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FIXED POINT THEOREMS FOR UPPER SEMICONTINUOUS MULTIVALUED OPERATORS IN LOCALLY CONVEX SPACES

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Abstract. In the paper, we will discus some τ-topological properties of the set $\mathcal{F}(S_0,T,H) := \{x \in X : S_0x \in Tx + Hx\}$, where T is multivalued operator, S_0 and H are two single valued operators acting on a Hausdorff locally convex space X and τ is a weaker Hausdorff locally convex topology on X. Moreover, when X is a τ-angelic space with the so-called τ-Krein-Šmulian property, we will provide some new variants of fixed point theorems for multivalued operators. Our results are formulated in terms of τ-upper semicontinuity, τ-S_0-demicom-pactness and families of axiomatic measures of noncompactenss.

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1. INTRODUCTION

In 1966, W. V. Petryshyn [16] introduced the concept of demicompact nonlinear operators and gave an iterative methods for the construction of fixed points on Hilbert space. In [11, 17] the authors used this notion to give new fixed point theorems for demicompact-k-set contractions defined on Banach space X. In other direction the demicompactness concept was used to provide several results on Fredholm and spectral theories (see [2, 6, 18]). In 2014 B. Krichen [13] gave a generalization of this concept by introducing the class of relative demicompact linear operator with respect to a given linear operator. This definition asserts that if X is a Banach space $T: \mathcal{D}(T) \subset X \to X$, and $S_0: \mathcal{D}(S_0) \subset X \to X$ are two linear operator with $\mathcal{D}(T) \subset \mathcal{D}(S_0)$. Then T is said to be S_0-demicompact (or relative demicompact with respect to S_0) if every bounded $(x_n)_n$ in $\mathcal{D}(T)$ such that $(S_0x_n - Tx_n)$ converges in X, have a convergent subsequence. In [14] B. Krichen and D. O'Regan studied some topological properties of the set $\mathcal{F}(S_0,T,z) := \{x \in X,\, S_0x \in Tx + z\}$, where T is a nonlinear multivalued mapping and S_0 is single valued mapping on Banach space X. Their results are formulated in term of the concept of weakly relative demicompactness.

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In this paper we generalize the results of [14], to discuss some topological properties of the set

$$\mathcal{F}(S_0,T,H) := \{x \in X,\ S_0x \in Tx + Hx\},$$

where T is a multivalued operator, S_0 and H are two single valued operators acting on a Hausdorff locally convex space X such that $D(T) \subset D(S_0) \cap D(H)$. Then, we present some new fixed point theorems for τ-upper semicontinuous multivalued operators on X, where τ is a weaker Hausdorff locally convex topology on X.

Firstly, let us recall some notations and basic concepts. Let $(X, \{|\cdot|_p\}_{p\in\Lambda})$ be a Hausdorff locally convex vector space endowed with a family of seminorms $\{|\cdot|_p\}_{p\in\Lambda}$ generating its topology and let τ be a weaker Hausdorff locally convex topology on X. We denote by $\overset{\tau}{\to}$ the convergence in (X,τ) and by $\to$ the convergence in $(X, \{|\cdot|_p\}_{p\in\Lambda})$. We mean by τ-compact, (resp. τ-closed) sets, compact, (resp. closed) sets with respect to the topology τ. We also denote by $\mathcal{B}(X)$ the family of all nonempty bounded subsets of X (with respect to the topology generated by $\{|\cdot|_p\}_{p\in\Lambda}$).

Now, let us consider the following axiomatic definition of a family of measures of noncompactenss in a Hausdorff locally convex vector space.

Definition 1 (Definition 2.1 in [4])**.** A family of functions $\phi_{p_\tau}\colon \mathcal{B}(X) \to \mathbb{R}^+$, $(p \in \Lambda)$ is said to be a Φ^τ_Λ-measures of noncompactenss in X (Φ^τ_Λ-MNC, in short) if for each $p \in \Lambda$, it satisfies the following conditions:

- (i) $\phi_{p_\tau}(\overline{\text{conv}}^\tau(M)) \le \phi_{p_\tau}(M)$ for each $M \in \mathcal{B}(X)$, where $\overline{\text{conv}}^\tau(M)$ is the closure of the convex hull of M in (X,τ);
- (ii) $M_1 \subseteq M_2 \Rightarrow \phi_{p_\tau}(M_1) \le \phi_{p_\tau}(M_2)$, where $M_1, M_2 \in \mathcal{B}(X)$;
- (iii) $\phi_{p_\tau}(\{x\} \cup M) = \phi_{p_\tau}(M)$ for any $x \in X$ and $M \in \mathcal{B}(X)$;
- (iv) $\phi_{p_\tau}(M) = 0$ implies M is relatively τ-compact in X;
- (v) if $(M_n)_n$ is a sequence of τ-closed sets of $\mathcal{B}(X)$ such that $M_{n+1} \subset M_n$, $n = 1,2,\dots$ and $\lim\limits_{n\to\infty} \phi_{p_\tau}(M_n) = 0$ for each $p \in \Lambda$, then $M_\infty = \bigcap_{n=1}^\infty M_n$ is nonempty relatively τ-compact subset of X.

The family Φ^τ_Λ-MNC is called:

- Positively homogeneous, if for each $p \in \Lambda$, $\phi_{p_\tau}(\lambda M) = \lambda\phi_{p_\tau}(M)$, $\lambda > 0$, where $M \in \mathcal{B}(X)$.
- Subadditive, if for each $p \in \Lambda$, $\phi_{p_\tau}(M_1 + M_2) \le \phi_{p_\tau}(M_1) + \phi_{p_\tau}(M_2)$, where $M_1, M_2 \in \mathcal{B}(X)$.

Example 1 ([7]). The family of measures of weak noncompactenss in a locally convex space X, which is defined by:

$$\omega_p(M) = \inf\{r > 0 : \exists W \in \mathcal{W}(X) \text{ such that } M \subseteq W + B_p(0,r)\}, \qquad p \in \Lambda$$

is positively homogeneous and subadditive $\Phi^{\sigma(X,X^*)}_\Lambda$-MNC. Here, $B_p(0,r)$ is the closed ball centered at 0 with radius r, $\sigma(X,X^*)$ is the weak topology of X and $\mathcal{W}(X)$ is the set of all nonempty relatively weakly compact subsets of X. This formula is based on the notion of single measure of weak noncompactenss introduced by De Blasi [7].

Definition 2. Let M be a nonempty subset of X and let $\Phi^{\tau}_{\Lambda} := \{\phi_{p_\tau}, p \in \Lambda\}$ be a family of Φ^{τ}_{Λ}-MNC in X. An operator $A\colon M \to X$ is said Φ^{τ}_{Λ}-contraction if for any bounded subset S of M, $A(S) \in \mathcal{B}(X)$, and for each $p \in \Lambda$, there exists a constant $\beta_p \in [0,1)$ such that $\phi_{p_\tau}(A(S)) \le \beta_p \phi_{p_\tau}(S)$. The operator A is called Φ^{τ}_{Λ}-condensing if for any bounded subset S of M, $A(S) \in \mathcal{B}(X)$, and for each $p \in \Lambda$ such that $\phi_{p_\tau}(S) > 0$, $\phi_{p_\tau}(A(S)) < \phi_{p_\tau}(S)$.

Definition 3 (Definition p. 30 in [8])**.** A topological (Hausdorff) space X is called angelic (or has countably determined compactness) if for every relatively countably compact subset M of X, the following holds:

(i) M is relatively compact.
(ii) For each $x \in \overline{M}$, there is a sequence in M which converges to x.

Remark 1.

(i) Note that all metrizable locally convex spaces endowed with their weak topology are angelic. (See Eberlein-Šmulian theorem [15].)
(ii) In angelic spaces the classes of compact, countably compact, and sequentially compact sets coincide (see [8, p. 31]).

Definition 4 (Definition 2.4 in [4])**.** Let M be a nonempty subset of X. An operator $A\colon M \to X$ is said to be τ-closed on M if for each sequence $(x_n)_n \in M$ such that $x_n \xrightarrow{\tau} x$ and $Ax_n \xrightarrow{\tau} y$, then $x \in M$ and $y = Ax$.

Definition 5 (Definition 2.5 in [5])**.** Let M be a nonempty subset of X, $A\colon M \to X$ be an operator. We say that A is τ-sequentially continuous on M if for each sequence $(x_n)_n \subset M$ with $x_n \xrightarrow{\tau} x$ and $x \in M$, we have that $Ax_n \xrightarrow{\tau} Ax$.

Remark 2.

(i) Clearly, every τ-sequentially continuous operator is τ-closed, but the converse is not true.
(ii) If X is angelic, then any τ-sequentially continuous operator on a τ-compact set is τ-continuous.

Definition 6 (Definition 2.6 in [4])**.** We say that X has the τ-Krein-Šmulian property (τ-KS, in short) if the closed convex hull of a τ-compact set is τ-compact.

Remark 3.

(i) Each Fréchet space X has the τ-KS property, here τ is the weak topology $\sigma(X, X^*)$ defined on X, particularly for each Banach space.
(ii) Let X be a Hausdorff locally convex space with the τ-KS property, and let M be a nonempty relatively τ-compact subset of X. Using property *(ii)* of Definition 1, we obtain that, $\phi_{p_\tau}(M) \le \phi_{p_\tau}(\overline{\text{conv}}^{\tau}(M))$, hence $\phi_{p_\tau}(M) = 0$. In this case, we say that Φ^{τ}_{Λ}-MNC is regular.

Let X be a Hausdorff locally convex space, we define

$$\mathcal{P}(X) = \{M \subseteq X : M \text{ is nonempty}\}.$$
$$\mathcal{P}_{cl,cv}(X) = \{M \subseteq X : M \text{ is nonempty closed convex }\}.$$
$$\mathcal{P}_{\tau-cl,cv}(X) = \{M \subseteq X : M \text{is nonempty}\, \tau\text{-closed convex }\}.$$

Let M be a nonempty subset of X and $A\colon M \to \mathcal{P}(X)$ be a multivalued mapping. We denote by $\mathcal{R}(A) = \bigcup_{y\in M} A(y)$ and $Gr(A) = \{(x,y) \in M \times X, x \in A(y)\}$ the range and the graph of A respectively.

Definition 7. Let $A\colon M \to \mathcal{P}(X)$ be a multivalued operator. We say that

(i) A is τ-compact if the set $\mathcal{R}(A)$ is relatively τ-compact in X.
(ii) A is τ-closed if its graph $Gr(T)$ is τ-closed in $X \times X$.
(iii) A has a τ-sequentially closed graph if for every sequence $(x_n)_n \subset M$ with $x_n \xrightarrow{\tau} x$ in M and for every sequence $y_n \in A(x_n)$, $y_n \xrightarrow{\tau} y$ in X implies $y \in A(x)$.

Definition 8. Let $A\colon M \to \mathcal{P}(X)$ be a multivalued operator.

(i) A is called τ-upper semicontinuous if for each τ-closed set $C \subset M$, the set $A^{-1}(C)$ is τ-closed in X, such that
$$A^{-1}(C) = \{x \in X,\ A(x) \cap C \neq \varnothing\}.$$
(ii) We say that A is sequentially τ-upper semicontinuous if $A^{-1}(C)$ is sequentially τ-closed.

Remark 4. Clearly, every τ-upper semicontinuous multivalued operator is sequentially τ-upper semicontinuous, but the converse is not true.

The following lemma provides a sequential characterization of an upper semicontinuous multi-valued mapping.

Lemma 1. *A multi-valued map T is upper semi-continuous at a point $x \in X$ if, and only if, for every sequence $\{x_n\}_{n=0}^{\infty}$ in X which converges to x, and for any open set $V \subset X$ such that $T(x) \subset V$, there exists $n_0 \in \mathbb{N}$ with $T(x_n) \subset V$, for all $n \geq n_0$.*

The following lemma [10] will be used later.

Lemma 2. *Assume that $T\colon X \to \mathcal{P}(X)$ is an upper semi-continuous multi-valued operator. Then the graph $Gr(T)$ is a closed subset of $X \times X$.*

2. RELATIVE τ-DEMICOMPACT NONLINEAR OPERATORS

In this section X is a Hausdorff locally convex topological vector space, τ is a weaker Hausdorff locally convex vector topology on X and S_0 is a single-valued operator from $\mathcal{D}(S_0) \subset X$ into X. Now, we introduce a new concept of τ-S_0-demicompact.

Definition 9. Let X be a topological vector space. Let $T\colon \mathcal{D}(T)\subset X\longrightarrow X$ be a single-valued operator with $\mathcal{D}(T)\subset\mathcal{D}(S_0)$. Now T is called τ-S_0-demicompact if whenever $S_0x_n - Tx_n$ is τ-convergent and $(x_n)_n$ is contained in a bounded set of X, then the sequence $(x_n)_n$ has a τ-convergent subsequence in $\mathcal{D}(T)$. If $S_0 = I$, T is simply said to be τ-demicompact.

Definition 10. A single-valued operator $T\colon \mathcal{D}(T)\subset X\longrightarrow X$ with $\mathcal{D}(T)\subset\mathcal{D}(S_0)$ is said to be τ-S_0-semiclosed if for any τ-closed subset $V\subset$ of X, the set $(S_0-T)V$ is τ-closed.

Remark 5. We note that there is no relationship between the concepts of τ-S_0-semiclosedness and τ-S_0-demicompactness. Consider the map

$$\begin{cases} T\colon \mathbb{R} & \longrightarrow \mathbb{R} \\ \quad x & \longrightarrow \dfrac{x|x|}{1+|x|} \end{cases}$$

Obviously, $(Id_{\mathbb{R}}-T)(\mathbb{R}) =]-1,1[$, so it follows that T is not $Id_{\mathbb{R}}$-semiclosed. However, we know that in finite dimensional spaces, every bounded sequence has a weakly convergent subsequence. Therefore, the operator T is weakly $Id_{\mathbb{R}}$-demicompact.

For the following, let T be multivalued operator, S_0 and H are two single-valued operators from $\mathcal{D}(S_0),\mathcal{D}(H)$ into X respectively, with $\mathcal{D}(S_0)\subset\mathcal{D}(H)$ and $\mathcal{D}(T)\subset\mathcal{D}(S_0)\cap\mathcal{D}(H)$. We denote by $\mathcal{F}(S_0,T,H)$ the set of solutions of

$$S_0x\in Tx+Hx, \tag{2.1}$$

where, $Tx+Hx=\{y+Hx : y\in Tx\}$, and x is a solution if the relation (2.1) holds. Note we do not assume existence or uniqueness so the set $\mathcal{F}(S_0,T,H)$ might be empty or contain many elements.

Remark 6. Let X be a Hausdorff locally convex topological vector space. Assume that S_0 and H are continuous single-valued operators. Then, the subset $\mathcal{F}(S_0,T,H)$ is closed.

If X is endowed with its τ-topology, then we have the following.

Theorem 1. *Let X be a Hausdorff topological vector space (locally convex topological vector space). Assume that T is a multi-valued operator with τ-sequentially closed graph and S_0 and H are τ-sequentially continuous operators. Then, the subset $\mathcal{F}(S_0,T,H)$ is τ-sequentially closed.*

Proof. Let $(x_n)_n$ be a sequence of $\mathcal{F}(S_0,T,H)$ such that $x_n\xrightarrow{\tau}x$. Then, there exists a sequence $(y_n)_n$ of $(Tx_n)_n$ such that $S_0x_n = y_n + Hx_n$ for every $n\in\mathbb{N}$. From the τ-sequential continuity of S_0 and H it follows that $S_0x_n\xrightarrow{\tau}S_0x$ and $Hx_n\xrightarrow{\tau}Hx$ and then $(x_n,y_n)\xrightarrow{\tau}(x,S_0x-Hx)$. Since T has a τ-sequentially closed graph, it follows that $x\in\mathcal{F}(S_0,T,H)$ and so, $\mathcal{F}(S_0,T,H)$ is τ-sequentially closed. $\square$

Notice that if S_0 is only τ-closed, then we have the following:

Theorem 2. *Let X be a Hausdorff topological vector space (locally convex topological vector space). Assume that compact sets in (X,τ) are angelic. If T is a τ-compact multi-valued operator and $S_0 - H$ is a τ-closed operator, then the subset $\mathcal{F}(S_0,T,H)$ is τ-sequentially closed.*

Proof. Let $(x_n)_n \subset \mathcal{D}(T)$ be a sequence of $\mathcal{F}(S_0,T,H)$ such that $x_n \xrightarrow{\tau} x$. Then, there exists a sequence $(y_n)_n$ of $(Tx_n)_n$ such that $S_0x_n = y_n + Hx_n$ for every $n \in \mathbb{N}$. Since $\{y_n : n \in \mathbb{N}\} \subset \mathcal{R}(T)$, it follows that $\phi_{p_\tau}(\{y_n : n \in \mathbb{N}\}) = 0$. Since compact sets in (X,τ) are angelic, there exists a subsequence $(y_{\varphi(n)})_n$ such that $(S_0 - H)x_{\varphi(n)} \xrightarrow{\tau} y$. The τ-closedness of $S_0 - H$ shows that $x \in \mathcal{D}(T)$ and $S_0x \in Tx + Hx$. Then, $x \in \mathcal{F}(S_0,T,H)$ and consequently, $\mathcal{F}(S_0,T,H)$ is τ-sequentially closed. □

Theorem 3. *Let X be a Hausdorff locally convex topological vector space such that compact sets in (X,τ) are angelic and let T be a τ-compact operator with τ-sequentially closed graph. Assume that $S_0 - H$ is τ-closed and H is a S_0-τ-demicompact mapping. Then the set $\mathcal{F}(S_0,T,H)$ is relatively τ-compact.*

Proof. Let $(x_n)_n \subset \mathcal{D}(T)$ be a sequence of $\mathcal{F}(S_0,T,H)$. Then, $S_0x_n - Hx_n \in Tx_n \subset \mathcal{R}(T)$. Since $\mathcal{R}(T)$ is τ-compact, there exists a subsequence $x_{\varphi(n)} \subset \mathcal{D}(T)$ such that $S_0x_n - Hx_n \xrightarrow{\tau} y$, $y \in X$. Using the S_0-τ-demicompactness of H, we deduce the existence of a subsequence $(x_{\varphi\circ\psi(n)})_n$ of $(x_{\varphi(n)})_n$ such that $x_{\varphi\circ\psi(n)}$ τ-converges to some $x \in X$. From the τ-closedness of $S_0 - H$, we obtain $x \in \mathcal{D}(T)$ and $y = S_0x - Hx$. Now, since $Gr(T)$ is τ-sequentially closed, we deduce that $x \in \mathcal{F}(S_0,T,H)$ and so the result follows from the angelicity of compact sets in (X,τ). □

Now, we give a sufficient condition to an nonlinear operator to be τ-relative demicompact with respect to a given nonlinear operator.

Theorem 4. *Let $T : \mathcal{D}(T) \subset X \longrightarrow X$, $S_0 : \mathcal{D}(S_0) \subset X \longrightarrow X$ be two single-valued operators with $\mathcal{D}(T) \subset \mathcal{D}(S_0)$ such that $S_0 - T$ have a τ-sequentially closed graph. Assume that for every τ-closed convex, bounded set D, the multi-valued map*

$$F_D\colon X \longrightarrow \mathcal{P}(X)$$
$$y \longrightarrow D \cap (S_0 - T)^{-1}y,$$

is compact-valued and τ-upper-semicontinuous. Then, the operator T is τ-S_0-demicompact.

Proof. Let $y_n := (S_0 - T)x_n$ be a τ-converging sequence and assume that $(x_n)_n$ is included in the τ-closed, and bounded set D. The τ-upper-semicontinuity of $y \longmapsto D \cap (S_0 - T)^{-1}y$ implies that $(x_n)_n$ τ-converges to the set $D \cap (S_0 - T)^{-1}y_0$ where $y_0 = \lim y_n$, with respect to the topology τ. The τ-compactness of $D \cap (S_0 - T)^{-1}y_0$ implies that there exists a subsequence of $(x_n)_n$ that converges τ-to an element $x \in D \cap (S_0 - T)^{-1}y_0$. Therefore, T is τ- S_0-demicompact. □

Remark 7. The above results extended the result in [14] in Hausdorff locally convex space endowed with topology τ.

3. Fixed point theorems

Let us give the following two important lemmas needed in our considerations.

Lemma 3. *Let (X,τ) be an angelic space, and let M be a τ-compact subset of X. Then, any sequentially τ-upper semicontinuous multivalued mapping $A : M \to \mathcal{P}(X)$ is τ-upper semicontinuous.*

Proof. Let C be a τ-closed subset of X. We have that A is sequentially τ-upper semicontinuous, then $A^{-1}(C)$ is τ-sequentially closed in X. Since C is τ-compact and $\overline{A^{-1}(C)}^{\tau} \subset C$, so $\overline{A^{-1}(C)}^{\tau}$ is τ-compact.

Now, we show that $\overline{A^{-1}(C)}^{\tau}$ is τ-closed. Let $x \in \overline{A^{-1}(C)}^{\tau}$, by the angelicity of X, there exists a sequence $(x_n)_n \in A^{-1}(C)$ such that $x_n \xrightarrow{\tau} x$. Also, since $A^{-1}(C)$ is τ-sequentially closed, we have $x \in A^{-1}(C)$. Hence, $\overline{A^{-1}(C)}^{\tau} = A^{-1}(C)$. Then, $A^{-1}(C)$ is τ-closed and the multivalued operator A is τ-upper semicontinuous. □

Lemma 4. *Assume that (X,τ) is angelic space. Let K and M be two τ-compact subsets of X. Then, every τ-sequentially closed multivalued mapping $A : K \to \mathcal{P}(M)$ is τ-upper semicontinuous.*

Proof. In view of Lemma 3 it is sufficient to prove that $A : K \to \mathcal{P}(M)$ is sequentially τ-upper semicontinuous. We have that $G_r(A) \subset K \times M$. Thus, $\overline{G_r(A)}^{\tau} \subset K \times M$. By hypothesis, we obtain that $G_r(A)$ is τ-compact. Using the angelicity of the space X, we get $G_r(A)$ is sequentially τ-compact. Consequentially, $G_r(A)$ is sequentially τ-closed.

Let $C \subset K$ be a τ-closed set. Now, we claim that $A^{-1}(C)$ is sequentially τ-closed. To prove that, let $(x_n)_n \in A^{-1}(C)$ with $x_n \xrightarrow{\tau} x$. Since $A(x_n) \cap C \neq \varnothing$, there exists $(y_n)_n \in A(x_n) \cap C$ and $y_n \in A(x_n) \subset M$. Since M is τ-compact, we may assume without loss of generality that $y_n \xrightarrow{\tau} y$, $y \in C$. Then, for every n, we have $(x_n, y_n)_n \in G_r(A)$ and

$$(x_n, y_n) \xrightarrow{\tau} (x,y),$$

since $G_r(A)$ is sequentially τ-closed subset of $X \times X$, so $(x,y) \in G_r(A)$. Thus, $y \in A(x) \cap C$. $x \in A^{-1}(C)$ and so $y \in A^{-1}(C)$. This prove that $A^{-1}(C)$ is sequentially τ-closed. □

Now, we are ready to stat our first result.

Theorem 5. *Let X be a Hausdorff locally convex space. Assume that (X,τ) is angelic space. Let M be a τ-compact convex subset of X. Suppose that the multivalued operator $A : M \to \mathcal{P}_{cv,\tau-cl}(M)$ is sequentially τ-upper semicontinuous. Then, A has a fixed point in M.*

Proof. From Lemma 3 it follows that A is τ-upper semicontinuous and Fan-Glicksberg fixed point theorem [9] guarantees that A has a fixed point in M. □

Remark 8. Not that Theorem 5 generalize Arino-Gautier-Penot fixed point theorem [3, see Remark 1] and Theorem 2.1 [1].

The following result is a version of Himmelberg fixed point theorem [12] for sequentially τ-upper semicontinuous multivalued operator in Hausdorff locally convex space.

Theorem 6. *Let X be a Hausdorff locally convex space. Assume that (X,τ) is angelic space with τ-KS property. Let M be a nonempty τ-closed convex subset of X. Assume that $A: M \to \mathcal{P}_{cv,\tau-cl}(M)$ is sequentially τ-upper semicontinuous multivalued operator. If $A(M)$ is relatively τ-compact, then A has a fixed point in M.*

Proof. Since $A(M)$ is relatively τ-compact and by the τ-KS property of X, we infer that the subset $K := \overline{conv^{\tau}}(A(M))$ is also τ-compact and $A(K) \subset K$. Using Theorem 5, we deduce that A has a fixed point in $K \subset M$. □

Corollary 1. *Let X be a Hausdorff locally convex space. Assume that (X,τ) is angelic space with τ-KS property. Let M be a τ-closed, convex subset of X. Let $A: M \to \mathcal{P}_{cv,\tau-cl}(M)$ be a multivalued operator with a τ-sequentially closed graph and $A(M)$ is relatively τ-compact. Then, A has a fixed point in M.*

Proof. Let $K := \overline{conv^{\tau}}(A(M))$ and by hypothesis is a τ-compact subset of M and $A(K) \subset K$. Then, by Lemma 4 A is sequentially τ-upper semicontinuous. By Theorem 5 the multivalued operator A has a fixed point in $K \subset M$. □

Corollary 2. *(X,τ) is angelic with τ-KS property. Let M be a τ-closed, bounded and convex subset of X. Assume that $A: M \to \mathcal{P}_{cv,\tau-cl}(M)$ has a τ-sequentially closed graph and Φ^{τ}_{Λ}-contraction (or Φ^{τ}_{Λ}-condensing). Then, A has a fixed point.*

Now, we will study the existence of solution of Eq. (2.1) under some conditions for the operators T, S_0, and H.

Theorem 7. *Let X be a Hausdorff locally convex space. Assume that (X,τ) is angelic and M is a τ-compact convex subset of X. Assume that $T: M \to \mathcal{P}_{cv}(X)$ is a multivalued operator, and S_0, H are two operators on X, such that $M \subset D(S_0) \cap D(H)$. Suppose that:*

(i) T have a τ-sequentially closed graph,
(ii) S_0 is invertible,
(iii) S_0 and H are τ-sequentially continuous, and
(iv) (iv) $(T+H)(M) \subset S_0(M)$.

Then, $\mathcal{F}(S_0,T,H)$ is nonempty τ-sequentially closed subset of X.

Proof. In view of assumptions (ii) and (iv) the multivalued operator $\psi: M \to \mathcal{P}_{cv}(M)$ defined by $\psi(x) = S_0^{-1}(T+H)(x)$, $x \in M$ is well defined. Now, we show that

ψ has a τ-sequentially closed graph. Let $(x_n)_n$ be a sequence in M with $x_n \xrightarrow{\tau} x$, and let $y_n \in \psi(x_n)$ with $y_n \xrightarrow{\tau} y$. Since M is τ-compact then $x \in M$, and $y_n = S_0^{-1}(T+H)x_n$ which imply that $S_0 y_n = Tx_n + Hx_n$, and by assumption (iii) we get

$$Tx_n \xrightarrow{\tau} S_0 y - Hx.$$

Since T is τ-sequentially closed graph, we get $S_0 y - Hx \in Tx$ and so $y \in \psi(x)$. This prove that ψ has τ-sequentially closed graph. Hence, by Lemma 4 and Theorem 5 the multivalued operator $\psi : M \to \mathcal{P}_{cv,\tau-cl}(M)$ has a fixed point in M i.e, there exists $x \in M$ such that $S_0 x \in Tx + Hx$. Hence, $\mathcal{F}(S_0, T, H)$ is nonempty and by Theorem 1 it is τ-sequentially closed. This achieve the proof. □

Theorem 8. *Let X be a Hausdorff locally convex space. Assume that (X,τ) is angelic space with τ-KS property. Let M is τ-closed convex subset of X. Assume that $T : M \to \mathcal{P}_{cv}(M)$ is a multivalued operator on X and S_0, H are two singled valued operators on X, such that $M \subset \mathcal{D}(S_0) \cap \mathcal{D}(H)$. Suppose that the following assumptions verified*

(i) T have a τ-sequentially closed graph and τ-compact,

(ii) S_0 is invertible,

(iii) S_0 and H are τ-sequentially continuous, and

(iv) $(T+H)(C) \subset S_0(C)$ for each τ-compact subset C of M.

Then, $\mathcal{F}(S_0, T, H)$ is nonempty sequentially τ-closed subset of X.

Proof. Let $K := \overline{\text{conv.}}^{\tau}(T(M))$. The τ-KS property guarantees that K is τ-compact. Using assumptions (ii) and (iv), we show that the multivalued operator $\psi : K \to \mathcal{P}_{cv}(K)$ by $\psi(x) = S_0^{-1}(T+H)(x), x \in K$ is well defined, and as we shown in the proof of Theorem 7, ψ has a τ-sequentially closed graph. By Corollary1, the operator ψ has a fixed point in K, so there exists $x \in K$ such that $x = S_0^{-1}(T+H)x$. Then, $\mathcal{F}(S_0, T, H)$ is nonempty and by Theorem 2 it is τ-sequentially closed subset of X. □

Theorem 9. *Let X be a Hausdorff locally convex space. Assume that (X,τ) is angelic with τ-KS property. Consider $\Phi_\Lambda^\tau = \{\phi_{p_\tau}, p \in \Lambda\}$ is a regular and subadditive Φ_Λ^τ-MNC. Let M be a nonempty, τ-closed, bounded and convex subset of X, and let T and H be two single valued operators such that $M \subset D(T) \subset D(H)$. If T is Φ_Λ^τ-condensing and H is τ-compact, then the following assertions hold*

(i) For every bounded subset D of X, $D \cap \mathcal{F}(I,T,H)$ is relatively τ-compact.

(ii) If the operators T and H are τ-closed, then $\mathcal{F}(I,T,H)$ is nonempty subset of X.

Proof.

(i) Let D be a bounded subset of X. Since $x \in \mathcal{F}(I,T,H)$ means that $x = Tx + Hx$, it follows that

$$\mathcal{F}(I,T,H) \cap D \subset T(\mathcal{F}(I,T,H) \cap D) + H(\mathcal{F}(I,T,H) \cap D).$$

From the τ-compactness of H and properties of Φ_Λ^τ-MNC, we deduce that

$$\phi_{p_\tau}(\mathcal{F}(I,T,H)\cap D) \leq \phi_{p_\tau}(T(\mathcal{F}(I,T,H)\cap D)).$$

If $\phi_{p_\tau}(\mathcal{F}(I,T,H)\cap D) > 0$, then since T is Φ_Λ^τ-condensing, we have

$$\phi_{p_\tau}(\mathcal{F}(I,T,H)\cap D) \leq \phi_{p_\tau}(T(\mathcal{F}(I,T,H)\cap D)) < \phi_{p_\tau}(\mathcal{F}(I,T,H)\cap D),$$

which is a contradiction. Thus $\phi_{p_\tau}(\mathcal{F}(I,T,H)\cap D) = 0$ so $\mathcal{F}(I,T,H)\cap D$ is relatively τ-compact.

(ii) Clearly, the operator $T+H$ has a τ-sequentially closed graph. Let B be a bounded subset of M. Then, we have

$$\phi_{p_\tau}(T+H)(B) \leq \phi_{p_\tau}(T(B)) + \phi_{p_\tau}(H(B)),$$

Since H is τ-compact, we get

$$\phi_{p_\tau}(T+H)(B) < \phi_{p_\tau}(H(B)),$$

Hence, if $\phi_{p_\tau}(B) > 0$, we obtain

$$\phi_{p_\tau}(T+H)(B) < \phi_{p_\tau}(B),$$

this show that $T+H$ is Φ_Λ^τ-condensing. Using Corollary 1, there exists $x\in M$ such that $x = Tx+Hx$. So, $\mathcal{F}(I,T,H)$ is nonempty.

□

Theorem 10. *Assume that (X,τ) is angelic with τ-KS property. Let M be a nonempty τ-closed, convex of X, and let $T : \mathcal{D}(T)\subset X \longrightarrow X$, $S_0 : \mathcal{D}(S_0)\subset X \longrightarrow X$ be two single-valued τ-closed operators with $M\subset \mathcal{D}(T)\subset \mathcal{D}(S_0)$ such that S_0-T is τ-closed. If T is τ-S_0-demicompact, then for every τ-closed, convex bounded set D of M, the multi-valued map*

$$F_D : M \longrightarrow \mathcal{P}(M),$$
$$y \longrightarrow D\cap (S_0-T)^{-1}y,$$

has a fixed point on M.

Proof. Suppose T is τ-S_0-demicompact. First we show F_D is τ-upper-semicontinuous. Let $(y_n)_n$ be a sequence with $y_n \xrightarrow{\tau} y$ and Q a τ-open set such that $F_D(y)\subset Q$. From Lemma 1 it suffices to show the existence of $n_0\in\mathbb{N}$ such that $F_D(y_n)\subset Q$, $\forall n\geq n_0$. If not, then there exists a subsequence $(x_{\varphi(n)})_n$ of $(x_n)_n$ (here $x_n\in D\cap (S_0-T)^{-1}y_n$) such that $x_{\varphi(n)}\in D\cap(S_0-T)^{-1}y_n$ and $x_{\varphi(n)}\notin Q$. Note $S_0x_{\varphi(n)} - Tx_{\varphi(n)}$ τ-converges to y. Since T is τ-S_0-demicompact we infer that there exists a subsequence $(x_{\varphi\circ\psi(n)})_n$ of $(x_{\varphi(n)})_n$ such that $(x_{\varphi\circ\psi(n)})_n \xrightarrow{\tau} x$, $x\in X$. Since D is a τ-closed and convex subset of X we deduce that $x\in D$. Moreover, taking into account that $(S_0-T)x_{\varphi\circ\psi(n)} \xrightarrow{\tau} y$ and S_0-T is a τ-closed mappings, we deduce that $y=(S_0-T)x$. Consequently, $x\in D\cap(S_0-T)^{-1}y = F_D(y)$. Therefore, $x_{\varphi\circ\psi(n)}\in Q$

for n large enough, a contradiction to the construction of $(x_{\varphi(n)})_n$. Thus F_D is τ-upper-semicontinuous. Next fix $y \in X$. Since T is τ-S_0-demicompact then if $(x_n)_n$ is a sequence in $D \cap (S_0 - T)^{-1}y$, then it has a τ-converging subsequence, and so the τ-closedness of $S_0 - T$ implies that the limit is also in $D \cap (S_0 - T)^{-1}y$. Thus, $D \cap (S_0 - T)^{-1}y$ is τ-compact. By Himmelberg fixed point theorem [12], there exist $x \in M$ such that $x \in F_D(x)$. Then $\mathcal{F}(S_0, T, I)$ is nonempty subset. □

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