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ON THE EXISTENCE OF SOLUTIONS FOR AN INFINITE SYSTEM OF INTEGRAL EQUATIONS

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Abstract. The paper is devoted to prove the existence of solutions for an infinite system of non-linear integral equations. This system is investigated in the WC-Banach algebra $C(I,c_0)$, the space of all continuous functions acting from an interval I into the sequence space c_0. Making use of the measure of weak noncompactness and the weak topology, we establish some fixed point theorems for the sum and the product of nonlinear weakly sequentially continuous operators acting on WC-Banach algebra involving w-contractive operators.

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1. Introduction

In this paper we are mainly concerned with the existence results for the following system of nonlinear integral equations [1, 11]

$$x_n(t) = a_n(x_n(t)) \int_0^1 b(t,s) f_n(s, x_n(s), x_{n+1}(s), \dots) ds + c_n(x_n(t)), \tag{1.1}$$

where $n = 1, 2, \dots$ and $t \in I = [0,1]$. The infinite system of equations (1.1) will be investigated in the Banach space $C_0 = C(I, c_0)$ consisting of all functions acting from the interval I into the sequence space c_0, which are continuous on I. Obviously, the norm in the space C_0 has the form

$$\|x\|_{C_0} = \|(x_1, x_2, \dots)\|_{C_0} = \sup_{t\in I}\{\sup[|x_n(t)| : n = 1, 2, \dots]\}.$$

Note that the previous system (1.1) may be written in the form

$$Ax.Bx + Cx = x, \tag{1.2}$$

where A, B, and C are three nonlinear operators defined on subsets of a Banach algebra.

The aim of the paper is to present new results in concern with the existence of fixed points of operators dealing with Banach algebras. More precisely, we will consider operators having the form (1.2), where A, B, and C are defined on subsets of a given Banach algebra and satisfy some conditions expressed in terms of a generalized Lipschitz continuity or the contractivity with respect to the De Blasi measure of weak noncompactness. This study is carried out in the frame work of the so-called WC-Banach algebra i.e., a Banach algebra in which the product of two weakly compact subsets is weakly compact. This concept has been introduced quite recently in [3] and it turned out to be very fruitful when investigating some problems of operators theory under weak topology [9].

Notice that it was shown in [10] that if the operator A involved in (1.2) is weakly compact and satisfy condition $(\mathcal{H})$ (this condition allows us to distinguish some classes of operators which transform each weakly convergent sequence in a Banach space X into a sequence containing a subsequence being strongly convergent in X) and B and C are w-contractive then the equation (1.2) has at least one solution (see [10, Theorem 3.2]). However, the condition (ii) in Theorem 3.2 in [10] is too restrictive and can be relaxed just by supposing the operator A is w-contractive. This result is developed in Theorem 4 (see section 3).

The remaining of the paper is organized as follows. After some preliminaries in section 2, we establish in section 3 some fixed point theorems for the sum and the product of nonlinear weakly sequentially continuous operators acting on WC-Banach algebra involving w-contractive operators. Finally, we provide in section 4 an example indicating that our results are still valid in the field of nonlinear integral equations.

2. Preliminaries

Throughout the paper we denote by $\mathbb{R}$ the set of real numbers. The symbol $\mathbb{N}$ stands for the set of natural numbers. By the symbol X we will denote a Banach space endowed with the norm $\|.\|$. For any $r > 0$, B_r denotes the closed ball in X centered at 0_X with radius r and $\mathcal{D}(A)$ denotes the domain of an operator A. Also Ω_X is the collection of all nonempty bounded subsets of X and $\mathcal{K}^w$ is the subfamily of Ω_X consisting of all weakly compact subsets of X. Now, $\rightharpoonup$ denotes the weak convergence and $\to$ denotes the strong convergence in X, respectively.

Further, let us recall the concept of the De Blasi measure of weak noncompactness [6] being the function $\omega\colon \Omega_X \to [0,+\infty)$, defined in the following way

$$\omega(M) = \inf\{r > 0 : \text{ there exits } K \in \mathcal{K}^w \text{ such that } M \subseteq K + B_r\},$$

for all $M \in \Omega_X$. For convenience we recall some basic properties of $\omega(.)$ needed below [6].

Lemma 1. *Let M_1, M_2 be two elements of Ω_X. Then, the following conditions are satisfied:*

(1) $M_1 \subseteq M_2$ implies $\omega(M_1) \leq \omega(M_2)$.
(2) $\omega(M_1) = 0$ if, and only if, M_1 is relatively weakly compact.
(3) $\omega(\overline{M_1}^w) = \omega(M_1)$, where $\overline{M_1}^w$ is the weak closure of the subset M_1.

(4) $\omega(M_1 \cup M_2) = \max\{\omega(M_1), \omega(M_2)\}$.
(5) $\omega(\lambda M_1) = |\lambda|\omega(M_1)$ *for all* $\lambda \in \mathbb{R}$.
(6) $\omega(\mathrm{co}(M_1)) = \omega(M_1)$, *where* $\mathrm{co}(M_1)$ *denotes the convex hull of* M_1.
(7) $\omega(M_1 + M_2) \leq \omega(M_1) + \omega(M_2)$.
(8) *if* $(M_n)_{n\geq 1}$ *is a decreasing sequence of nonempty bounded and weakly closed subsets of* X *with* $\lim_{n\to\infty} \omega(M_n) = 0$, *then* $M_\infty := \cap_{n=1}^{\infty} M_n$ *is nonempty and* $\omega(M_\infty) = 0$.

Definition 1 ([9, Definition 1.3.3])**.** An operator $A\colon \mathcal{D}(A) \subseteq X \to X$ is said to be weakly sequentially continuous on $\mathcal{D}(A)$ if for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(A)$, $x_n \rightharpoonup x$ implies $Ax_n \rightharpoonup Ax$.

Definition 2 ([9, Definition 1.4.3])**.** An operator $A\colon \mathcal{D}(A) \subseteq X \to X$ is said to be ω-contractive (or ω-α-contraction) if it maps bounded sets into bounded sets, and there exists some $\alpha \in [0,1)$ such that $\omega(A(S)) \leq \alpha\omega(S)$ for all bounded subsets $S \subseteq \mathcal{D}(A)$.

An operator $A\colon \mathcal{D}(A) \subseteq X \to X$ is said to be ω-condensing if it maps bounded sets into bounded sets, and $\omega(A(S)) < \omega(S)$ for all bounded sets $S \subseteq \mathcal{D}(A)$ with $\omega(S) > 0$.

Example 1 ([9, Corollary 2.3.2]). If $A\colon \mathcal{D}(A) \subseteq X \to X$ is Lipschitzian with a Lipschitz constant $\alpha \in [0,1)$ and is weakly sequentially continuous on X, then A is ω-α-contraction.

3. Fixed Point Theorems in WC-Banach Algebras

First, let us recall other definitions which will be used in our study.

Definition 3 ([3, Definition 2.2])**.** Let X be a Banach algebra. We say that X is a WC-Banach algebra, if the product $K^w.K'^w$ of arbitrary weakly compact subsets K^w, K'^w of X is weakly compact.

It is interesting to work with these WC-Banach algebras and find important characterizations under the weak topology setting. Some results in this direction were established in [3, 4]. We provide the following definition as a way to highlight the most important of these results [4].

Definition 4 ([4, Definition 3.1])**.** We will say that the Banach algebra X satisfies condition $(\mathcal{P})$ if for every two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X such that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$ for some $x, y \in X$, we have $x_n.y_n \rightharpoonup x.y$.

Recently, J. Banas and L. Olszowy [2] have shown the equivalence between WC-Banach algebra and a Banach algebra satisfying $(\mathcal{P})$.

Theorem 1 ([2, Theorem 2.9])**.** *A Banach algebra* X *satisfies condition* $(\mathcal{P})$ *if and only if* X *is the* WC*-Banach algebra.*

Example 2. Clearly, every finite dimensional Banach algebra is a WC-Banach algebra. If X is a WC-Banach algebra, then the set $\mathcal{C}(\mathcal{K}, X)$ (here $\mathcal{K}$ is a compact

Hausdorff space) of all continuous functions from $\mathcal{K}$ to X is also a WC-Banach algebra. The proof is based on Dobrakov's Theorem [7].

Theorem 2 ([7, Theorem 9])**.** *Let $\mathcal{K}$ be a compact Hausdorff space and X be a Banach space. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{C}(\mathcal{K},X)$, and $f \in \mathcal{C}(\mathcal{K},X)$. Then, $(f_n)_{n\in\mathbb{N}}$ is weakly convergent to f if and only if $(f_n(t))_{n\in\mathbb{N}}$ is weakly convergent to $f(t)$ for each $t \in \mathcal{K}$.*

Example 3 ([1, Example 2.9]). Now, let us consider the classical Banach sequence space c_0 consisting of all real (or complex) sequence converging to zero and equipped with the standard supremum (maximum) norm. In other words

$$c_0 = \left\{ x = (x_n)_{n\in\mathbb{N}} \ ; \ \lim_{n\to\infty} x_n = 0 \right\} \quad \text{and} \quad \|x\| = \sup\{|x_n| : \ n \in \mathbb{N}\}.$$

We define the product of two elements $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ in c_0 in the classical way:

$$x \cdot y = (x_n)_{n\in\mathbb{N}} \cdot (y_n)_{n\in\mathbb{N}} = (x_n y_n)_{n\in\mathbb{N}}.$$

Observe that for $x, y \in c_0$ we have

$$\begin{aligned} \|x \cdot y\| &= \sup\{|x_n y_n| : \ n \in \mathbb{N}\} \\ &\le \|x\| \sup\{|y_n| : \ n \in \mathbb{N}\} \\ &= \|x\| \|y\|. \end{aligned}$$

Thus c_0 is a Banach algebra (normalized).

We show that c_0 is the WC-Banach algebra. To this end let us recall [8] that in the Banach space c_0 the sequence $(x_k)_{k\in\mathbb{N}} = ((x_n^k)_{n\in\mathbb{N}})_{k\in\mathbb{N}}$ (denoted $((x_k^n))$ in the sequel), where $x_k = (x_n^k)_{n\in\mathbb{N}} \in c_0$ for any $k = 1,2,\ldots$, is convergent to an element $x = (x_n)_{n\in\mathbb{N}} \in c_0$ if and only if the sequence $(x_k)_{k\in\mathbb{N}}$ is bounded in c_0 and $\lim_{k\to\infty} x_n^k = x_n$ for any $n = 1,2,\ldots$. In other words, the sequence $(x_k)_{k\in\mathbb{N}} = ((x_k^n))$ is weakly convergent to $x = (x_n)_{n\in\mathbb{N}} \in c_0$ if only if the sequence $(x_k)_{k\in\mathbb{N}}$ is bounded in c_0 and is coordinatewise convergent to $x = (x_n)_{n\in\mathbb{N}}$.

Further, let us assume that W and W' are weakly compact subsets of the space c_0. Consider the product $W \cdot W'$. Let us take an arbitrary sequence $(z_k)_{k\in\mathbb{N}} \subset W \cdot W'$, $z_k = (z_n^k)_{n\in\mathbb{N}}$ for any $k = 1,2,\ldots$. This means that there exist two sequences $(x_k)_{k\in\mathbb{N}} \subset W$, $(y_k)_{k\in\mathbb{N}} \subset W'$ such that $z_k = x_k \cdot y_k$ for any $k = 1,2,\ldots$. Since the sets W and W' are weakly compact, without loss of generality we can assume that $x_k \rightharpoonup x = (x_n)_{n\in\mathbb{N}} \in W$ and $y_k \rightharpoonup y = (y_n)_{n\in\mathbb{N}} \in W'$ as $k \to \infty$. If we denote $x_k = (x_n^k)_{n\in\mathbb{N}}$, $y_k = (y_n^k)_{n\in\mathbb{N}}$ for each $k = 1,2,\ldots$, then in view of the above quoted characterization of the weak convergence in c_0 we deduce that $\lim_{k\to\infty} x_n^k = x_n$ for any $n = 1,2,\ldots$. This implies that $\lim_{k\to\infty} z_n^k = \lim_{k\to\infty} x_n^k y_n^k = x_n y_n$ for $n = 1,2,\ldots$. Obviously, the sequence $(z_k)_{k\in\mathbb{N}} = (x_k)_{k\in\mathbb{N}} \cdot (y_k)_{k\in\mathbb{N}}$ is bounded in c_0.

Thus we showed that the sequence $(z_k)_{k\in\mathbb{N}}$ is weakly convergent to the element $z = x\cdot y \in W\cdot W'$. This allows us to infer that the set $W\cdot W'$ is weakly compact in the space c_0.

To present the main fixed point results of this section we need the following theorems.

Theorem 3 ([5, Theorem 3.2])**.** *Let S be a nonempty, bounded, closed, and convex subset of a Banach space X and let $A\colon S\to S$ be a weakly sequentially continuous mapping. If A is ω-condensing, then it has, at least, a fixed point in S.*

Lemma 2 ([3, Lemma 2.4])**.** *Let M_1 and M_2 be two bounded subsets of a WC-Banach algebra X. Then, we have the following inequality*

$$\omega(M_1.M_2) \le \|M_2\|\omega(M_1) + \|M_1\|\omega(M_2) + \omega(M_1)\omega(M_2).$$

Now, we are ready to state and prove the main result of this section.

Theorem 4. *Let S be a nonempty, bounded, closed, and convex subset of a WC-Banach algebra X and let A, $C\colon X\to X$ and $B\colon S\to X$ be three weakly sequentially continuous operators, satisfying the following conditions:*

(i) A is regular and $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$.
(ii) A, B, and C are w-contractive with constants α, β, and γ, respectively.
(iii) For each $y\in S$ the following implication holds

$$x = Ax.By + Cx \Rightarrow x \in S.$$

(iv) The following inequality is satisfied

$$\beta\Delta + \alpha\beta w(S) + \delta\alpha + \gamma < 1,$$

where, $\Delta = \|A(S)\|$ and $\delta = \|B(S)\|$.

Under the above assumptions the operator equation $x = Ax.Bx + Cx$ has at least one solution in the set S.

Proof. From assumption (i), it follows that for each $y\in S$, there is a unique $x_y\in X$ such that

$$\left(\frac{I-C}{A}\right)x_y = By,$$

or equivalently

$$Ax_y.By + Cx_y = x_y.$$

Since hypothesis (iii) holds, then $x_y\in S$. Hence, the operator $T = \left(\frac{I-C}{A}\right)^{-1}B\colon S\to S$ is well defined. At first, let us notice that after converting of the equality $T = \left(\frac{I-C}{A}\right)^{-1}B$, we obtain

$$T = AT.B + CT. \tag{3.1}$$

Let us now prove that T is ω-condensing. Let M be a subset of S with $\omega(M) > 0$ and taking into account the properties of the De Blasi measure of weak noncompactness, we get

$$\omega(TM) \leq \omega(A(TM).BM) + \omega(C(TM)).$$

The properties of ω in Lemma 2 and assumption (ii) on A, B and C yield

$$\omega(TM) \leq \delta\alpha\omega(TM) + \Delta\beta\omega(M) + \alpha\beta\omega(TM)\omega(M) + \gamma\omega(TM). \tag{3.2}$$

Note that from the inequality (3.2) and the fact that $\omega(TM)\omega(M) \leq \omega(S)\omega(M)$, one can deduce that

$$\omega(TM) \leq k_1\omega(M) \quad \text{where } k_1 = \frac{\Delta\beta + \alpha\beta\omega(S)}{1 - \delta\alpha - \gamma},$$

and from the inequality (3.2) and the fact that $\omega(TM)\omega(M) \leq \omega(TM)\omega(S)$, we can deduce that

$$\omega(TM) \leq k_2\omega(M) \quad \text{where } k_2 = \frac{\Delta\beta}{1 - \alpha\beta\omega(S) - \delta\alpha - \gamma}.$$

Then, we have

$$\omega(TM) \leq \min\{k_1, k_2\}\omega(M).$$

Using assumption (v), we have

$$\min\{k_1, k_2\} < 1,$$

and this prove that, the operator T is ω-condensing. Now, we prove that T is weakly sequentially continuous. For this purpose, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in S which converges weakly to some $x \in S$. In view of the equality (3.1) we obtain:

$$\omega(\{Tx_n :\ n \in \mathbb{N}\}) \leq \omega(\{A(Tx_n)Bx_n :\ n \in \mathbb{N}\}) + \omega(\{C(Tx_n) :\ n \in \mathbb{N}\}).$$

In the above estimate we have used that the set $\{Bx_n :\ n \in \mathbb{N}\}$ is relatively weakly compact (since $\{x_n :\ n \in \mathbb{N}\}$ is relatively weakly compact). Further, observe that according to assumption (ii) the operators A and C are w-contractive with constants α and γ, respectively. So, we obtain

$$\begin{aligned}\omega(\{Tx_n :\ n \in \mathbb{N}\}) &\leq \delta\alpha\omega(\{Tx_n :\ n \in \mathbb{N}\}) + \gamma\omega(\{Tx_n :\ n \in \mathbb{N}\}) \\ &\leq (\delta\alpha + \gamma)\omega(\{Tx_n :\ n \in \mathbb{N}\}) \\ &< \omega(\{Tx_n :\ n \in \mathbb{N}\}).\end{aligned}$$

Observe that conducting the same reasoning as above, on the basis of assumption (iv) we deduce that $\omega(\{Tx_n :\ n \in \mathbb{N}\}) = 0$. Thus the set $\{Tx_n :\ n \in \mathbb{N}\}$ is relatively weakly compact. Consequently, there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $Tx_{n_i} \rightharpoonup y$.

Going back to equality (3.1), to the weak sequential continuity of A, B and C and in view of condition $(\mathcal{P})$ we deduce that $Tx_{n_i} = A(Tx_{n_i}).Bx_{n_i} + C(Tx_{n_i}) \rightharpoonup Ay.Bx + Cy$. Then, $y = Ay.Bx + Cy$ so, $y = Tx$. Consequently, $Tx_{n_i} \rightharpoonup Tx$.

Now, we claim that $Tx_n \rightharpoonup Tx$. Suppose the contrary, then there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and a weak neighborhood V^w of Tx, such that $Tx_{n_i} \notin V^w$ for all $i \in \mathbb{N}$. Since $(x_{n_i})_{i\in\mathbb{N}}$ converges weakly to x then arguing as before, we may extract a subsequence $(x_{n_{i_j}})_{j\in\mathbb{N}}$ of $(x_{n_i})_{i\in\mathbb{N}}$, such that $Tx_{n_{i_j}} \rightharpoonup Tx$, which is absurd, since $Tx_{n_{i_j}} \notin V^w$, for all $j \in \mathbb{N}$. As a result, T is weakly sequentially continuous.

Next, taking into account the above proved properties of the operator T and utilizing Theorem 3 we infer that the operator T has at least one fixed point x in the set S i.e., there exists $x \in S$ such that $x = Tx$. Hence, we obtain

$$x = \left(\frac{I-C}{A}\right)^{-1} Bx,$$

and consequently

$$\left(\frac{I-C}{A}\right) x = Bx.$$

From the above equality we get

$$x - Cx = Ax.Bx.$$

Finally, we have

$$x = Ax.Bx + Cx.$$

This completes the proof of our theorem. □

The above theorem yields the following corollaries.

Corollary 1 ([10]). *Let S be a nonempty, bounded, closed, and convex subset of a WC-Banach algebra X and let A, $C\colon X \to X$ and $B\colon S \to X$ be three weakly sequentially continuous operators, satisfying the following conditions:*

(i) A is regular and $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$.
(ii) $A(S)$,$B(S)$, and $C(S)$ are relatively weakly compact.
(iii) For each $y \in S$ the following implication holds

$$x = Ax.By + Cx \Rightarrow x \in S.$$

Under the above assumptions the operator equation $x = Ax.Bx + Cx$ has at least one solution in the set S.

Now, we formulate the final result of this section.

Theorem 5. *Let S be a nonempty, bounded, closed, and convex subset of a WC-Banach algebra X and let A, $C\colon X \to X$ and $B\colon S \to X$ be three weakly sequentially continuous operators, satisfying the following conditions:*

(i) A and C are Lipschitzian with the Lipschitz constants α and γ, respectively, where $\alpha, \gamma \in [0,1)$.
(ii) A is regular.
(iii) $B(S)$ is relatively weakly compact.

(iv) *For each $y \in S$ the following implication holds*

$$x = Ax.By + Cx \Rightarrow x \in S.$$

(v) *The following inequality is satisfied*

$$\delta\alpha + \gamma < 1,$$

where, $\Delta = \|A(S)\|$ *and* $\delta = \|B(S)\|$.

Under the above assumptions the operator equation $x = Ax.Bx + Cx$ has at least one solution in the set S.

Proof. Let y be fixed in S and let's define the mapping

$$\begin{cases} \varphi_y : X \longrightarrow X, \\ \qquad x \longrightarrow \varphi_y(x) = Ax.By + Cx. \end{cases}$$

Let $x_1, x_2 \in X$. The use of assumption (i) leads to

$$\begin{aligned} \|\varphi_y(x_1) - \varphi_y(x_2)\| &\leq \|Ax_1.B_y - Ax_2.B_y\| + \|Cx_1 - Cx_2\| \\ &\leq \|Ax_1 - Ax_2\|\|By\| + \|Cx_1 - Cx_2\| \\ &\leq (\delta\alpha + \gamma)\|x_1 - x_2\|. \end{aligned}$$

Now, an application of Banach's fixed point theorem [9] shows the existence of a unique point $x_y \in X$, such that

$$\varphi_y(x_y) = x_y.$$

Hence, the operator $T := \left(\frac{I-C}{A}\right)^{-1} B \colon S \to X$ is well defined. Moreover, the use of assumption (iv) allows us to have $T(S) \subset S$. Using arguments similar to those used in the proof of Theorem 4, we can deduce that the operator T is weakly sequentially continuous. By using Theorem 3, it is sufficient to check that T is ω-condensing. In order to achieve this, let M be a subset of S with $\omega(M) > 0$.

Using equality (3.1), we have

$$\omega(TM) \leq \omega(A(TM).BM) + \omega(C(TM)).$$

Making use of Lemmas 1, 2, and Example 1, together with the assumptions on A, B, and C enables us to have

$$\begin{aligned} \omega(T(M)) &\leq \omega(A(T(M))B(M)) + \omega(C(T(M))) \\ &\leq \delta\alpha\omega(T(M)) + \gamma\omega(T(M)) \\ &\leq (\delta\alpha + \gamma)\omega(T(M)). \end{aligned}$$

From the assumption (v), we have

$$\omega(T(M)) < \omega(T(M)).$$

Hence, $T(M)$ is relatively weakly compact, and in particular, T is ω-condensing. $\square$

4. Application to infinite systems of integral equations

This section is dedicated to show the applicability of the theory developed in the previous section to prove a result on the existence of solutions of the following infinite system of nonlinear integral equations [1, 2]

$$x_n(t) = a_n(x_n(t)) \int_0^1 b(t,s) f_n(s, x_n(s), x_{n+1}(s), \ldots) ds + c_n(x_n(t)), \tag{4.1}$$

where $n = 1, 2, \ldots$ and $t \in I = [0,1]$. The infinite system of equations (4.1) will be investigated in the Banach space $C_0 = C(I, c_0)$ consisting of all functions acting from the interval I into the sequence space c_0, which are continuous on I. Obviously, the norm in the space C_0 has the form

$$\|x\|_{C_0} = \|(x_1, x_2, \ldots)\|_{C_0} = \sup_{t \in I}\{\sup[|x_n(t)| : \ n = 1, 2, \ldots]\}.$$

Moreover, let us recall that by $C(I)$ we will denote the space $C(I, \mathbb{R})$ equipped with the norm $\|x\|_{C(I)} = \sup\{|x(t)| : \ t \in I\}$. Let us mention that C_0 forms a WC-Banach algebra (cf. Example 2). Now, we formulate a few assumptions under which we will investigate the infinite system (4.1).

(i) For any $n \geq 1$, the function $a_n \colon \mathbb{R} \to (0, \infty)$ is Lipschitzian with the constant $\alpha \in [0,1)$.

(ii) For any sequence $(x_n)_{n \geq 1}$ with $x_n \to 0$ as $n \to \infty$ we have that $a_n x_n \to 0$ as $n \to \infty$.

(iii) The function $c_n \colon \mathbb{R} \to \mathbb{R}$ is Lipschitzian with the Lipschitz constant $\gamma \in [0,1)$ and $c_n(0) = 0$ for $n = 1, 2, \ldots$.

(iv) There exist two constants M_1 and $M_2 > 0$ such that $a_n(x) \leq M_1$ and $|c_n(x)| \leq M_2$, for any $x \in \mathbb{R}$ and for $n = 1, 2, \ldots$.

(v) The function f_n acts from the set $I \times \mathbb{R}^\infty$ into $\mathbb{R}$ for any $n = 1, 2, \ldots$. Moreover, we assume that there exist two sequences $(k_n)_{n \geq 1}$, $(l_n)_{n \geq 1}$ with positive terms such that $k_n \to 0$ as $n \to \infty$, $(l_n)_{n \geq 1}$ is bounded and the following inequality is satisfied

$$|f_n(t, x_n, x_{n+1}, \ldots)| \leq k_n + l_n \sup\{|x_i| : \ i \geq n\}$$

for any $t \in I$, $x = (x_n)_{n \geq 1} \in c_0$ and for $n = 1, 2, \ldots$.

(vi) The family of functions $\{f_n\}_{\geq 1}$ is uniformly equicontinuous on the set $I \times c_0$. This means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $n \geq 1$ and $t \in I$, and for all $x = (x_n)_{n \geq 1}$, $y = (y_n)_{n \geq 1} \in c_0$ with $\|x - y\|_{c_0} \leq \delta$ we have that

$$|f_n(t, x_n, x_{n+1}, \ldots) - f_n(t, y_n, y_{n+1}, \ldots)| \leq \varepsilon.$$

(vii) The function $b(t,s) = b \colon \ I \times I \to \mathbb{R}$ is continuous in t uniformly with respect to the variable $s \in I$ and is integrable with respect to s for any $t \in I$.

Remark 1. Observe that from assumptions (i) and (iii) it follows that the functions a_n and c_n $(n = 1, 2, \ldots)$ are continuous on $\mathbb{R}$.

Remark 2. From the above formulated assumption we deduce that $K < \infty$ and $L < \infty$, where the constants K and L are defined by the equalities:

$$K = \sup\{k_n :\ n = 1,2,\dots\}, \qquad L = \sup\{l_n :\ n = 1,2,\dots\}.$$

Remark 3. In view of assumption (vii) the function

$$\overline{b}(t) = \int_0^1 b(t,s)ds$$

is well defined on I. Observe that this function is continuous on the interval I. In fact, for a fixed $\varepsilon > 0$ and for arbitrary $t_1, t_2 \in I$ such that $|t_2 - t_1| \le \varepsilon$, we have

$$\begin{aligned}|\overline{b}(t_2) - \overline{b}(t_1)| &\le \int_0^1 |b(t_2,s) - b(t_1,s)|ds \\ &\le \int_0^1 \mu_1(b,\varepsilon)ds = \mu_1(b,\varepsilon),\end{aligned}$$

where the function $\mu_1(b,\varepsilon)$ denotes the modulus of continuity of the function $t \to b(t,s)$ defined by the formula

$$\mu_1(b,\varepsilon) = \sup\{|b(t_2,s) - b(t_1,s)| :\ t_1,t_2,s \in I,\ |t_2 - t_1| \le \varepsilon\}.$$

Obviously, in view of assumption (vii) we have that $\mu_1(b,\varepsilon) \to 0$ as $\varepsilon \to 0$. This shows that the function $\overline{b} = \overline{b}(t)$ defined above is continuous on I. Taking into account the above statement we can define the finite constant $\overline{B}$ by putting

$$\overline{B} = \sup\left\{ \int_0^1 |b(t,s)|ds :\ t \in I \right\}.$$

Now, we formulate our further assumptions. To this end, let us put

$$M = \max\{M_1, M_2\}.$$

(viii) The following inequality holds

$$M\overline{B}L < 1.$$

where the constants L and $\overline{B}$ were defined earlier.

For further purposes we define the number r_0 by putting

$$r_0 = \frac{MK\overline{B} + M}{1 - ML\overline{B}}. \tag{4.2}$$

Finally, we assume that

(ix) $\alpha\left(\dfrac{K\overline{B} + L\overline{B}M}{1 - ML\overline{B}}\right) + \gamma < 1.$

Before formulating our main result we provide an auxiliary lemma which will be useful in our investigations.

Lemma 3 ([1, Lemma 4.6]). *Let the function $x(t) = (x_1(t), x_2(t), \dots) = (x_n(t))_{n\geq 1}$ be an element of the space $C_0 = C(I, c_0)$. Then*

$$\lim_{n\to\infty} \|x_n\|_{C(I)} = 0.$$

Theorem 6. *Under assumptions $(i)-(ix)$ the infinite system of integral equations (4.1) has at least one solution $x(t) = (x_n(t))_{n\geq 1}$ in the space $C_0 = C(I, c_0)$.*

Proof. Let us define the subset S of $C_0 = C(I, c_0)$ by

$$S = B_{r_0} := \{x \in C_0 : \|x\| \leq r_0\},$$

where r_0 is a number described by equality (4.2). Obviously, S is a nonempty, closed, convex, and bounded subset of $C_0 = C(I, c_0)$. To make lecture of the infinite system of equations let us consider three operators A, B, and C defined on C_0 as follow

$$\begin{cases} (Ax)(t) = (a_n(x_n(t)))_{n\geq 1} = (a_1(x_1(t)), a_2(x_2(t)), \dots), \\ (Bx)(t) = \left(\int\limits_0^1 b(t,s) f_n(s, x_n(s), x_{n+1}(s), \dots) ds \right)_{n\geq 1}, \\ (Cx)(t) = (c_n(x_n(t)))_{n\geq 1} = (c_1(x_1(t)), c_2(x_2(t)), \dots), \end{cases} .$$

for $t \in I = [0, 1]$. We show that these operators satisfy the assumptions of Theorem 5. We start with investigations concerning the operator A. At first, let us observe that assumption (i) guarantees that A is regular. This means that there is satisfied assumption (ii) of Theorem 5.

Next let us fix arbitrary $x = (x_n)_{n\geq 1}$ and $y = (y_n)_{n\geq 1} \in S$. If we take an arbitrary $t \in I$, then we get

$$\begin{aligned} \|A(x)(t) - A(y)(t)\|_{c_0} &= \|(a_1(x_1)(t) - a_1(y_1)(t), a_2(x_2)(t) - a_2(y_2)(t), \dots)\|_{c_0} \\ &\leq \sup\{|a_n(x_n)(t) - a_n(y_n)(t)| : \ n = 1, 2, \dots\} \\ &\leq \alpha \|x(t) - y(t)\|_{c_0}. \end{aligned}$$

From the last inequality and taking the supremum over t, we obtain

$$\|Ax - Ay\|_{C_0} \leq \alpha \|x - y\|_{C_0}.$$

This shows that the operator A is Lipschitzian with a Lipschitzian constant α.

Now, we will verify that A transforms the space C_0 into itself. Indeed, let us take a function $x(t) = (x_n(t))_{n\geq 1} \in C_0$. This means, that for each fixed $t \in I$ we have that $x_n(t) \to 0$ when $n \to \infty$. Hence, in view of assumption (ii) we get that $a_n(x_n(t)) \to 0$ as $n \to \infty$. This proves our claim.

Further on, we show that the operator A is weakly sequentially continuous. Let us take a sequence $(x_n)_{n\geq 1} \subset C_0$ which is weakly convergent to a function $x \in C_0$. This means by [7] that if we denote

$$x_n(t) = (x_1^n(t), x_2^n(t), x_3^n(t), \dots)$$

for $n = 1, 2, \ldots$ and for an arbitrary $t \in I$ and if we denote $x(t) = (x_1(t), x_2(t), x_3(t), \ldots)$, then we have $x_1^n(t) \to x_1(t), \ldots, x_k^n(t) \to x_k(t), \ldots$ for any $t \in I$, if $n \to \infty$. Now, let us consider the sequence $(Ax_n)_{n\geq 1}$ i.e.,

$$\begin{aligned}(Ax_n)_{n\geq 1} &= (A(x_1^n, x_2^n, x_3^n, \ldots))_{n\geq 1} = (a_1(x_1^n), a_2(x_2^n), a_3(x_3^n), \ldots)_{n\geq 1} \\ &= (a_k(x_k^n))_{n\geq 1}.\end{aligned}$$

Then, for an arbitrarily fixed $t \in I$ we obtain

$$\begin{aligned}((Ax_n)(t))_{n\geq 1} &= (a_1(x_1^n(t)), a_2(x_2^n(t)), a_3(x_3^n(t)), \ldots)_{n\geq 1} \\ &= (a_k(x_k^n(t)))_{n\geq 1}.\end{aligned}$$

Since, according to our assumptions we have that $x_k^n(t) \to x_k(t)$ as $n \to \infty$, $(k = 1, 2, \ldots)$, this implies that $a_k(x_k^n(t)) \to a_k(x_k(t))$, $(k = 1, 2, \ldots)$, which is a simple consequence of the continuity of each function a_k, $(k = 1, 2, \ldots)$, on the set $\mathbb{R}$ (cf. Remark 1). Since $(Ax_n)_{n\geq 1}$ is bounded by M_1, then we can again apply Dobrakov's theorem (see Theorem 2), A is weakly sequentially continuous.

In a similar way we can show that the operator C is well-defined, Lipschitzian with the Lipschitz constants γ, and weakly sequentially continuous.

In what follows we will consider the operator B. To this end let us take the set S. At he beginning we show that B transforms the set S into the space C_0. Thus, take an arbitrary function $x(t) = (x_n(t))_{n\geq 1} \in S$. Fix arbitrarily $n \geq 1$ and a number $t \in I$. Then, keeping in mind assumptions (v) and (vii), we obtain:

$$\begin{aligned}|(Bx)(t)| &\leq \int_0^1 |b(t,s)||f_n(s, x_n(s), x_{n+1}(s), \ldots)|ds \\ &\leq \int_0^1 |b(t,s)|\{k_n + l_n \sup[|x_i(s)| :\ i \geq n]\}ds \\ &\leq k_n \int_0^1 |b(t,s)|ds + l_n \int_0^1 |b(t,s)| \sup[|x_i(s)| :\ i \geq n]ds \\ &\leq k_n\overline{B} + l_n \sup_{i\geq n}\{\sup[|x_i(t)| :\ t \in I]\}\overline{B},\end{aligned}$$

where $\overline{B}$ was defined in Remark 3.

Consequently, in view of assumption (v) and Remark 2, we get

$$|(Bx)(t)| \leq k_n\overline{B} + L\overline{B} \sup_{i\geq n}\{\sup[|x_i(t)| :\ t \in I]\},$$

for any $n \geq 1$ and for each $t \in I$. The above estimate implies the following one

$$|(Bx)(t)| \leq k_n\overline{B} + L\overline{B} \sup_{i\geq n} \|x_i\|_{C(I)}. \tag{4.3}$$

Now, taking into account that $k_n \to 0$ as $n \to \infty$ and keeping in mind Lemma 3 we conclude from estimate (4.3) that the operator B transforms the set S into the C_0.

Moreover, from estimate (4.3) we infer the following inequality

$$\|Bx\|_{C_0} \leq K\overline{B} + L\overline{B}\|x\|_{C_0}.$$

Consequently (since $x \in S$), we obtain

$$\|Bx\|_{C_0} \leq K\overline{B} + L\overline{B}r_0. \tag{4.4}$$

Further on, we show that the operator B is weakly sequentially continuous on the set S. To do it, let $(x_n)_{n\geq 1}$ be any sequence in S weakly converging to a function $x \in S$. Then, by using Dobrakov's theorem [7], we get for all $t \in I$, $x_k^n(t) \to x_k(t)$, if $n \to \infty$, $(k = 1,2,\dots)$. Now, fix $\varepsilon > 0$. This means that for any $t \in I$, we have the following inequalities:

$$\begin{aligned}\|Bx_n(t) - Bx(t)\|_{c_0} &= \max_{k\geq 1}\left\{\left|\int_0^1 b(t,s)(f_k(s,x_k^n(s),x_{k+1}^n(s),\dots)\right.\right.\\ &\qquad \left.-f_k(s,x_k(s),x_{k+1}(s),\dots))ds|\right\}\\ &\leq \max_{k\geq 1}\left\{\int_0^1 |b(t,s)||f_k(s,x_k^n(s),x_{k+1}^n(s),\dots)\right.\\ &\qquad \left.-f_k(s,x_k(s),x_{k+1}(s),\dots)|ds\right\}.\end{aligned}$$

Let $\delta > 0$. Since $x_k^n(t) \to x_k(t)$, then $\forall k \geq 1$ there exists $n_0 \geq 1$ such that for all $n \geq n_0$ and for all $t \in I$,

$$|x_k^n(t) - x_k(t)| \leq \delta, \qquad \forall\, n \geq n_0.$$

Hence, in view of assumption (vi), we obtain for any $t \in I$

$$\begin{aligned}\|Bx_n(t) - Bx(t)\|_{c_0} &\leq \int_0^1 |b(t,s)|\varepsilon ds\\ &\leq \overline{B}\varepsilon, \qquad \forall\, n \geq n_0.\end{aligned}$$

This shows that $Bx_n(t) \to Bx(t)$ in c_0, in particular, $Bx_n(t) \rightharpoonup Bx(t)$ in c_0. Since $(Bx_n)_{n\geq 1}$ is bounded by $K\overline{B} + L\overline{B}r_0$, then we can again apply Dobrakov's theorem [7] to obtain $Bx_n \rightharpoonup Bx$. We conclude that the operator B is weakly sequentially continuous.

Now, we are going to prove that for all $t \in I$, the set $\{Bx(t) :\ x \in S\}$ is relatively compact in c_0. At first let $t_1, t_2 \in I$, $n \geq 1$, and for a function $x \in S$, on the basis of the estimate from Remark 3, we have:

$$|(Bx)(t_2) - (Bx)(t_1)| \leq \int_0^1 |b(t_2,s) - b(t_1,s)||f_n(s,x_n(s),x_{n+1}(s),\dots)|ds$$

$$\leq \int_0^1 \mu_1(b,|t_2-t_1|)(K+Lr_0)ds$$
$$\leq (K+Lr_0)\mu_1(b,|t_2-t_1|).$$

This implies that the family of functions $\{Bx(t): \ x\in S\}$ is equicontinuous on the space c_0. Further, let $(Bx_k(t))_{k\geq 1}$ be a sequence of $B(S)(t)$. From estimate (4.4), we have

$$\sup_{n\geq 1}|Bx_n^k(t)| \leq \|Bx_k\|_{C_0} \leq K\overline{B}+L\overline{B}r_0.$$

for all $t\in I$ and for all $k\geq 1$. Then, the sequence $(Bx_n^k(t))_{k\geq 1}$ is equibounded in $\mathbb{R}$, then using Bolzano-Weirstrass's Theorem, there is a renamed subsequence such that

$$\lim_{k\to\infty} Bx_n^k(t) = g_n(t), \quad \text{for all } n\geq 1 \text{ and } t\in I.$$

Then, this means that the following condition is satisfied:

For all $t\in I$ and $\varepsilon>0$, there exists $k_0\geq 1$ such that for all $k\geq k_0$, we have

$$|Bx_n^k(t)-g_n(t)| \leq \frac{\varepsilon}{2}, \quad \text{for all } n\geq 1.$$

Taking into account that $\lim_{n\to\infty} Bx_n^{k_0}(t)=0$ then, for all $t\in I$, there exists $n_0\geq 1$ such that for all $n\geq n_0$, we have

$$|Bx_n^{k_0}(t)| \leq \frac{\varepsilon}{2}.$$

Now, for $t\in I$ and $n\geq n_0$, we have

$$|g_n(t)| \leq |g_n(t)-Bx_n^{k_0}(t)|+|Bx_n^{k_0}(t)| \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

It follows that $(g_n(t))_{n\geq 1}\in c_0$. Then, $B(S)(t)$ is relatively compact in c_0. From the above established facts and Arzelá-Ascoli criterion [8] for the relative compactness, we deduce that the set $B(S)$ is relatively compact in C_0, in particular, $B(S)$ is relatively weakly compact in C_0. This means that there is satisfied assumption (iii) of Theorem 5.

In our next step we show that there is satisfied assumption (iv) of Theorem 5. To this end let us fix arbitrarily $y\in S$. Next, assume that an element $x\in C_0$ satisfies the equality

$$x = Ax.By+Cx.$$

This yields

$$\|x\|_{C_0} \leq \|Ax\|_{C_0}\|By\|_{C_0}+\|Cx\|_{C_0}.$$

Further, by hypothesis (iv) we obtain

$$\|x\|_{C_0} \leq M_1\|By\|_{C_0}+M_2$$
$$\leq M(K\overline{B}+L\overline{B}r_0+1)$$

Hence, in view of assumption $(viii)$ we obtain $\|x\|_{C_0} \leq r_0$. Finally, let us notice in view of equality (4.3) and hypothesis (ix) that

$$\begin{aligned}\alpha\|B(S)\|_{C_0}+\gamma &\leq \alpha\left(K\overline{B}+L\overline{B}\frac{MK\overline{B}+M}{1-ML\overline{B}}\right)+\gamma\\ &\leq \alpha\left(\frac{K\overline{B}+L\overline{B}M}{1-ML\overline{B}}\right)+\gamma<1.\end{aligned}$$

Hence, the hypothesis (v) of Theorem 5 is satisfied which achieves the proof. □

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