all } x, y, z \in N, \]
and so
\[ \theta(x) \theta(y) h(z) = 0, \text{ for all } x, y, z \in N. \]

That is
\[ \theta(x) N h(z) = 0, \text{ for all } x, z \in N. \]

By the primeness of $N$ gives $h = 0$. If $h = 0$, then $d(x) \theta(y) = 0$, for all $x, y \in N$ by the hypothesis. Again using the primeness of $N$, we get $d = 0$. This completes the proof. □
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